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A CHARACTERIZATION OF ALMOST-EINSTEIN REAL HYPERSURFACES OF QUATERNIONIC PROJECTIVE SPACE

By

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Abstract. Almost-Einstein real hypersurfaces of quaternionic projective space, as defined in [3], can be characterized by a condition involving their curvature and Ricci tensors.

1. Introduction

Let M be a connected real hypersurface of a quaternionic projective space QP^m, $m \geq 3$, with metric g of constant quaternionic sectional curvature 4. let ξ be the unit local normal vector field on M and $\{J_1, J_2, J_3\}$ a local basis of the quaternionic structure of QP^m, [2]. Then $U_i = -J_i\xi$, $i = 1, 2, 3$, are tangent to M. Let us denote by $\boldsymbol{D}^\perp = \mathrm{Span}\{U_1, U_2, U_3\}$ and by $\boldsymbol{D}$ its orthogonal complement in TM.

Let A be the Weingarten endomorphism of M and S its Ricci tensor. M is said to be almost-Einstein, [3], if

$$SX = aX + b\sum_{i=1}^{3} g(AX, U_i)U_i \tag{1.1}$$

for any $X \in TM$, where a and b are constant. In [3] we studied such real hypersurfaces obtaining

THEOREM A. *Let M be an almost-Einstein real hypersurface of QP^m, $m \geq 2$. Then it is an open subset of one of the following*:

i) *a geodesic hypersphere.*

ii) *a tube of radius r over QP^k, $0 < k < m-1$, $0 < r < \pi/2$ and* $\cot^2(r) = (4k+2)/(4m-4k-2)$.

iii) *a tube of radius r over CP^m, $0 < r < \pi/4$ and* $\cot^2(2r) = 1/(m-1)$.

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Among the real hypersurfaces appearing in Theorem A, only the geodesic hyperspheres of radius r, $0 < r < \pi/2$ and $\cot^2(r) = 1/(2m)$ are Einstein.

Recently, in [4] we studied real hypersurfaces of QP^m, $m \geq 2$, such that $\sigma(R(X, Y)SZ) = 0$, for any X, Y, Z tangent to M, where σ denotes the cyclic sum and R the curvature tensor of M. Concretely we obtained

Theorem B. *A real hypersurface M of QP^m, $m \geq 2$, satisfies $\sigma(R(X, Y)SZ) = 0$, for any X, Y, Z tangent to M if and only if it is Einstein.*

In the present paper we propose to study a weaker condition than the one appearing in Theorem B. Concretely we shall consider real hypersurfaces M of QP^m, $m \geq 3$, satisfying

$$\sigma(R(X, Y)SZ) = 0 \quad \text{for any } X, Y, Z \in \boldsymbol{D} \tag{1.2}$$

It is easy to see, bearing in mind the first identity of Bianchi, that all almost-Einstein real hypersurfaces of QP^m satisfy (1.2). Our purpose is to obtain the converse. That is, we shall prove the following

Theorem. *A real hypersurface M of QP^m, $m \geq 3$, satisfies* (1.2) *if and only if it is almost-Einstein.*

2. Preliminaries

Let X be a vector field tangent to M. We write $J_iX = \Phi_iX + f_i(X)\xi$, $i = 1, 2, 3$, where Φ_iX denotes the tangential component of J_iX and $f_i(X) = g(X, U_i)$. From this, [3], we have

$$g(\Phi_iX, Y) + g(X, \Phi_iY) = 0, \quad \Phi_iU_i = 0, \quad \Phi_jU_k = -\Phi_kU_j = U_t \tag{2.1}$$

for any X, Y tangent to M, $i = 1, 2, 3$, (j, k, t) being a cyclic permutation of (1, 2, 3). We also obtain

$$\Phi_i\Phi_jX = -\Phi_j\Phi_iX = \Phi_kX \tag{2.2}$$

for any $X \in \boldsymbol{D}$, where (i, j, k) is a cyclic permutation of (1, 2, 3).

From the expression of the curvature tensor of QP^m, [2], the equation of Gauss is given by

$$\begin{aligned} R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + \sum_{i=1}^{3}\{g(\Phi_iY, Z)\Phi_iX - g(\Phi_iX, Z)\Phi_iY \\ + 2g(X, \Phi_iY)\Phi_iZ\} + g(AY, Z)AX - g(AX, Z)AY \end{aligned} \tag{2.3}$$

for any X, Y, Z tangent to M. This implies that the Ricci tensor of M has the following expression:

$$SX = (4m+7)X - 3\sum_{i=1}^{3} f_i(X)U_i + HX \tag{2.4}$$

for any X tangent to M, where $H = (\text{trace } A)A - A^2$.

3. Proof of the Theorem

Along this paragraph M will denote a real hypersurface of QP^m, $m \geq 3$, satisfying (1.2).

From (2.4) and the first identity of Bianchi, (1.2) is equivalent to have $\sigma(R(X,Y)HZ) = 0$ for any $X, Y, Z \in \boldsymbol{D}$.

Let $\{E_1, \dots, E_{4m-4}\}$ be a local orthonormal frame of $\boldsymbol{D}$ at any point of M. The following computations are made locally on a neighbourhood of any point of M.

If from (2.3) we develop $\sigma(R(X,Y)HZ) = 0$ and take $Z = E_j$, $Y = \Phi_1 E_j$, $j = 1, \dots, 4m-4$, we have

$$\begin{aligned}
&-(g(E_j, HE_j) + g(\Phi_1 E_j, H\Phi_1 E_j))\Phi_1 X - (g(\Phi_3 E_j, HE_j) \\
&+ g(\Phi_2 E_j, H\Phi_1 E_j))\Phi_2 X + (g(\Phi_2 E_j, HE_j) - g(\Phi_3 E_j, H\Phi_1 E_j))\Phi_3 X \\
&+ 2\Phi_1 HX + (g(\Phi_1 X, HE_j) - g(HX, \Phi_1 E_j))E_j + (g(HX, E_j) \\
&+ g(\Phi_1 X, H\Phi_1 E_j))\Phi_1 E_j + (2g(HX, \Phi_3 E_j) + g(\Phi_2 X, H\Phi_1 E_j) \\
&- g(\Phi_3 X, HE_j))\Phi_2 E_j + (g(\Phi_2 X, HE_j) + g(\Phi_3 X, H\Phi_1 E_j) \\
&- 2g(HX, \Phi_2 E_j))\Phi_3 E_j - 2g(X, E_j)\Phi_1 HE_j \\
&- 2g(X, \Phi_3 E_j)\Phi_2 HE_j + 2g(X, \Phi_2 E_j)\Phi_3 HE_j \\
&+ 2g(\Phi_1 X, E_j)\Phi_1 H\Phi_1 E_j + 2g(\Phi_2 X, E_j)\Phi_2 H\Phi_1 E_j \\
&+ 2g(\Phi_3 X, E_j)\Phi_3 H\Phi_1 E_j = 0
\end{aligned} \tag{3.1}$$

for any $X \in \boldsymbol{D}$.

Now we prepare the following Lemmas

Lemma 1. *$g(HX, \Phi_i X) = 0$ for any $X \in \boldsymbol{D}$, $i = 1, 2, 3$.*

PROOF. We take the scalar product of (3.1) and X and take summation on j. Then we obtain

$$(8m-16)g(\Phi_1 HX, X) = 0 \tag{3.2}$$

for any $X \in \boldsymbol{D}$. As $m \geq 3$, (3.2) implies $g(HX, \Phi_1 X) = 0$.

If we develop $\sigma(R(X, \Phi_i E_j)HE_j) = 0$, $i = 2, 3$, we also obtain $g(HX, \Phi_2 X) = g(HX, \Phi_3 X) = 0$, finishing the proof.

Let us denote by $Q(X) = \mathrm{Span}\{X, \Phi_1 X, \Phi_2 X, \Phi_3 X\}$ for any $X \in TM$.

LEMMA 2. *$g(X, HZ) = 0$ for any unit $X, Z \in \boldsymbol{D}$ such that $Q(X) \perp Q(Z)$.*

PROOF. Let us consider $X, Y \in \boldsymbol{D}$. From Lemma 1 and polarization we have

$$g(H\Phi_i X, Y) = g(\Phi_i HX, Y) \quad i = 1, 2, 3 \tag{3.3}$$

for any $X, Y \in \boldsymbol{D}$. Taking in (3.3) $Y = \Phi_i Z$, $i = 1, 2, 3$ we obtain

$$g(H\Phi_i X, \Phi_i Z) = g(HX, Z) \quad i = 1, 2, 3 \tag{3.4}$$

Take the scalar product of (3.1) and Z and then summation on j. We have

(3.5)
$$g(H\Phi_1 X, Z) + (4m-7)g(\Phi_1 HX, Z) + g(\Phi_2 X, H\Phi_3 Z) - g(\Phi_3 X, H\Phi_2 Z) = 0$$

for any unit $X, Z \in \boldsymbol{D}$ such that $Q(X) \perp Q(Z)$. If in (3.5) we exchange Z by $\Phi_1 Z$ and apply (3.4) we obtain

$$(4m-4)g(HX, Z) = 0 \tag{3.6}$$

Now as $m \geq 3$ the result follows.

LEMMA 3. *$g(HX, X) = g(HY, Y)$ for any nonnull $X, Y \in \boldsymbol{D}$.*

PROOF. Let us take a unit $X \in \boldsymbol{D}$ and consider the scalar product of (3.1) and $\Phi_1 X$. After taking summation on j we have

$$\begin{aligned}(3.12)\qquad & (8m-14)g(HX, X) + 2g(H\Phi_1 X, \Phi_1 X) + 2g(H\Phi_2 X, \Phi_2 X) \\ & + 2g(H\Phi_3 X, \Phi_3 X) - \sum_j \{g(E_j, HE_j) + g(\Phi_1 E_j, H\Phi_1 E_j)\} = 0\end{aligned}$$

If in (3.12) we change X by $\Phi_1 X$ and substract we have

$$(8m-16)g(HX, X) = (8m-16)g(H\Phi_1 X, \Phi_1 X) \tag{3.13}$$

As $m \geq 3$, we obtain $g(HX, X) = g(H\Phi_1 X, \Phi_1 X)$. Similarly we can obtain

$$g(HX, X) = g(H\Phi_i X, \Phi_i X) \quad i = 1, 2, 3 \tag{3.14}$$

Now from (3.12) and (3.14) we get

$$(4m - 4)g(HX, X) = \sum_j g(HE_j, E_j) \tag{3.15}$$

and this finishes the proof.

Lemma 4. *$g(HU_i, X) = 0$, $i = 1, 2, 3$, for any $X \in \boldsymbol{D}$.*

Proof. Let us take the scalar product of (3.1) and U_1 and sum on j. Thus we have

$$g(\Phi_2 X, HU_2) + g(\Phi_3 X, HU_3) = 0 \tag{3.16}$$

Similarly we can obtain

$$g(\Phi_1 X, HU_1) + g(\Phi_3 X, HU_3) = 0 \tag{3.17}$$

and

$$g(\Phi_1 X, HU_1) + g(\Phi_2 X, HU_2) = 0 \tag{3.18}$$

From (3.16), (3.17) and (3.18) we get

$$g(\Phi_i X, HU_i) = 0, \quad i = 1, 2, 3 \tag{3.19}$$

and changing X by $\Phi_i X$ we obtain the result.

Now we have that any $X \in \boldsymbol{D}$ is principal for H and has the same eigenvalue. Moreover $g(H\boldsymbol{D}, \boldsymbol{D}^{\perp}) = \{0\}$. But $HA = AH$. Thus we can find an orthonormal basis of $T_x M$, for any $x \in M$, such that it diagonalizes simultaneously both H and A. But from the above Lemmas we must have $g(A\boldsymbol{D}, \boldsymbol{D}^{\perp}) = \{0\}$. Thus M, [1], must be congruent to an open subset of either a geodesic hypersphere or a tube of radius r, $0 < r < \pi/2$, over QP^k, $k \in \{1, \dots, m-2\}$ or a tube of radius r, $0 < r < \pi/4$, over CP^m.

All geodesic hyperspheres only have a principal curvature on $\boldsymbol{D}$, [3]. Thus from the first identity of Bianchi they satisfy (1.2).

A tube of radius r, $0 < r < \pi/2$, over QP^k, $k \in \{1, \dots, m-2\}$, has two distinct principal curvatures on $\boldsymbol{D}$, $\cot(r)$ with multiplicity $4(m-k-1)$ and $-\tan(r)$ with multiplicity $4k$, and a unique principal curvature on $\boldsymbol{D}^{\perp}$, $2\cot(2r)$, [3]. Let us suppose that it satisfies (1.2). Thus from Lemma 3 every vector field of D must have the same eigenvalue for H. Take $X \in D$ such that $AX = \cot(r)X$ and $Z \in D$ such that $AZ = -\tan(r)Z$. Then $HX = ((4m-4k-2)\cot^2 r - (4k+3))X$ and $HZ = ((4k+2)\tan^2 r - (4m-4k-1))Z$. This implies that $\cot^2(r) = (4k+2)/(4m-4k-2)$.

A similar argument applied to a tube of radius r, $0 < r < \pi/4$, over CP^m, whose principal curvatures are $\cot(r)$ and $-\tan(r)$ on D both with multiplicity $2(m-1)$ and $2\cot(2r)$ with multiplicity 1 and $-2\tan(2r)$ with multiplicity 2 on $D^{\perp}$ implies that (1.2) is satisfied only if $\cot^2(2r) = 1/(m-1)$.

Thus we have proved that a real hypersurface of QP^m, $m \geq 3$, satisfies (1.2) if and only if it is one appearing in Theorem A. This finishes the proof.

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