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PL APPROXIMATIONS OF FIBER PRESERVING HOMEOMORPHISMS OF VECTOR BUNDLES

By

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Abstract. We investigate the group of f.p. homeomorphisms of an n-dimensional vector bundle ξ. In the case $n \geq 5$ and the base space of ξ is countable dimensional, we show that every f.p. stable homeomorphism of ξ can be approximated by f.p. *PL* homeomorphisms with respect to the majorant topology. As an application we can show that if the base space is compact, then the group of f.p. *PL* homeomorphisms of ξ with the uniform topology has the mapping absorption property for maps from countable dimensional metric spaces into the group of f.p. homeomorphisms of ξ which are PL on the unit open ball.

1. Introduction

In [3] and [9] it is shown that in the case $n \geq 5$ any stable homeomorphism $f: \boldsymbol{R}^n \to \boldsymbol{R}^n$ is approximated by *PL* homeomorphisms. In this paper we extend this result to the case of f.p. stable homeomorphisms of vector bundles. To state the main results precisely, first we introduce some notations in the vector bundle setting.

Let $\xi \equiv (\pi: E \to X)$ be an n-dimensional real vector bundle with a Riemannian metric. Since each fiber $E_x = \pi^{-1}(x) (x \in X)$ is a real vector space, it admits the natural piecewise linear structure. Let $C_{\geq 0}(X)$ (resp. $C_{>0}(X)$) denote the set of all continuous functions from X to $[0, \infty)$ (resp. $(0, \infty)$). We define $B_x(a) \equiv \{u \in E_x : \|u\|_x \leq a(x)\}$ for $x \in X$ and $B(a) \equiv \bigcup_{x \in X} B_x(a)$. Similarly $O_x(a) \equiv \{u \in E_x : \|u\|_x < a(x)\}$ and $O(a) \equiv \bigcup_{x \in X} O_x(a)$.

Let $\mathscr{H}(\xi)$ be the group of fiber preserving (f.p.) homeomorphisms $f: E \to E$ (i.e., $\pi f = \pi$). By $\mathscr{H}^P(\xi)$ we denote the subgroup of $\mathscr{H}(\xi)$ consisting of the f.p.

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homeomorphisms of E which satisfy a condition P. In this paper we will consider the following conditions:

(i) $PL: f \in \mathscr{H}(\xi)$ is PL is for each $x \in X$, $f_x \equiv f|_{E_x}: E_x \to E_x$ is PL. More generally, it is said that f is PL on an open set $U \subset E$ if each f_x is PL on $U_x = U \cap E_x$.

(i)′ $PL_1: f \in \mathscr{H}(\xi)$ is PL_1 if f is PL on $O(1)$.

(ii) $S: f \in \mathscr{H}(\xi)$ is stable if $f = f_1 \dots f_k$, where each f_i is PL on an open neighborhood U_i of (the image $s_i(X)$ of) some section $s_i: X \to E$. In the case X is a one point and f is orientation preserving, then this definition coincides with the usual one (cf. [9, p 195, Exercise 4.11.1]).

(iii) $U: f \in \mathscr{H}(\xi)$ is an f.p. uniform homeomorphism if both f and f^{-1} are f.p. uniformly continuous, where an f.p. map $g: E \to E$ is an f.p. uniformly continuous if for each map $\varepsilon \in C_{>0}(x)$ there is a map $\delta \in C_{>0}(X)$ such that if $u, v \in E_x$ and $\|u - v\|_x < \delta(x)$ then $\|g(u) - g(v)\|_x < \varepsilon(x)$.

When $\mathscr{H}(\xi)$ (or $\mathscr{H}^P(\xi)$) is given a topology τ, it is denoted by $\mathscr{H}_\tau(\xi)$ (or $\mathscr{H}_\tau^P(\xi)$). We will be concerned with the following topologies:

(i) The f.p. majorant topology m: The neighborhood base of $f \in \mathscr{H}_m(\xi)$ is given by $N(f, \varepsilon) = \{g \in \mathscr{H}(\xi) : \|f - g\| < \varepsilon\}$ $(\varepsilon \in C_{>0}(E))$. Here, $\|f - g\| < \varepsilon$ means that $\|f(u) - g(u)\|_x < \varepsilon(u)$ for each $u \in E_x$.

(ii) The f.p. uniform topology u: The neighborhood base of $f \in \mathscr{H}_u(\xi)$ is given by $N(f, \varepsilon) = \{g \in \mathscr{H}(\xi) : \|f - g\| < \varepsilon\}$ $(\varepsilon \in C_{>0}(X))$. Here, $\|f - g\| < \varepsilon$ means that $\|f(u) - g(u)\|_x < \varepsilon(x)$ for each $u \in E_x$. Note that, if X is not totally bounded, then $N(f, \varepsilon)$ $(\varepsilon > 0)$ does not satisfy the axiom of neighborhood base, that is, for $g \in N(f, \varepsilon)$ there exists no $\delta > 0$ such that $N(g, \delta) \subset N(f, \varepsilon)$. This is the reason that we need to use functions $\varepsilon: X \to (0, \infty)$.

(iii) The compact-open topology c: The neighborhood base of $f \in \mathscr{H}_c(\xi)$ is given by $N(f, a, \varepsilon) = \{g \in \mathscr{H}(\xi) : \|f - g\| < \varepsilon \text{ on } B(a)\}$ $(a, \varepsilon \in C_{>0}(X))$. Here, $\|f - g\| < \varepsilon$ on $B(a)$ means that $\|f(u) - g(u)\|_x < \varepsilon(x)$ for each $u \in B_x(a)$.

We refer to [9, Ch 5. §6] for topologies on groups of non-f.p. homeomorphisms.

It should be remarked that $\mathscr{H}_m(\xi)$, $\mathscr{H}_u^U(\xi)$ and $\mathscr{H}_c(\xi)$ are topological groups, but $\mathscr{H}_u(\xi)$ is not a topological group even if X is one point. (See Fact 7, 8, 9, 10 in §2 and compare with [9, p 272, Exercise 5.6.2].)

The following is the main result of the paper.

THEOREM. *Let $\xi \equiv (\pi : E \to X)$ be an n-dimensional vector bundle over X. If X is a countable dimensional metrizable space and $n \geq 5$, then $\mathscr{H}_m^{PL}(\xi)$ is dense in $\mathscr{H}_m^S(\xi)$ (i.e., if $g: E \to E$ is a f.p. stable homeomorphism, then for any map*

$\varepsilon : E \to (0, \infty)$, *there is a f.p. PL homeomorphism* $f : E \to E$ *such that* $\|f - g\| < \varepsilon$).

The proof of Theorem is the f.p. version of the $\boldsymbol{R}^n$-case [3] & [9]. The key lemma is lemma 4 in §3, in which we will extend the basic engulfing lemma in $\boldsymbol{R}^n$ to a vector bundle with a base space of countable dimension.

This theorem derives a kind of absorption property of $\mathscr{H}^{PL}(\xi)$ in $\mathscr{H}(\xi)$. This is an important notion in infinite dimensional topology which has been studied in many references with many variations (cf. [1], [2], [4], [5], [7], [10], [11] etc.). In this paper we consider the following mapping absorption property. A subspace B of a space Y (or simply a pair (Y, B) of spaces) is said to have the mapping absorption property for a class $\mathscr{C}$ of spaces if for any $Z \in \mathscr{C}$ and any map $f : Z \to Y$ there is a homotopy $f_t : Z \to Y$ $(0 \le t \le 1)$ such that $f_0 = f$ and $f_t(z) \subset B$ for $0 < t \le 1$. We will consider the class of countable dimensional metrizable spaces. A metrizable space is countable dimensional if it is a countable union of 0-dimensional subspaces (cf. [6]).

Corollary. *Let* $\xi \equiv (\pi : E \to X)$ *be an n-dimensional vector bundle over* X. *If* $n \ge 5$ *and* X *is a countable dimensional compact metrizable space, then the following pairs have the mapping absorption property for maps from countable dimensional metrizable spaces*:

(i) $(\mathscr{H}_u^{PL_1}(\xi),\ \mathscr{H}_u^{PL}(\xi))$,

(ii) $(\mathscr{H}_u^{PL_1, U}(\xi),\ \mathscr{H}_u^{PL, U}(\xi))$.

In [7] it is shown that for any compact PL manifold M^n ($n \ne 4$; if $n = 5$ suppose $\partial M = \emptyset$), the closure $\overline{\mathscr{H}_c^{PL}(M)}$ of $\mathscr{H}_c^{PL}(M)$ is a union of some components of $\mathscr{H}_c(M)$ and $\mathscr{H}_c^{PL}(M)$ has the finite dimensional compact absorption property in $\overline{\mathscr{H}_c^{PL}(M)}$ (see [7] for the definition). It should be noted that the proof is based on the uniform local contractibility of $\mathscr{H}_c(M)$ and $\mathscr{H}_c^{PL}(M)$. As for a non-compact manifold M, $\mathscr{H}_m(M)$ is always not locally path-connected and $\mathscr{H}_c(M)$, $\mathscr{H}_u(M)$ are not in general locally path-connected. In the special case that M is the interior of a compact manifold $\overline{M}$, for example $M = \boldsymbol{R}^n$, $\mathscr{H}_c(M)$ and $\mathscr{H}_u(M)$ are locally contractible, where the uniform topology u is induced by the metric of M which is the restriction of a metric of $\overline{M}$ (cf. [9, Ch 5. §6]). The author has no references for the local contractibility of $\mathscr{H}_u^U(\boldsymbol{R}^n)$ with respect to the usual metric.

In contrast to [7], we will reduce Corollary to Theorem by crossing the countable dimensional space $\times(0, 1]$ to ξ. More precisely, given any map

$f_0 : Y \to \mathscr{H}(\xi)$, we have the corresponding f.p. homeomorphism $F : Y \times E \to Y \times E$ over $Y \times X$ defined by $F(y,u) = (y, f_0(y)(u))$. Then the f.p. PL approximation of F gives a map $f_1 : Y \to \mathscr{H}^{PL}(\xi)$ which approximates f_0. The additional $(0,1]$ factor is necessary to obtain a homotopy f_t. In Corollary we replace the majorant topology m by the uniform topology u and assume that X is compact in order to ensure that f_t is continuous. The stable condition S is replaced by the rather restricted condition PL_1 to ensure that F is stable.

Finally we list some remaining problems:

(1) By the stable homeomorphism theorem ([8]), every (orientation preserving) homeomorphism of $\boldsymbol{R}^n$ is a stable homeomorphism. In the f.p. case, is any f.p. homeomorphism of vector bundle a f.p. stable homeomorphism (i.e., $\mathscr{H}(\xi) = \mathscr{H}^S(\xi)$)?

(2) Can one omit any technical assumptions in Corollary (the compactness of the base space X, the condition PL_1, etc.)? If $\mathscr{H}(\xi) = \mathscr{H}^S(\xi)$ for any n-dimensional vector bundle with a countable dimensional base space, then we can omit the PL_1 condition in Corollary.

(3) Are the groups $\mathscr{H}_u(\xi)$ and $\mathscr{H}_u^{PL}(\xi)$ ANR's for any appropriate class of spaces?

2. Preliminaries on vector bundles and f.p. homeomorphisms

First we list some notations which are used throughout the paper. All spaces are assumed to be metrizable. The n-dimensional Euclidean space is denoted by $\boldsymbol{R}^n$. The standard inner product $\langle\ ,\ \rangle$ defines the norm $\|u\|$ of a vector $u \in \boldsymbol{R}^n$ and the angle $\theta(u,v)$ of vectors $u, v \in \boldsymbol{R}^n \setminus \{0\}$. Note that $\theta(u,v) + \theta(v,w) \geq \theta(u,w)$ for $u,v,w \in \boldsymbol{R}^n \setminus \{0\}$. For $a \geq 0$, we set $B(a) \equiv \{u \in \boldsymbol{R}^n : \|u\| \leq a\}$ and $O(a) \equiv \{u \in \boldsymbol{R}^n : \|u\| < a\}$.

Let $\xi \equiv (\pi : E \to X)$ be an n-dimensional real vector bundle. For a subset A of X, $E_A \equiv \pi^{-1}(A)$. A Riemannian metric on ξ is a family $\{\langle\ ,\ \rangle_x\}_{x \in X}$ such that

(i) $\langle\ ,\ \rangle_x$ is an inner product of E_x for each $x \in X$,

(ii) For any sections $s, t : X \to E$ of π the map $\langle s, t \rangle : X \to R$ defined by $\langle s,t \rangle(x) = \langle s(x), t(x) \rangle_x (x \in X)$ is continuous.

The inner product $\langle\ ,\ \rangle_x$ defines the norm $\|\ \|_x$ and the angle θ_x on the vector space E_x. If E is trivial over U, then, using the Gram-Schmidt orthogonalization, there is a fiber preserving homeomorphism $\phi : E_U \to U \times \boldsymbol{R}^n$ over U such that for each $y \in U$, $\phi_y : (E_y, \langle\ ,\ \rangle_y) \to (\boldsymbol{R}^n, \langle\ ,\ \rangle)$ is an isometry (i.e., a linear isomorphism which preserves the inner products).

For $a, b \in C_{\geq 0}(X)$, $a \leq b$ (resp. $a < b$) means that $a(x) \leq b(x)$ (resp. $a(x) < b(x)$) for each $x \in X$. For a section s of ξ and subset A of X we set

$B_x(s,a) \equiv \{u \in E_x : \|u - s(x)\|_x \le a(x)\}$, $B_A(s,a) \equiv \bigcup_{x \in A} B_x(s,a)$ and $O_x(s,a) \equiv \{u \in E_x : \|u - s(x)\|_x < a(x)\}$, $O_A(s,a) \equiv \bigcup_{x \in A} O_x(s,a)$. Note that for each $u \in E_x$ and an open neighborhood U of u in E, there exist a section s of ξ, a constant $\varepsilon > 0$ and an open neighborhood W of x in X such that $s(x) = u$ and $O_W(s,\varepsilon) \subset U$.

Below we will list some basic facts and preliminary lemmas on vector bundles and f.p. homeomorphisms, which will be used in the process of proofs of Theorem and Corollary.

FACT 1. *If $a, b \in C_{\ge 0}(X)$, $a < b$ and $B_x(b) \backslash O_x(a)$ is contained in an open set U in E, then there is an open neighborhood V of x in X such that $B_V(b) \backslash O_V(a) \subset U$ (i.e., for each $y \in V$, $B_y(b) \backslash O_y(a) \subset U$).*

PROOF. By an isometric local trivialization, we may assume $E = W \times \boldsymbol{R}^n$, W is an open neighborhood of x in X. Use the compactness of $B_x(b) \backslash O_x(a)$. □

FACT 2. *If $\{U_\lambda\}_{\lambda \in \Lambda}$ is a locally finite open cover of X and $\delta_\lambda > 0$ $(\lambda \in \Lambda)$, then there is a map $\delta \in C_{>0}(X)$ such that $\delta(x) < \delta_\lambda$ for each $x \in U_\lambda$.*

PROOF. Each $x \in X$ has an open neighborhood V_x which meets only finitely many U_λ. Take an $\varepsilon_x > 0$ with $\varepsilon_x < \min\{\delta_\lambda : V_x \cap U_\lambda \ne \emptyset\}$. We take a partition of unity $\{\alpha_x\}$ subordinated to $\{V_x\}$ and define $\delta = \sum \varepsilon_x \alpha_x$. Let $y \in U_\lambda$. If $\alpha_x(y) \ne 0$ then $y \in V_x \cap U_\lambda \ne \emptyset$ and $\varepsilon_x < \delta_\lambda$. Hence $\delta(y) = \sum \varepsilon_x \alpha_x(y) < \sum \delta_\lambda \alpha_x(y) = \delta_\lambda$. □

FACT 3. *For any map $\delta \in C_{\ge 0}(E)$, the function $F : X \times [0,\infty) \to [0,\infty)$ defined by $F(x,s) = \min \delta(B_x(s))$ is continuous.*

PROOF. Observe that the relation $X \times [0,\infty) \ni (x,s) \mapsto B_x(s) \subset E$ is continuous, that is, for each $(x,s) \in X \times [0,\infty)$,

(i) if U is an open subset of E such that $B_x(s) \subset U$, then there exists a neighborhood V of (x,s) in $X \times [0,\infty)$ such that $B_y(t) \subset U$ for each $(y,t) \in V$, and

(ii) if U is an open subset of E such that $B_x(s) \cap U \ne \emptyset$, then there exists a neighborhood V of (x,s) in $X \times [0,\infty)$ such that $B_y(t) \cap U \ne \emptyset$ for each $(y,t) \in V$.

In fact, (i) follows from Fact 1 and (ii) is easy. The continuity of δ follows from this observation. □

FACT 4. *Let $f : E \to E$ be a f.p. map. Then for each $a \in C_{>0}(X)$, f is f.p. uniformly continuous on $B(a)$ (i.e., for every $\varepsilon \in C_{>0}(X)$ there exists a $\delta \in C_{>0}(X)$ such that if $u, v \in B_x(a)$ and $\|u - v\|_x < \delta(x)$, then $\|f(u) - f(v)\|_x < \varepsilon(x)$).*

PROOF. Let $\varepsilon \in C_{>0}(X)$. We will show that for each $x \in X$ there exist a neighborhood V_x of x in X and $\delta_x > 0$ such that if $y \in V_x$, $u, v \in B_y(a)$ and $\|u - v\|_y < \delta_x$ then $\|f(u) - f(v)\|_y < \varepsilon(y)$. Let $x \in X$. For each $u \in B_x(a)$ take a section t_u of ξ such that $t_u(x) = f(u)$ and consider the open set $O(t_u, \varepsilon/2)$. There exists a section s_u of ξ, a neighborhood V_u of x in X and $\delta_u > 0$ such that $s_u(x) = u$ and $f(O_{V_u}(s_u, 2\delta_u)) \subset 0(t_u, \varepsilon/2)$. Since $B_x(a)$ is compact, there exist finite $u_i \in B_x(a)$ such that $B_x(a) \subset \bigcup_i O_{V_{u_i}}(s_{u_1}, \delta_{u_i})$. By Fact 1, there is a neighborhood V_x of x such that $B_{V_x}(a) \subset \bigcup_i O_{V_{u_i}}(s_{u_i}, \delta_{u_i})$. Let $\delta_x = \min \delta_{u_i}$. If $y \in V_x$, $u, v \in B_y(a)$ and $\|u - v\|_y < \delta_x$, then $u \in O_{V_{u_i}}(s_{u_i}, \delta_{u_i})$ for some i and since $\delta_x \leq \delta_{u_i}$ we have $v \in O_{V_{u_i}}(s_{u_i}, 2\delta_{u_i})$. Hence $f(u)$, $f(v) \in O(t_{u_i}, \varepsilon/2)$, so that $\|f(u) - f(v)\|_y < \varepsilon(y)$.

Finally we take a locally finite refinement $\{U_x\}$ of $\{V_x\}$ with $U_x \subset V_x$ and apply Fact 2 to obtain the desired $\delta \in C_{>0}(X)$. □

FACT 5. *If $f : E \to E$ is a f.p. map and $f_x \in \mathscr{H}(E_x)$ for each $x \in X$ then $f \in \mathscr{H}(\xi)$.*

PROOF. We may assume $E = X \times \boldsymbol{R}^n$. Then the continuity of the inverse map $f^{-1} : (x, u) \mapsto (x, f_x^{-1}(u))$ follows from the following facts:

(i) Let $\mathscr{C}(\boldsymbol{R}^n)$ be the space of continuous maps from $\boldsymbol{R}^n$ into itself with the compact-open topology c. The homeomorphism group $\mathscr{H}_c(\boldsymbol{R}^n)$ with the compact-open topology is a subspace of $\mathscr{C}(\boldsymbol{R}^n)$ and it is a topological group.

(ii) A function $g : X \to \mathscr{C}(\boldsymbol{R}^n)$ is continuous iff $X \times \boldsymbol{R}^n \ni (x, u) \mapsto g_x(u) \in \boldsymbol{R}^n$ is continuous. □

Let $\mathscr{U}$ be an open cover of E. We say that f.p. maps $f, g : E \to E$ are $\mathscr{U}$-close and write as $(f, g) \leq \mathscr{U}$ if for each $u \in E$ there is a $U \in \mathscr{U}$ such that $f(u), g(u) \in U$. Set $N(f, \mathscr{U}) \equiv \{g \in \mathscr{H}(\xi) : (f, g) \leq \mathscr{U}\}$. The next fact shows that $\{N(f, \mathscr{U}) : \mathscr{U}$ is an open cover of $E\}$ forms a neighborhood base of f in $\mathscr{H}_m(\xi)$.

FACT 6. *Let $f : E \to E$ be a f.p. map.*

(i) *For each open cover $\mathscr{U}$ of E there exists an $\varepsilon \in C_{>0}(E)$ such that $\|f - g\| < \varepsilon f$ implies $(f, g) \leq \mathscr{U}$.*

(ii) *For each $\varepsilon \in C_{>0}(E)$ there exists an open cover $\mathscr{U}$ of E such that $(f, g) \leq \mathscr{U}$ implies $\|f - g\| < \varepsilon f$.*

In the case $f \in \mathscr{H}(\xi)$, (i) means that $N(f, \varepsilon f) \subset N(f, \mathscr{U})$, and applying (ii) to εf^{-1}, we have an open cover $\mathscr{U}$ of E such that $N(f, \mathscr{U}) \subset N(f, \varepsilon)$.

PROOF. (i) Let $\mathscr{U}$ be any open cover of E. It suffices to show that there exists an $\varepsilon \in C_{>0}(E)$ such that if $v, w \in E_y$ and $\|v - w\|_y < \varepsilon(v)$ then $v, w \in U$ for some $U \in \mathscr{U}$. For each $u \in E_x$, choose a $U_u \in \mathscr{U}$ with $u \in U_u$. There exists a section s_u of ξ, an open neighborhood V_u of x in X and $\varepsilon_u > 0$ such that $s_u(x) = u$ and $O_{V_u}(s_u, 2\varepsilon_u) \subset U_u$. There exists a locally finite open cover $\{W_u\}$ of E such that $W_u \subset O_{V_u}(s_u, \varepsilon_u)$ for each u. By Fact 2, there exists an $\varepsilon \in C_{>0}(E)$ such that $\varepsilon < \varepsilon_u$ on each W_u. If $v, w \in E_y$ and $\|v - w\|_y < \varepsilon(v)$ then $v \in W_u \subset O_{V_u}(s_u, \varepsilon_u)$ for some $u \in E_x$ and $\|v - w\|_y < \varepsilon(v) < \varepsilon_u$ and $\|v - s_u(v)\|_y < \varepsilon_u$. Hence $\|w - s(y)\| < 2\varepsilon_u$ and $v, w \in O_{V_u}(s_u, 2\varepsilon_u) \subset U_u$.

(ii) Let $\varepsilon \in C_{>0}(E)$ be any map. It suffices to show that there exists an open cover $\{U_u\}$ of E such that if $v, w \in E_y$ and $v, w \in U_u$ then $\|v - w\|_y < \varepsilon(v)$. For each $u \in E_x$, choose $\varepsilon_u > 0$ such that $2\varepsilon_u < \varepsilon(u)$ and take an open neighborhood W_u of u in E such that $\inf \varepsilon(W_u) > 2\varepsilon_u$. Then there exists a section s_u of ξ, an open neighborhood V_u of x in X and $\delta_u > 0$ such that $s_u(x) = u$, $\delta_u < \varepsilon_u$ and $U_u \equiv O_{V_u}(s_u, \delta_u) \subset W_u$. If $v, w \in E_y$ and $v, w \in U_u$, we have $\|v - w\|_y < 2\delta_u < 2\varepsilon_u < \varepsilon(v)$. □

FACT 7. *$\mathscr{H}_m(\xi)$ is a topological group.*

PROOF. We must show that the composition and the inverse map are continuous. By Fact 6 we can use the neighborhood bases measured by open covers of E.

(i) Let $f, g \in \mathscr{H}_m(\xi)$ and $\mathscr{U}$ is an open cover of E. Since E is assumed to be metrizable, there is an open cover $\mathscr{V}$ which is a star refinement of $\mathscr{U}$ (cf. [6]). If $f' \in N(f, g^{-1}(\mathscr{V}))$ and $g' \in N(g, \mathscr{V})$ then $(gf, gf') \leq \mathscr{V}$ and $(gf', g'f') \leq \mathscr{V}$, hence $(gf, g'f') \leq \mathscr{U}$, which means $g'f' \in N(gf, \mathscr{U})$.

(ii) Let $f \in \mathscr{H}_m(\xi)$ and $\mathscr{U}$ be an open cover of E. If $g \in N(f, f(\mathscr{U}))$, then $(f^{-1}g, g^{-1}g) = (f^{-1}g, f^{-1}f) \leq \mathscr{U}$, hence $(f^{-1}, g^{-1}) \leq \mathscr{U}$, which means that $g^{-1} \in N(f^{-1}, \mathscr{U})$. □

FACT 8. *$\mathscr{H}_u^U(\xi)$ is a topological group.*

PROOF. We must show that the multiplication and the inverse map are continuous.

(i) Let $f, g \in \mathscr{H}_u^U(\xi)$ and $\varepsilon \in C_{>0}(X)$. Since g is f.p. uniformly continuous, there is a map $\delta \in C_{>0}(X)$ such that if $u, v \in E_x$ and $\|u - v\|_x < \delta(x)$ then $\|g(u) - g(v)\|_x < \varepsilon(x)/2$. Suppose $f' \in N(f, \delta)$ and $g' \in N(g, \varepsilon/2)$. Then $\|gf - gf'\| < \varepsilon/2$ and $\|gf' - g'f'\| < \varepsilon/2$, hence $\|gf - g'f'\| < \varepsilon$.

(ii) Let $f \in \mathscr{H}_u^U(\xi)$ and $\varepsilon \in C_{>0}(X)$. Since f^{-1} is f.p. uniformly continuous, there is a map $\delta \in C_{>0}(X)$ such that if $u, v \in E_x$ and $\|u - v\|_x < \delta(x)$ then $\|f^{-1}(u) - f^{-1}(v)\|_x < \varepsilon(x)$. For any $g \in N(f, \delta)$ we have $\|f^{-1}g - g^{-1}g\| = \|f^{-1}g - f^{-1}f\| < \varepsilon$, which implies $\|f^{-1} - g^{-1}\| < \varepsilon$. □

FACT 9. *$\mathscr{H}_u(\boldsymbol{R}^n)$ is not a topological group. Indeed, neither the composition nor the inverse is continuous.*

PROOF. (i) The case $n = 1$. Define $f \in \mathscr{H}(\boldsymbol{R})$ by

$$f(x) = \begin{cases} x & x \leq 2 \\ k(x-k) + k & k \leq x \leq k + \dfrac{1}{k+1} \\ \dfrac{1}{k}(x - (k+1)) + k + 1 & k + \dfrac{1}{k+1} \leq x \leq k+1 \end{cases} \qquad (k : \text{an integer}, k \geq 2)$$

For each $k \geq 1$, define $g_k \in \mathscr{H}(\boldsymbol{R})$ by

$$g_k(x) = x + \frac{1}{k+1} \quad (x \in \boldsymbol{R}).$$

Then $g_k \in N(id, 1/k)$. However, since $fg_k(k) = f(k + (1/k+1)) = k + k/(k+1)$ and $f(k) = k$, so $\|fg_k(k) - f(k)\| = k/(k+1) \geq 1/2$, we have $fg_k \notin N(f, 1/2)$. This means that the composition $g \mapsto fg$ is not continuous with respect to the topology u.

Observe that

$$f^{-1}(x) = \begin{cases} x & x \leq 2, \\ \dfrac{1}{k}(x-k) + k & k \leq x \leq k + \dfrac{k}{k+1}, \\ k(x - (k+1)) + k + 1 & k + \dfrac{k}{k+1} \leq x \leq k+1. \end{cases}$$

For each $k \geq 1$, define $f_k \in \mathscr{H}(\boldsymbol{R})$ by

$$f_k(x) = f(x) + \frac{1}{k+1}.$$

Then $\|f_k - f\| = 1/(k+1)$, so f_k converges to f. On the other hand, since $f_k^{-1}(x) = f^{-1}(x - 1/(k+1))$, we have

$$f^{-1}(k+1) - f_k^{-1}(k+1) = k+1 - \left(k + \frac{1}{k+1}\right) = \frac{k}{k+1},$$

whence $\|f_k^{-1} - f^{-1}\| \geq k/(k+1) \geq 1/2$. Therefore the inverse $g \mapsto g^{-1}$ is not continuous.

(ii) The general case: We have the same conclusion for $f \times id_{\boldsymbol{R}^{n-1}}$, $g_k \times id_{\boldsymbol{R}^{n-1}}$ and $f_k \times id_{\boldsymbol{R}^{n-1}}$. □

FACT 10. *$\mathscr{H}_c(\xi)$ is a topological group.*

PROOF. We must show that the composition and the inverse map are continuous.

(i) Let $f, g \in \mathscr{H}_c(\xi)$ and $a, \varepsilon \in C_{>0}(X)$. There exists a $b \in C_{>0}(X)$ such that $f(B(a)) \subset B(b)$. By Fact 4, there exists a $\delta \in C_{>0}(X)$ such that $\delta < 1$ and if $u, v \in B_x(b+1)$ and $\|u - v\|_x < \delta(x)$ then $\|g(u) - g(v)\|_x < \varepsilon(x)/2$. Suppose $f' \in N(f, a, \delta)$ and $g' \in N(g, b+1, \varepsilon/2)$. Then for each $u \in B_x(a)$ it follows that $f(u),\ f'(u) \in B_x(b+1)$ and $\|f(u) - f'(u)\| < \delta(x)$, hence $\|gf(u) - g'f'(u)\|_x \leq \|gf(u) - gf'(u)\|_x + \|gf'(u) - g'f'(u)\|_x < \varepsilon(x)$, which implies $g'f' \in N(gf, a, \varepsilon)$.

(ii) Let $f \in \mathscr{H}_c(\xi)$ and $a, \varepsilon \in C_{>0}(X)$. We will find $\delta, b \in C_{>0}(X)$ such that if $g \in N(f, b, \delta)$ then $g^{-1} \in N(f^{-1}, a, \varepsilon)$. Take $b, c \in C_{>0}(X)$ such that $f^{-1}(B(a+2)) \subset B(b)$ and $f(B(b)) \subset B(c)$. Applying Fact 4 to f^{-1} and $c+1$, we have $\delta \in C_{>0}(X)$ such that $\delta < 1, a$ and if $u, v \in B_x(c+1)$ and $\|u - v\|_x < \delta(x)$ then $\|f^{-1}(u) - f^{-1}(v)\|_x < \varepsilon(x)$. Suppose $g \in N(f, b, \delta)$. Since $\partial g(B_x(b)) \cap B_x(a) = \emptyset$ and $g(f^{-1}(0)) \in g(B_x(b)) \cap B_x(a) \neq \emptyset$, we have $B_x(a) \subset g(B_x(b))$, that is, for each $v \in B_x(a)$ there exists a $u \in B_x(b)$ such that $v = g(u)$. Since $g(u) \in B_x(c+1)$ and $\|f(u) - g(u)\|_x < \delta(x)$, it follows that $\|f^{-1}(v) - g^{-1}(v)\|_x = \|f^{-1}g(u) - u\|_x = \|f^{-1}g(u) - f^{-1}f(u)\|_x < \varepsilon(x)$, which means that $g^{-1} \in N(f^{-1}, a, \varepsilon)$. □

FACT 11. *For each $u \in E$ and $\varepsilon > 0$ there exist a neighborhood U of u in E and $\delta > 0$ such that if $v \in U_y$, $w \in E_y$, $|\|v\|_y - \|w\|_y| < \delta$ and $\theta_y(v, w) < \delta$ then $\|v - w\|_y < \varepsilon$.*

PROOF. As easily observed, for each $u \in \boldsymbol{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $v \in \boldsymbol{R}^n$, $|\|u\| - \|v\|| < \delta$ and $\theta(u, v) < \delta$, then $\|u - v\| < \varepsilon$. Then it follows that for each $u \in \boldsymbol{R}^n$ and $\varepsilon > 0$, there exist a neighborhood U of u in $\boldsymbol{R}^n$ and $\delta > 0$ such that if $v \in U$, $w \in \boldsymbol{R}^n$, $|\|v\| - \|w\|| < \delta$ and $\theta(v, w) < \delta$ then $\|v - w\| < \varepsilon$.

Let $u \in E_x$ and take an isometric local trivialization of E over an open neighborhood W of x in X, so that we may assume $E = W \times \boldsymbol{R}^n$ and $u = (x, u_0)$. By the above observation we obtain an open neighborhood U_0 of u_0 in $\boldsymbol{R}^n$ and $\delta > 0$. Define $U = W \times U_0$. □

LEMMA 1. *Let $\lambda : [0, \infty) \to (0, 1]$ be a function such that $\lambda(s) \to 0 (s \to \infty)$. Suppose $f \in \mathscr{H}^U(\xi)$ and $g \in \mathscr{H}(\xi)$. If $\|f(u) - g(u)\|_x \le \lambda(\|u\|_x)$ for any $u \in E_x$, then $g \in \mathscr{H}^U(\xi)$.*

PROOF. (1) The f.p. uniform continuity of g: Let $\varepsilon \in C_{>0}(X)$. By Fact 2 there exists a map $a \in C_{>0}(X)$ such that (*i*) if $s \ge a(x)$, then $\lambda(s) < \varepsilon(x)/3$. By Fact 4 and by the f.p. uniform continuity of f, there is a $\delta \in C_{>0}(X)$ with $\delta < 1$ and such that (ii) if $u, v \in B_x(a+1)$ and $\|u - v\|_x < \delta(x)$ then $\|g(u) - g(v)\|_x < \varepsilon(x)$ and (iii) if $u, v \in E_x$ and $\|u - v\|_x < \delta(x)$ then $\|f(u) - f(v)\|_x < \varepsilon(x)/3$.

Suppose $u, v \in E_x$ and $\|u - v\|_x \le \delta(x)$. If $u, v \notin B_x(a)$ then $\|f(u) - g(u)\|_x \le \lambda(\|u\|_x) < \varepsilon(x)/3$ by (i) and similarly $\|f(v) - g(v)\|_x < \varepsilon(x)/3$. Hence $\|g(u) - g(v)\|_x \le \|g(u) - f(u)\|_x + \|f(u) - f(v)\|_x + \|f(v) - g(v)\|_x < \varepsilon(x)$ by (iii). Otherwise, $u, v \in B_x(a+1)$ and the same conclusion follows from (ii).

(2) The f.p. uniform continuity of g^{-1}: Let $\varepsilon \in C_{>0}(X)$. Since f^{-1} is f.p. uniformly continuous, there exists a map $\delta_1 \in C_{>0}(X)$ with $\delta_1 < 1$ and such that (i) if $\|u - v\|_x < \delta_1(x)$ then $\|f^{-1}(u) - f^{-1}(v)\|_x < \varepsilon(x)/3$. Similarly to (1), we have a map $a \in C_{>0}(X)$ such that (ii) if $s \ge a(x)$ then $\lambda(s) < \delta_1(x)$. Choose a map $b \in C_{>0}(X)$ so that $g(B(a)) \subset B(b)$. By Fact 4 there exists a $\delta \in C_{>0}(X)$ such that $\delta < \delta_1$ and (iii) if $u, v \in B_x(b+1)$ and $\|u - v\|_x < \delta(x)$ then $\|g^{-1}(u) - g^{-1}(v)\|_x < \varepsilon(x)/3$.

Suppose $u, v \in E_x$ and $\|u - v\|_x < \delta(x)$. Since $\delta < 1$, we have either (a) $u, v \notin g(B_x(a))$ or (b) $u, v \in B_x(b+1)$. In the case (a), $g^{-1}(u) \notin B(a)$ by (i) and $\|fg^{-1}(u) - u\|_x < \delta_1(x)$ by (ii). Hence by (i) $\|g^{-1}(u) - f^{-1}(u)\|_x < \varepsilon(x)/3$. Similarly $\|g^{-1}(v) - f^{-1}(v)\|_x < \varepsilon(x)/3$. By (i) $\|f^{-1}(u) - f^{-1}(v)\|_x < \varepsilon(x)/3$. Hence $\|g^{-1}(u) - g^{-1}(v)\|_x < \varepsilon(x)$. In the case (b) the same conclusion follows from (iii). □

LEMMA 2. *For any map $\varepsilon \in C_{>0}(E)$ there exists a map $\delta \in C_{>0}(E)$ such that if $u, v \in E_x \backslash \{0\}$, $|\|u\|_x - \|v\|_x| < \delta(u)$ and $\theta_x(u, v) < \delta(u)$ then $\|u - v\|_x < \varepsilon(u)$.*

PROOF. For each $u \in E$ there exists an open neighborhood W_u of u such that $\varepsilon(v) > \varepsilon_u \equiv \varepsilon(u)/2 (v \in W_u)$. By Fact 11, there exist an open neighborhood U_u of u in W_u and $\delta_u > 0$ such that if $v \in (U_u)_y$, $w \in E_y$, $|\|v\|_y - \|w\|_y| < \delta_u$ and $\theta_y(v, w) < \delta_u$, then $\|v - w\|_y < \varepsilon_u$. Take a locally finite refinement $\{U_\lambda\}$ of $\{U_u\}$ and define $\delta_\lambda = \delta_{u(\lambda)}$, where $U_\lambda \subset U_{u(\lambda)}$. By Fact 2 there exists a map $\delta \in C_{>0}(E)$ such that $\delta < \delta_\lambda$ on U_λ. Now, suppose $v, w \in E_y$, $|\|v\|_y - \|w\|_y| < \delta(v)$ and $\theta_y(v, w) < \delta(v)$. Then, $v \in U_\lambda \subset U_{u(\lambda)} \subset W_{u(\lambda)}$ for some λ. Since $\delta(v) < \delta_\lambda = \delta_{u(\lambda)}$ and $\varepsilon_{u(\lambda)} < \varepsilon(v)$, $\|v - w\|_y < \varepsilon_{u(\lambda)} < \varepsilon(v)$. □

LEMMA 3. *For any map $\delta \in C_{>0}(E)$, there exists an increasing sequence $0 \equiv r_0 < r_1 < \cdots < r_i < \cdots$ in $C_{\geq 0}(X)$ such that for each $x \in X$, $r_i(x) \to \infty (i \to \infty)$ and $r_{i+2}(x) - r_i(x) < \min \delta(B_x(r_{i+2}))(i \geq 0)$.*

PROOF. We first show that for each $a \in C_{\geq 0}(X)$ there exists a $b \in C_{\geq 0}(X)$ such that $a < b$ and $b(x) - a(x) = \min \delta(B_x(b))$ for each $x \in X$. By Fact 3 the map $F : X \times [0, \infty) \to (0, \infty)$ defined by $F(x, s) = \min \delta(B_x(s))$ is continuous. Since $F(x, s) \geq F(x, t)$ for $0 \leq s \leq t$, we have a unique $s = s(x) > 0$ for each $x \in X$ such that $F(x, a(x) + s) = s$. The map $s : X \to (0, \infty)$ is continuous. In fact, for every sufficiently small $\varepsilon > 0$, $F(x, a(x) + s(x) - \varepsilon) > s(x) - \varepsilon$ and $F(x, a(x) + s(x) + \varepsilon) < s(x) + \varepsilon$. Hence, x has a neighborhood U in X such that $F(y, a(y) + s(x) - \varepsilon) > s(x) - \varepsilon$ and $F(y, a(y) + s(x) + \varepsilon) < s(x) + \varepsilon$ for each $y \in U$, which means that $s(x) - \varepsilon < s(y) < s(x) + \varepsilon$. The desired map b can be defined by $b(x) = a(x) + s(x)(x \in X)$.

By the repeated application of the above observation, we obtain an increasing sequence $0 \equiv a_0 < a_1 < \cdots < a_i < a_{i+1} < \cdots$ in $C_{\geq 0}(X)$ such that $a_{i+1}(x) - a_i(x) = \min \delta(B_x(a_{i+1}))$. Next we will show that $a_i(x) \to \infty (i \to \infty)$ for each $x \in X$. On the contrary, assume that $c = \sup a_i(x) < \infty$. Choose an integer $n \geq 1$ so that $n^{-1} < c^{-1} \min \delta(B_x(c))$. Since $a_i(x) - a_{i-1}(x) = \min \delta(B_x(a_i)) \geq \min \delta(B_x(c)) > c/n$ for each i, we have $a_n(x) = \sum_{i=1}^n (a_i(x) - a_{i-1}(x)) > c$, which is a contradiction.

Finally, let $r_{3k} = a_k$ and take maps r_{3k+1}, $r_{3k+2} \in C_{>0}(X)(K \geq 0)$ such that

(i) $r_{3k} < r_{3k+1} < r_{3k+2} < r_{3k+3}$,

(ii) $r_{3k+1} - r_{3k} < \frac{1}{2}(r_{3k+3} - r_{3k})$ and $r_{3k+3} - r_{3k+2} < \frac{1}{2}(r_{3k+6} - r_{3k+3})$.

Then we have

(a) $r_{3k+2}(x) - r_{3k}(x) < r_{3k+3}(x) - r_{3k}(x) = \min \delta(B_x(r_{3k+3})) \leq \min \delta(B_x(r_{3k+2}))$,

(b) $r_{3k+3}(x) - r_{3k+1}(x) < r_{3k+3}(x) - r_{3k}(x) = \min \delta(B_x(r_{3k+3}))$,

(c) $r_{3k+4}(x) - r_{3k+2}(x) = (r_{3k+4}(x) - r_{3k+3}(x)) + (r_{3k+3}(x) - r_{3k+2})(x)) < r_{3k+6} - r_{3k+3} = \min \delta(B_x(r_{3k+6})) \leq \min \delta(B_x(r_{3k+4}))$.

This completes the proof. □

3. The basic f.p. engulfing lemma

The purpose of this section is to extend the radial engulfing lemma [3, Theorem 2], [9, Lemma 4.11.2] to the vector bundle case.

LEMMA 4. *Suppose X is a countable dimensional metrizable space and $n \geq 5$. Let $g \in \mathscr{H}(\xi)$ and $a, a', b, b', \varepsilon \in C_{>0}(X)$ with $\varepsilon < a' < a < b < b'$. Then there is a f.p. PL isotopy $f_t : E \to E (0 \leq t \leq 1)$ such that*

(i) $f_0 = id$,

(ii) $f_t = id$ *on* $g(B(a')) \cup (E \backslash g(B(b')))(0 \leq t \leq 1)$,

(iii) $f_1(g(B(a))) \supset g(B(b))$ *and*

(iv) $\theta_x(g_x^{-1}(f_t)_x(u), g_x^{-1}(u)) < \varepsilon(x)$ *for* $u \in E_x \backslash \{g_x(0)\}$.

PROOF. We first take maps a_i, $b_i \in C_{>0}(X)$ $(i = 0, 1, \dots)$ such that $a' < \cdots < a_{i+1} < a_i < \cdots < a_1 = a$ and $b < \cdots < b_{i+1} < b_i < \cdots < b_1 < b_0 = b'$. Since X is countable dimensional, we can write $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is 0-dimensional.

Now fix $i \geq 1$ and let $x \in X_i$. There is an open neighborhood W_x of x in X and a f.p. homeomorphism $\phi : E_{W_x} \to W_x \times \boldsymbol{R}^n$ over W_x such that $\phi_y : E_y \to \boldsymbol{R}^n$ is an isometry for each $y \in W_x$. Let $\tilde{g} = \phi g \phi^{-1} : W_x \times \boldsymbol{R}^n \to W_x \times \boldsymbol{R}^n$. We take c, c' and $d, d' > 0$ with $a_{i+1}(x) < c' < c < a_i(x)$ and $b_i(x) < d < d' < b_{i-1}(x)$, and apply [9, Lemma 4.11.2] to $\tilde{g}_x = \phi_x g_x \phi_x^{-1} : \boldsymbol{R}^n \to \boldsymbol{R}^n$ and $0 < \varepsilon(x)/2^i < c' < c < d < d'$ to obtain a PL isotopy $h_t : \boldsymbol{R}^n \to \boldsymbol{R}^n (0 \leq t \leq 1)$ such that

(i) $h_0 = id$,

(ii) $h_t = id$ on $\tilde{g}_x(B(c')) \cup (\boldsymbol{R}^n \backslash \tilde{g}_x(O(d')))(0 \leq t \leq 1)$,

(iii) $h_1(\tilde{g}_x(O(c))) \supset \tilde{g}_x(B(d))$,

(iv) $\theta(\tilde{g}x^{-1}(h_t(u)), \tilde{g}_x^{-1}(u)) < \varepsilon(x)/2^i$ for $u \in \boldsymbol{R}^n \backslash \{\tilde{g}_x(0)\}$.

Applying Fact 1 in the product bundle $W_x \times \boldsymbol{R}^n$, we can find an open neighborhood V_x of x in W_x such that (ii)′ $\tilde{g}(B_{V_x}(a_{i+1})) \subset (id \times \tilde{g}_x)(B_{V_x}(c'))$ and $\tilde{g}(O_{V_x}(b_{i-1})) \supset (id \times \tilde{g}_x)(B_{V_x}(d'))$

(iii)′ $\tilde{g}(B_{V_x}(a_i)) \supset (id \times \tilde{g}_x)(B_{V_x}(c))$ and $\tilde{g}(B_{V_x}(b_i)) \subset (id \times \tilde{g}_x)(B_{V_x}(d))$.

Thus, the f.p. PL isotopy $\bar{h}_t^x = \phi^{-1}(id \times h_t)\phi : E_{V_x} \to E_{V_x}$ satisfies the following conditions:

(i) $\bar{h}_0^x = id$,

(ii) $\bar{h}_t^x = id$ on $g(B_{V_x}(a_{i+1})) \cup (EV_x \backslash g(O_{V_x}(b_{i-1})))(0 \leq t \leq 1)$,

(iii) $\bar{h}_1^x(g(O_{V_x}(a_i))) \supset g(B_{V_x}(b_i))$,

(iv)′ $\theta_x(g_x^{-1}(\bar{h}_t^x)_x(u),\ g_x^{-1}(u)) < \varepsilon(x)/2^i$ for $u \in g(B_x(b_{i-1})\backslash O_x(a_{i+1}))$.

Let $E_x^\times \equiv E_x \backslash \{0\}$ and $\Delta E^\times \equiv \bigcup_{x \in X} E_x^\times \times E_x^\times \subset E \times E$. The angle map $\theta : \Delta E^\times \to [0, \pi]$ defined by $\theta(u, v) = \theta_x(u, v)(u, v \in E_x^\times)$ is continuous. By Fact 1, if we replace V_x by a smaller one then

(iv)″ $\theta_y(g_y^{-1}(\bar{h}_t^x)_y(u), g_y^{-1}(u)) < \varepsilon(y)/2^i$ for $u \in g(B_{V_x}(b_{i-1})\backslash O_{V_x}(a_{i+1})) \cap E_y$.

Hence it follows from (ii) that

(iv) $\theta_y(g_y^{-1}(\bar{h}_t^x)_y(u), g_y^{-1}(u)) < \varepsilon(y)/2^i$ for $y \in V_x$ and $u \in E_y \backslash \{g_y(0)\}$.

Since $\{V_x\}_{x \in X_i}$ is an open cover of X_i in X and X_i is 0-dimensional, there is a disjoint open cover $\{U_\lambda\}$ of X_i in X such that each U_λ is contained in some $V_{x(\lambda)}$ (cf. [4]). We define $U_i = \bigcup_\lambda U_\lambda$ and a f.p. PL isotopy $g_t^i : E_{U_i} \to E_{U_i}$ by $g_t^i = \bar{h}_t^{x(\lambda)}$ on each E_{U_λ}. Then, for each $y \in U_i$

(i) $(g_0^i)_y = id$,

(ii) $(g_t^i)_y = id$ on $g_y(B_y(a_{i+1})) \cup (E_y \backslash g_y(B_y(b_{i-1})))$,

(iii) $(g_1^i)_y(g_y(B_y(a_i))) \supset g_y(B_y(b_i))$,

(iv) $\theta_y((g_y)^{-1}(g_t^i)_y(u), (g_y)^{-1}(u)) < \varepsilon(y)/2^i$ for any $u \in E_y \backslash \{g_y(0)\}$.

We have obtained an open cover $\{U_i\}$ of X. We can take locally finite open covers $\{W_i\}$ and $\{V_i\}$ of X such that the closure $\bar{V}_i \subset W_i$ and $\bar{W}_i \subset U_i$ for each i. Take maps $\alpha_i : X \to [0, 1](i \geq 1)$ such that $\alpha_i(V_i) = 1$ and $\alpha_i(X - W_i) = 0$. For each i, we define a f.p. PL isotopy $f_t^i : E \to E$ by $(f_t^i)_x = (g^i_{t\alpha_i(x)})_x$ if $x \in U_i$ and $(f_t^i)_x = id$ if $x \notin U_i$ (cf. Fact 5). Finally we define the f.p. PL isotopy $f_t : E \to E$ by $f_t(u) = \lim_{\ell \to \infty} f_t^\ell \ldots f_t^1(u)(u \in E)$.

We must verify that f_t is well-defined and satisfies the required conditions. Let $x \in X$. Then x has an open neighborhood U which meets at most a finite number of W_i. Choose an ℓ so that $U \cap W_i = \emptyset$ for $i > \ell$, whence $(f_t)_y = (f_t^\ell)_y \cdots (f_t^1)_y$ for each $y \in U$ because $(f_t^i)_y = id$ for $i > \ell$. Therefore f_t is well defined and continuous. Since each $(f_t^i)_x$ is a PL isotopy, $(f_t)_x = (f_t^\ell)_x \cdots (f_t^1)_x$ is also a PL isotopy.

(i) Obviously $(f_0)_x = (f_0^\ell)_x \cdots (f_0^1)_x = id$.

(ii) For each $i = 1, \ldots, \ell, (f_t^i)_x = id$ on $g_x(B_x(a_{i+1})) \cup (E_x \backslash g_x(O_x(b_{i-1})))$, $g_x(B_x(a')) \subset g_x(B_x(a_{i+1}))$ and $g_x(B_x(b')) \supset g_x(B_x(b_{i-1}))$. Then it follows that $(f_t)_x = id$ on $g_x(B_x(a')) \cup (E_x \backslash g_x(B_x(b')))$.

(iii) Note that $x \in V_k$ for some $1 \leq k \leq \ell$. Since $(f_t^i)_x = id$ on $g_x(B_x(a_{i+1}))$ and $g_x(B_x(a_{i+1})) \supset g_x(B_x(a_k))$ for $i = 1, \ldots, k-1$, we have $(f_1^{k-1})_x \cdots$

$(f_1^0)_x(g_x(B_x(a_k))) = g_x(B_x(a_k)))$. Then $(f_1^k)_x(g_x(B_x(a_k))) = (g_1^k)_x(g_x(B_x(a_k))) \supset g_x(B_x(b_k))$. For $i = k+1, \dots, \ell$, $(f_t^i)_x = id$ on $E_x \backslash g_x(B_x(b_{i-1}))$ and $g_x(B_x(B_k)) \supset g_x(B_x(b_{i-1}))$, whence $(f_t^i)_x = id$ on $E_x \backslash g_x(B_x(b_k))$, which implies that $(f_t^i)_x(g_x(B_x(b_k))) = g_x(B_x(b_k))$. Therefore $(f_1)_x(g_x(B_x(a))) \supset (f_1)_x(g_x(B_x(a_k))) = (f_1^\ell)_x \cdots (f_1^k)_x(g_x(B_x(a_k))) \supset (f_1^\ell)_x \cdots (f_1^{k+1})_x(g_x(B_x(b_k))) = g_x(B_x(b_k)) \supset g_x(B_x(b))$.

(iv) For $u \in E_x \backslash \{g_x(0)\}$, let $u_0 = u$ and $u_i = (f_t^i)_x \cdots (f_t^1)_x(u) \neq g_x(0)$ for $i \geq 1$. Then $\theta_x(g_x^{-1}(u_i), g_x^{-1}(u_{i-1})) < \varepsilon(x)/2^i$ since $u_i = (f_t^i)_x(u_{i-1}) = (g^i_{t\alpha_i(x)})_x(u_{i-1})$ for $x \in U_i$. It follows that $\theta_x(g_x^{-1}(f_t)_x(u), g_x^{-1}(u)) = \theta_x(g_x^{-1}(u_\ell), g_x^{-1}(u_0)) \leq \sum_{i=1}^{\ell} \theta_x(g_x^{-1}(u_i), g_x^{-1}(u_{i-1})) < \varepsilon(x)$. This completes the proof. □

4. PL approximations of f.p. stable homeomorphisms

This section contains the proofs of the Main Theorem and Corollary. The theorem follows from the next lemma.

LEMMA 5 (cf. [9, Lemma 4.11.1]). *Let $g \in \mathscr{H}(\xi)$ and $a \in C_{>0}(X)$ such that g is PL on $O(a)$. Let $h : O(a) \to E$ be a f.p. homeomorphism such that $h(0) = 0$, $\theta_x(h(u), u) = 0$ for $u \in O_x(a) \backslash \{0\}$ and, for any $R \in C_{>0}(X)$, there is an $r \in C_{>0}(X)$ such that $r < a$, $r < R$ and $h(B(r)) = B(R)$. Then, for any $\varepsilon \in C_{>0}(O(a))$ there is a f.p. PL homeomorphism $f : g(O(a)) \to E$ such that $\|fg - gh\| < \varepsilon$.*

PROOF. By Facts 6 and 7, we have an $\varepsilon' \in C_{>0}(E)$ such that for $f \in \mathscr{H}(\xi)$, $\|f - id\| < \varepsilon'$ implies $\|gf - g\| < \varepsilon h^{-1}$. By Lemma 2, there is a $\delta \in C_{>0}(E)$ such that if $u, v \in E_x \backslash \{0\}$, $|\|u\|_x - \|v\|_x| < \delta(u)$ and $\delta_x(u, v) < \delta(u)$ then $\|u - v\|_x < \varepsilon'(u)$. We will construct a f.p. *PL* homeomorphism $f : g(O(a)) \to E$ such that $f(g(0)) = g(0)$ and if $u \in O_x(a)$ then (i) $|\|g^{-1}fg(u)\|_x - \|h(u)\|_x| < \delta h(u)$ and (ii) $\theta_x(g^{-1}fg(u), h(u)) < \delta h(u)$ for $u \neq 0$, whence $\|g^{-1}fg(u) - h(u)\|_x < \varepsilon' h(u)$. Then $\|g^{-1}fgh^{-1} - id\| < \varepsilon'$, which implies that $\|fgh^{-1} - g\| < \varepsilon h^{-1}$, hence $\|fg - gh\| < \varepsilon$.

By Lemma 3, there exists an increasing sequence of maps $0 = R_o < R_1 < \cdots < R_i < \cdots$ in $C_{>0}(X)$ such that for each $x \in X$, $R_i(x) \to \infty (i \to \infty)$ and $R_{i+2}(x) - R_i(x) < \delta_i(x) \equiv \min \delta(B_x(R_{i+2}))(i \geq 0)$. Define $r_i \in C_{>0}(X)$ by $h(B(r_i)) = B(R_i)$. Then, $0 = r_0 < r_1 < \cdots < r_i < \cdots < a, r_i < R_i$ and for each $x \in X$, $r_i(x) \to a(x)(i \to \infty)$ and $\delta_{i+1}(x) \leq \delta_i(x)(i \geq 0)$.

By the repeated application of Lemma 4, we obtain a sequence of f.p. *PL* homeomorphisms $f_k : E \to E(k \geq 1)$ which satisfies the following conditions:

(1) $f_k = id$ on $g_{k-1}(B(r_{k-1})) \cup (E \backslash g_{k-1}(B(R_{k+1})))$,

(2) $g_k(B(r_k)) \supset g_{k-1}(B(R_k))$,

(3) $\theta_x(g_{k-1}^{-1} f_k(u), g_{k-1}^{-1}(u)) < \delta_k(X)/2$ for $u \in E_x \backslash \{g_{k-1}(0)\}$,

where $g_0 = g$ and $g_k = f_k \dots f_1 g$ for $k \geq 1$. Then, by the induction on k, we have

(4) $f_k \dots f_1 = id$ on $E \backslash g(B(R_{k+1}))$ and $g_{k-1}(B(R_k)) = g(B(R_k))$.

In fact, the case $k = 1$ comes from (1) and if $f_{k-1} \dots f_1 = id$ on $E \backslash g(B(R_k))$, then $g_{k-1}(B(R_k)) = g(B(R_k))$ and $g_{k-1}(B(R_{k+1})) = g(B(R_{k+1}))$, whence $f_k \dots f_1 = id$ on $E \backslash g(B(R_{k+1}))$ by (1).

By the induction on ℓ, we have also

(5) $f_\ell = id$ on $g_k(B(r_k))$ and $g_\ell(B(r_\ell)) \supset g_k(B(r_k))$ for $\ell > k$.

In fact, the case $\ell = k + 1$ comes from (1) and, if $f_j = id$ on $g_k(B(r_k))$ for $k < j \leq \ell$, then $g_\ell(B(r_\ell)) \supset g_\ell(B(r_k)) = g_k(B(r_k))$, where $f_{\ell+1} = id$ on $g_k(B(r_k))$ by (1).

Define $f : g(O(a)) \to E$ by $f = \lim_{k\to\infty} f_k \dots f_1$. By (5), $f = f_k \dots f_1$ on $g(B(r_k))$, so that f is well-defined. It follows from (2) and (4) that

(6) $fg(B(r_k)) = g_k(B(r_k)) \supset g_{k-1}(B(R_k)) = g(B(R_k))$,

so f is an onto f.p. homeomorphism. For each $x \in X$ and $k \geq 1$, on the open set $g_x(O_x(r_k))$ of $g_x(O_x(a))$, $f_x = (f_k)_x \cdots (f_1)_x$ is a PL map, hence $f_x : g_x(O_x(a)) \to E_x$ is a PL homeomorphism. Since $f_k(g(0)) = g(0) (k \geq 1)$ by (5), we have $f(g(0)) = g(0)$.

Below we will verify (i) and (ii). Let $u \in O(a)$. Then $u \in B_x(r_{k+1}) \backslash B_x(r_k)$ for some $k \geq 0$. Since $h(u) \in B_x(R_{k+1})$, it follows that $\delta h(u) \geq \delta_k(x)$.

(i) Since $fg(B(r_k)) = g_k(B(r_k)) \subset g_k(B(R_{k+1})) = g(B(R_{k+1}))$, it follows from (6) that $fg(u) \in (E \backslash g(B(R_k))) \cap g(B(R_{k+2}))$, hence $g^{-1}fg(u) \in (E \backslash B(R_k)) \cap B(R_{k+2})$. Thus $R_k(x) < \|g^{-1}fg(u)\|_x \leq R_{k+2}(x)$. On the other hand, since $h(u) \in B(R_{k+1}) \backslash B(R_k)$, we have $R_k(x) < \|h(u)\|_x \leq R_{k+1}(x)$. Therefore, $|\|g^{-1}fg(u)\|_x - \|h(u)\|_x| < R_{k+2}(x) - R_k(x) < \delta_k(x) \leq \delta h(u)$.

Before verifying (ii), we note that

(7) if $v \in E_x \backslash B_x(r_k)$ then $\theta_x(g^{-1}gk(v), v) < \delta_k(x)/2$.

In fact, the case $k = 0$ is trivial. For $k \geq 1$, $g_k(v) \notin g_k(B(r_k)) \supset g(B(R_k))$ by (6), so $g^{-1}g_k(v) = g_{k-1}^{-1}g_k(v) = g_{k-1}^{-1}f_k g_{k-1}(v)$ by (4). From (3) it follows that $\theta_x(g^{-1}g_k(v), v) = \theta_x(g_{k-1}^{-1}f_k g_{k-1}(v),\ g_{k-1}^{-1}g_{k-1}(v)) < \theta_k(x)/2$.

(ii) Since $u \in B_x(r_{k+1})$, we have $fg(u) = g_{k+1}(u) = f_{k+1}g_k(u)$ by (5). Since $g_k^{-1}f_{k+1}g_k(B(r_k)) = B(r_k)$ by (5), we can apply (7) to $v = g_k^{-1}f_{k+1}g_k(u) \notin B(r_k)$. Then $\theta_x(g^{-1}fg(u), v) = \theta_x(g^{-1}f_{k+1}g_k(u), v) = \theta_x(g^{-1}g_k(v), v) < \delta_k(x)/2$. On the other hand, $\theta_x(v, u) = \theta_x(g_k^{-1}f_{k+1}g_k(u), g_k^{-1}g_k(u)) < \delta_{k+1}(x)/2$ by (3) and $\theta_x(h(u), u) = 0$. Therefore, $\theta_x(g^{-1}fg(u), h(u)) < \delta_k(x) \leq \delta(h(u))$. This completes the proof. □

PROOF OF THEOREM. By Fact 7 we may assume that g is PL on $O(s,a)$ for some section s of ξ and $a \in C_{>0}(X)$. Since the translation $E \ni u \mapsto u + s(\pi(u)) \in E$ is a f.p. PL homeomorphism, it suffices to consider the case that s is the zero section.

Take a f.p. homeomorphism $h : O(a) \to E$ as in Lemma 5. For example, define h by

$$h(u) = \left(\frac{1}{a(x) - \|u\|_x} + 1\right)u \quad (u \in O_x(a)).$$

By Lemma 5, there is a f.p. PL homeomorphism $f_2 : g(O(a)) \to E$ such that $f_2 g|_{O(a)}$ approximates gh. Applying Lemma 5 to $g = id$, we have a f.p. PL homeomorphism $f_1 : O(a) \to E$ which approximates h. By Fact 7, if f_1 is sufficiently close to h, then hf_1^{-1} is close to id. Then, $f = f_2 g f_1^{-1} : E \to E$ is the required f.p. PL homeomorphism. □

PROOF OF COROLLARY. (i) Suppose Y is a countable dimensional metrizable space and $g : Y \to \mathscr{H}_u^{PL_1}(\xi)$ is a map. Consider the vector bundle $id \times \pi : Y \times (0,1] \times E \to Y \times (0,1] \times X$ with the natural Riemannian metric induced from ξ. The base space $Y \times (0,1] \times X$ is countable dimensional metrizable space and g induces the f.p. PL_1 homeomorphism $G : Y \times (0,1] \times E \to Y \times (0,1] \times E$ defined by $G(y,t,u) = (y,t,g(y)(u))$ (cf. Fact 5).

Let $\lambda(s) = 1/(1+s)$ for $s \in [0,\infty)$. Then $\lambda(s) \to 0 (s \to \infty)$. We define $\delta : Y \times (0,1] \times E \to (0,\infty)$ by $\delta(y,t,u) = \min\{\lambda(\|u\|), t\}$. By Theorem there is a f.p. PL homeomorphism $F : Y \times (0,1] \times E \to Y \times (0,1] \times E$ such that $\|F - G\| < \delta$.

Define $f_0 = g$ and $f_t : Y \to \mathscr{H}_u^{PL_1}(\xi)$ by $f_t(y)(u) = \pi_E F(y,t,u)$ for $0 < t \le 1$, where $\pi_E : Y \times (0,1] \times E \to E$ is the projection. We will verify that the homotopy f is continuous. Let $y \in Y$, $t \in [0,1]$ and $\varepsilon > 0$. Note that the base space X is compact. We must find an open neighborhood U of y in Y and $\alpha > 0$ such that if $z \in U$ and $|t-s| < \alpha$ then $\|f_t(y)(u) - f_s(z)(u)\|_x < \varepsilon$ for each $u \in E_x$.

(1) The case that $0 < t \le 1$: Since $\lambda(s) \to 0 (s \to \infty)$, there is $a > 0$ such that $\lambda(a) < \varepsilon/3$. Since F is continuous and $B(a)$ is compact, there exist a neighborhood V of y in Y and $\alpha > 0$ such that if $z \in V$, $|t-s| < \alpha$ and $u \in B_x(a)$, then $\|f_t(y)(u) - f_s(z)(u)\|_x < \varepsilon/3$. This fact can be shown by noting that $f : Y \times (0,1] \to \mathscr{H}(\xi)$ (with the compact-open topology) is continuous. Here, for completeness we give a more elementary verification:

For each $u \in B(a)$, take a neighborhood O_u of $f_t(y)(u)$ in E of the form: $O_u = O(s_u, \varepsilon/6)$, where s_u is a section of ξ with $s_u(x) = f_t(y)(u) = \pi_E F(t,y,u)$, here $x = \pi(u)$. Then there exists a neighborhood V_u of y in Y, W_u of t in $(0,1]$ and U_u of u in E such that $\pi_E F(V_u \times W_u \times U_u) \subset O_u$. Since $B(a)$ is compact, there are finite $u_i \in B(a)$ such that $B(a) \subset \bigcup_i U_{u_i}$. Set $V = \bigcap_i V_{u_i}$ and take $\alpha > 0$ such that $(t-\alpha, t+\alpha) \subset \bigcap_i W_{u_i}$. If $z \in V$ and $|t-s| < \alpha$ and $v \in B_x(a)$, then $v \in U_{u_i}$ for some u_i. Then $f_s(z)(v)$, $f_t(y)(v) \in \pi_E F(V_{u_i} \times W_{u_i} \times U_{u_i}) \subset O_{u_i} = O(s_{u_i}, \varepsilon/6)$ and hence $\|f_s(z)(v) - f_t(y)(v)\|_x < \varepsilon/3$.

Since $g : Y \to \mathscr{H}_u(\xi)$ is continuous, there is a neighborhood W of y in Y such that for $z \in W$, $\|g(y)(u) - g(z)(u)\|_x < \varepsilon/3 (u \in E_x)$. Now let $z \in U = V \cap W$, $|t-s| < \alpha$ and $u \in E_x$. In case $u \in B(a)$, $\|f_t(y)(u) - f_s(z)(u)\|_x < \varepsilon/3 < \varepsilon$. In case $u \notin B(a)$, it follows that $\delta(z,s,u) \leq \lambda(\|u\|_x) < \lambda(a) < \varepsilon/3$. Since $\|F - G\| < \delta$, $\|g(z)(u) - f_s(z)(u)\|_x < \delta(z,s,u) < \varepsilon/3$. Similarly $\|g(y)(u) - f_t(y)(u)\|_x < \varepsilon/3$. Since $\|g(y)(u) - g(z)(u)\|_x < \varepsilon/3$, we have $\|f_t(y)(u) - f_s(z)(u)\|_x < \varepsilon$.

(2) The case $t = 0$: Since $f_0 = g : Y \to \mathscr{H}_u^S(\xi)$ is continuous, there is a neighborhood U of y in Y such that for $z \in U$, $\|g(y)(u) - g(z)(u)\|_x < \varepsilon/2$. Let $0 < s < \alpha = \varepsilon/2$. Since $\|F - G\| < \delta$, it follows that $\|g(z)(u) - f_s(z)(u)\|_x < \delta(z,s,u) \leq s < \varepsilon/2$. Therefore, if $z \in U$ and $0 \leq s < \alpha$ then $\|f_0(y)(u) - f_s(z)(u)\|_x < \varepsilon$. This completes the proof of (i).

(ii) It suffices to show that in the proof of (i), if $g(y) \in \mathscr{H}_u^{PL_1,U}(\xi)$ then $f_t(y) \in \mathscr{H}_u^{PL_1,U}(\xi)$. This follows from Lemma 1 since $\|g(y)(u) - f_t(y)(u)\|_x \leq \delta(y,t,u) \leq \lambda(\|u\|_x)$ for any $u \in E_x$. □

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