TSUKUBA J. MATH.
Vol. 21 No. 1 (1997), 153-168

STRUCTURE THEOREMS OF THE SCALAR CURVATURE EQUATION ON SUBDOMAINS OF A COMPACT RIEMANNIAN MANIFOLD

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1. Introduction

Let (M, g) be a Riemannian manifold with $\dim M = n \geq 3$, Δ_g the Laplacian of g, S_g the scalar curvature of g and L_g the conformal Laplacian of g, i.e. $L_g := -a_n\Delta_g + S_g$ with $a_n = 4(n-1)/(n-2)$. Let u be a positive smooth function on M, and define a conformal metric by $\tilde{g} := u^{4/(n-2)}g$. Then its scalar curvature is given by $S_{\tilde{g}} = u^{-q}L_g u$, where $q = (n+2)/(n-2) = 4/(n-2)+1$. Hence, a smooth function f on M can be realized as the scalar curvature of some metric which is pointwise conformal to g if and only if there is a smooth solution u of the equation

$$\begin{cases} L_g u = f u^q \\ u > 0 \end{cases} \quad \text{on } M.$$

Throughout this paper, we refer to this equation as "the equation (f, M)".

Now, we are interested in the structure of the moduli space of (complete) conformal metrics on M with scalar curvature f. In this work, we study the equation (f, M) in the case when (M, g) is a subdomain of a compact Riemannian manifold $(\overline{M}, \bar{g})$. More precisely, we consider mainly the case when $\lambda_1(L_{\bar{g}}) > 0$, (M, g) is the complement $\overline{M}\backslash\Sigma$ of a compact submanifold Σ, and f is nonpositive.

Under this assumption, Mazzeo [12] proved that, when $d = \dim\Sigma \leq (n-2)/2$ and $f \equiv 0$ on M, "the full solution space of scalar flat complete conformal metrics on M is parametrized by the space of strictly positive measures on Σ." This fact means that Σ is the Martin boundary of the Laplacian with respect to a scalar flat complete conformal metric on M.

When f has a compact support, any conformal metric $u^{q-1}g$ on M with scalar curvature f is bounded above by some scalar flat conformal metric $\varphi^{q-1}g$

Received February 6, 1995.
Revised August 30, 1995.

on M. Moreover, if $\lim_{x\to\Sigma} u(x) = +\infty$, then such φ can be chosen to satisfy $\lim_{x\to\Sigma}(u(x)/\varphi(x)) = 1$. (We describe this in Section 6. See also [9, Section 4].)

Also when supp$f$ is not compact, if f satisfies a certain condition, then there are infinitely many conformal metrics on M with scalar curvature f each of which behaves asymptotically to a scalar flat conformal metric on M (see [6], [10], [7]). However, in this case, the space of (complete) conformal metrics with scalar curvature f is more complicated in general. For example if $S_g \geq 0$, f is negative outside a compact set and the equation (f, M) possesses a positive subsolution, then there is a maximal solution of the equation (f, M) which does not behave asymptotically to any solution of the equation $(0, M)$, where we call a solution U of the equation (f, M) is maximal if and only if $U \geq u$ holds for any solution u of (f, M) (see [11], [3], [14], [4], [5], [9]).

In Section 2, we prove the following uniqueness theorem for solutions of the maximal order.

THEOREM 1. *Let (M, g) be an open Riemannian manifold $(n = \dim M \geq 3)$. Let f be a nonpositive smooth function on M, U the maximal solution of the equation (f, M), and u a solution of the equation (f, M). If $u \sim U$ on M, then $u \equiv U$ on M.*

Here and throughout this paper, we use the notation "$f_0 \sim f$" to mean that the condition $C_1 f \leq f_0 \leq C_2 f$ holds for some positive constants C_1 and C_2. We also denote the distance function to a submanifold Σ (resp. a point p) by r_Σ (resp. r_p).

Next, in Section 3, we prove the following lower estimate for solutions of the equation (f, M) whose order is higher than that of the standard solutions $\gamma G_\Sigma := \gamma \int_\Sigma G(\cdot, y) d\sigma_y$ of $(0, M)$, where γ is a positive number, G is the Green function of $L_{\bar{g}}$ and $d\sigma$ is the volume element of Σ.

THEOREM 2. *Let $(\overline{M}, \bar{g})$ be a compact Riemannian manifold $(n = \dim \overline{M} \geq 3)$ with $\lambda_1(L_{\bar{g}}) > 0$, Σ a compact submanifold $(d = \dim \Sigma < n - 2)$, and $(M, g) := (\overline{M} \backslash \Sigma, \bar{g}|_{\overline{M}\backslash\Sigma})$. Let f be a nonpositive smooth function on M. Suppose f satisfies $f \sim -r_\Sigma^l$ near Σ for a nonnegative number $l > 2 - 4d/(n-2)$, and suppose u is a solution of the equation (f, M) such that $u(x)/r_\Sigma(x)^{d-n+2} \to +\infty$ as $x \to \Sigma$. Then u satisfies the estimate $u \geq C r_\Sigma^{-(l+2)/(q-1)}$ for some positive constant C, and the metric $u^{q-1} g$ is complete. In particular, u is the maximal solution of the equation (f, M).*

Remark here that $(r_\Sigma(x)^{d-n+2})^{q-1}$ is the order of the ratio of the scalar flat conformal metric $G_\Sigma^{q-1}g$ to the original metric g (see the proof of Theorem 2).

The assertions of Theorems 1 and/or 2 are known in some cases (see [11], [14], [4], [1], [13] for $d = n-1$; [5] for $(M,g) = (\boldsymbol{R}^n, g_0)$; [2], [12], [7] for 1 with $f \sim -1$). However, our proof of Theorem 1 is quite simple, although we are concerned with more general situation. By virtue of the weakness of the assumption of Theorem 1, we can show, in Theorem 2, the uniqueness of solutions satisfying $\lim_{x\to\Sigma}(u(x)/r_\Sigma(x)^{d-n+2}) = +\infty$ from only the rough estimate $u \geq Cr_\Sigma^{-(l+2)/(q-1)}$.

These theorems enable us, in Section 4, to generalize the structure theorem on the Euclidean space $(\boldsymbol{R}^n, g_0)$ (which is conformal to $\boldsymbol{S}^n\backslash\{p\}$ with the standard metric) proved by Cheng-Ni [5] to the following

Theorem 3. *Let $(\overline{M}, \bar{g})$ be a compact Riemannian manifold ($n = \dim \overline{M} \geq 3$) with $\lambda_1(L_{\bar{g}}) > 0$, p a point in $\overline{M}$, and $(M,g) := (\overline{M}\backslash\{p\}, \bar{g}|_{\overline{M}\backslash\{p\}})$. Set $G_p(x) := G(p,x)$, where G is the Green function of $L_{\bar{g}}$. Let f be a nonpositive smooth function on M. If f satisfies $f \sim -r_p^l$ near p for a number $l > 2$, then, for any $\gamma \in (0, +\infty]$, the equation (f, M) possesses a unique solution u_γ such that $u_\gamma(x)/G_p(x) \to \gamma$ as $x \to p$ and the metric $u_\gamma^{q-1}g$ is complete. Conversely, any solution u of the equation (f, M) coincides with u_γ for some γ. Namely, the space of complete conformal metrics on M with scalar curvature f is parametrized by* $(0, +\infty]$.

Now, the following question arises naturally. "For any $(M,g) = (\overline{M}\backslash\Sigma, \bar{g}|_{\overline{M}\backslash\Sigma})$ and f, does any solution u of the equation (f, M) satisfying $\lim_{x\to\Sigma}u(x) = +\infty$ coincide with either the maximal solution U or a solution asymptotic to some solution of $(0, M)$?" In Section 5, we show that the answer to this question is "No" in general. The simplest case we can observe this is the case when Σ is a finite number (larger than 1) of points. We can also construct solutions of the equation $(-1, M)$ which do not behave like any solution of the equation $(0, M)$ near a subset of Σ even when Σ is connected. These observations teach us that the space of solutions of the equation (f, M) has some more complicated structure in general.

The author wish to thank Professor N. Nakauchi for his interest in this work and helpful advice.

2. The uniqueness of solutions of the maximal order

In this section, we prove Theorem 1. For this purpose and to use later, we recall here the following well-known formula without a proof.

METHOD OF SUPERSOLUTIONS AND SUBSOLUTIONS. *Let (M, g) be a Riemannian manifold ($n = \dim M \geq 3$), and f a smooth function on M. If there exist a supersolution u_+ and a subsolution u_- of the equation (f, M) such that $0 < u_- \leq u_+$, then the equation (f, M) possesses a smooth solution u satisfying $u_- \leq u \leq u_+$.*

PROOF OF THEOREM 1. Set $\beta := \sup_M(U/u)$. Then, by the assumption $u \sim U$, we have $\beta < +\infty$.

Suppose $u \not\equiv U$. Then $\beta > 1$ and, by the strong maximal principle, it holds that $u < U$. It is easy to see that $\gamma := \{(\beta^q - 1)/(\beta - 1)^q\}^{1/(q-1)} > \beta/(\beta - 1)$. Set

$$v_+ := \gamma(\beta - 1)u, \quad v_- := \gamma(U - u).$$

Then clearly $v_\pm > 0$ and

$$v_+ - v_- = \gamma(\beta u - U) \geq 0.$$

Moreover, we get

$$v_+^{-q} L_g v_+ = \{\gamma(\beta - 1)\}^{1-q} u^{-q} L_g u = \{\gamma(\beta - 1)\}^{1-q} f \geq \beta^{1-q} f \geq f$$

and

$$\begin{aligned} v_-^{-q} L_g v_- &= \gamma^{1-q}(U - u)^{-q} L_g(U - u) = \gamma^{1-q}(U - u)^{-q} f(U^q - u^q) \\ &= \frac{U^q - u^q}{\gamma^{q-1}(U - u)^q} f \leq \frac{\beta^q - 1}{\gamma^{q-1}(\beta - 1)^q} f = f, \end{aligned}$$

namely, v_+ (resp. v_-) is a supersolution (resp. subsolution) of the equation (f, M), where we use the inequality

$$\frac{t^q - 1}{(t - 1)^q} \geq \frac{\beta^q - 1}{(\beta - 1)^q} \quad \text{for } t \in (1, \beta].$$

Therefore, by the method of supersolutions and subsolutions, the equation (f, M) possesses a solution v satisfying $v_+ \geq v \geq v_-$.

Now, by the definition of β, there exists a sequence $\{x_i\}_{i\in N}$ of points in M such that $\lim_{i\to+\infty}(U(x_i)/u(x_i)) = \beta$. Hence we get

$$\frac{v(x_i)}{u(x_i)} \geq \frac{v_-(x_i)}{u(x_i)} = \gamma\left(\frac{U(x_i)}{u(x_i)} - 1\right) \to \gamma(\beta - 1) > \beta \quad \text{as } i \to +\infty$$

from which it follows that, for any i large enough,

$$\frac{v(x_i)}{u(x_i)} > \beta \geq \frac{U(x_i)}{u(x_i)},$$

namely, $v(x_i) > U(x_i)$. This contradicts the assumption that U is maximal. Therefore we have $u \equiv U$. q.e.d.

Now, by the proofs of [9, Theorem II, III and IV], we know the order of the maximal solution U of the equation (f, M) in various cases. Combining this and Theorem 1, we immediately get the following corollaries.

Corollary 2.1. *Let (M, g) be a complete, noncompact, simply connected Riemannian manifold ($n = \dim M \geq 3$) with nonpositive curvature whose Ricci curvature Ric_g satisfies $Ric_g/(n-1) \geq -A^2/(r_p^2 + \varepsilon^2)$ for positive numbers A and ε such that $A^2 \leq (n-2)/n$. Let f be a nonpositive smooth function on M satisfying $f \sim -r_p^{-l}$ near infinity for some point $p \in M$ and a number $l > 2$, u a solution of the equation (f, M). If u satisfies $u \geq C r_p^{(l-2)/(q-1)}$ for a positive constant C, then u is the maximal solution of the equation (f, M).*

Corollary 2.2. *Let (M, g) be a complete, noncompact, simply connected Riemannian manifold ($n = \dim M \geq 3$) whose sectional curvature K_g and Ricci curvature Ric_g satisfy $K_g \leq -B^2$ and $Ric_g/(n-1) \geq -A^2$ for positive numbers A and B such that $(A/B)^2 \leq (n-1)^2/n(n-2)$. Let f be a nonpositive smooth function on M satisfying $f \sim -e^{-lr_p}$ near infinity for some point $p \in M$ and a nonnegative number l, u a solution of the equation (f, M). If u satisfies $u \geq Ce^{lr_p/(q-1)}$ for a positive constant C, then u is the maximal solution of the equation (f, M).*

Corollary 2.3. *Let $(\overline{M}, \bar{g})$ be a compact Riemannian manifold ($n = \dim \overline{M} \geq 3$) with $\lambda_1(L_{\bar{g}}) \geq 0$, Σ a compact submanifold ($d = \dim \Sigma$) of $\overline{M}$, and $(M, g) := (\overline{M}\backslash\Sigma, \bar{g}|_{\overline{M}\backslash\Sigma})$. Let f be a nonpositive smooth function on M satisfying $f \sim -r_\Sigma^l$ near Σ for a nonnegative number $l > 2 - 4d/(n-2)$, u a solution of the* *equation* (f, M). *If* u *satisfies* $u \geq Cr_\Sigma^{-(l+2)/(q-1)}$ *for a positive constant* C, *then* u *is the maximal solution of the equation* (f, M).

3. A lower estimate for solutions of high order

In this section, we prove Theorem 2. First, we prepare a key lemma. Its proof is quite similar to that of [9, Theorem V], but some more delicate. In common with Loewner-Nirenberg [11, Theorem 4], we make use of a family of (sub-)solutions. However, our lemma applies to more complicated cases.

LEMMA 3.1. *Let* (M, g) *be an open Riemannian manifold* $(n = \dim M \geq 3)$, $\{\Omega_i\}_{i \in N}$ *a sequence of relatively compact domains of* M *which satisfies* $\Omega_i \subset\subset \Omega_{i+1}$ *and* $\cup_{i \in N}\Omega_i = M$. *Let* f *be a nonpositive smooth function on* M, φ *a solution of the equation* $(0, M)$, *and* u_1 *a solution of the equation* (f, M) *satisfying* $u_1 \leq \varphi$ *on* M *and* $\lim_{i \to +\infty}\{\inf_{M \setminus \Omega_i}(u_1/\varphi)\} = 1$. *Suppose* u_+ *is a supersolution of the equation* (f, M) *satisfying* $\lim_{i \to +\infty}\{\inf_{M \setminus \Omega_i}(u_+/\varphi)\} = +\infty$. *Then it holds that*

$$u_+ \geq (q-1)q^{-q/(q-1)}\left(1 - \frac{u_1}{\varphi}\right)^{-1/(q-1)}\varphi \quad \text{where } u_1 \geq \frac{q-1}{q}\varphi.$$

PROOF. Put $w_1 := 1 - u_1/\varphi$. Then w_1 is nonnegative, $\lim_{i \to +\infty}(\sup_{M \setminus \Omega_i} w_1) = 0$ and $u_1 = \varphi(1 - w_1)$.

Let γ be the supremum of γ_0's such that, for any number $\mu \in [1, \gamma_0]$, the equation (f, M) possesses a solution u_μ satisfying $\mu\varphi(1 - \mu^{q-1}w_1) \leq u_\mu \leq \mu\varphi$. Clearly $\gamma \geq 1$.

Suppose $\gamma < +\infty$. Then, for any $\mu \in [1, \gamma)$, the equation (f, M) possesses a solution u_μ as above. Since f is nonpositive, it follows from [9, Lemma 2.2] that $\{u_\mu\}_{1 \leq \mu < \gamma}$ is monotonically increasing and bounded above by $\gamma\varphi$. Therefore, if we set $u_\gamma := \lim_{\mu \to \gamma} u_\mu$, then u_γ is a solution of the equation (f, M) with the same properties as above.

Put $w_\gamma := \gamma - u_\gamma/\varphi$. Then it is clear that w_γ is nonnegative, $\max_M w_\gamma < \gamma$, and $w_\gamma \leq \gamma^q w_1$. Choose a positive number δ satisfying $1 < \delta < \gamma/\max_M w_\gamma$, and, for any $\varepsilon \in (1, \delta^{1/(q-1)}]$, set $u_{\varepsilon\gamma-} := \varepsilon\varphi(\gamma - \varepsilon^{q-1}w_\gamma)$. Then we get

$$L_g u_{\varepsilon\gamma-} = \varepsilon^q L_g u_\gamma = \varepsilon^q f u_\gamma^q = f u_{\varepsilon\gamma-}^q \left(\frac{\gamma - w_\gamma}{\gamma - \varepsilon^{q-1}w_\gamma}\right)^q \leq f u_{\varepsilon\gamma-}^q \quad \text{on } M,$$

namely, $u_{\varepsilon\gamma-}$ is a subsolution of the equation (f, M). On the other hand, if we set

$u_{\varepsilon\gamma+} := \varepsilon\gamma\varphi$, then we have

$$L_g u_{\varepsilon\gamma+} = 0 \geq f u_{\varepsilon\gamma+}^q \quad \text{on } M,$$

namely, $u_{\varepsilon\gamma+}$ is a supersolution of the equation (f, M). Since $u_{\varepsilon\gamma+} \geq u_{\varepsilon\gamma-} > 0$, by the method of supersolutions and subsolutions, the equation (f, M) possesses a solution $u_{\varepsilon\gamma}$ satisfying $u_{\varepsilon\gamma+} \geq u_{\varepsilon\gamma} \geq u_{\varepsilon\gamma-}$. It is clear that

$$\varepsilon\gamma\varphi \geq u_{\varepsilon\gamma} \geq \varepsilon\varphi(\gamma - \varepsilon^{q-1} w_\gamma) \geq \varepsilon\varphi(\gamma - \varepsilon^{q-1}\gamma^q w_1) = \varepsilon\gamma\varphi\{1 - (\varepsilon\gamma)^{q-1} w_1\}.$$

This contradicts the definition of γ since $\delta^{1/(q-1)}\gamma > \gamma$.

Hence we conclude that $\gamma = +\infty$. Namely, for any $\mu \geq 1$, the equation (f, M) possesses a solution u_μ satisfying $\mu\varphi(1 - \mu^{q-1} w_1) \leq u_\mu \leq \mu\varphi$.

Now, by [9, Lemma 2.2] again, it holds that $u_+ \geq u_\mu$ for any $\mu \geq 1$. Therefore we have

$$u_+ \geq \sup_{\mu \geq 1}\{\mu(1 - \mu^{q-1} w_1)\}\varphi.$$

By using the equality

$$\sup_{t \geq 1}\{t(1 - t^{q-1} a)\} = (q-1) q^{-q/(q-1)} a^{-1/(q-1)} \quad \text{for } a \in \left(0, \frac{1}{q}\right],$$

we get our assertion. q.e.d.

Proof of Theorem 2. Let $G(x, y)$ be the Green function of $L_{\bar{g}}$, and set $G_\Sigma(x) := \int_\Sigma G(x, y) d\sigma_y$, where $d\sigma$ is the volume element of Σ with respect to the induced metric. Clearly $L_g G_\Sigma \equiv 0$ on M and, by [10, Proposition 2], there exist positive constants C_1 and C_2 such that

$$C_1 r_\Sigma^{d-n+2} \leq G_\Sigma \leq C_2 r_\Sigma^{d-n+2}.$$

In the case when $2 - 4d/(n-2) < l < n - (n+2)d/(n-2)$, by [10, Theorem 2 (a) and its Remark] (see also Delanoë [6, Theorem 5]), the equation (f, M) possesses a solution u_1 satisfying

$$C_3 G_\Sigma(1 - C_4 r_\Sigma^\alpha) \leq u_1 \leq C_3 G_\Sigma$$

for some positive constants C_3, C_4 and $\alpha := l - 2 + 4d/(n-2)$. In [10], we assumed $d \leq (n-2)/2$ for this fact. However, if we do not assert the completeness of the metric $u_1^{q-1} g$, then the same consequence as above holds also when $(n-2)/2 < d < n-2$. Apply Lemma 3.1 to this u_1 and $\varphi = C_3 G_\Sigma$. Then, since $w_1 = 1 - u_1/\varphi \leq C_4 r_\Sigma^\alpha$, any supersolution u_+ of the equation (f, M) such that $u_+(x)/G_\Sigma(x) \to +\infty$ as $x \to \Sigma$ satisfies the estimate

$$\begin{aligned} u_+ &\geq (q-1)q^{-q/(q-1)}(C_4 r_\Sigma^\alpha)^{-1/(q-1)} C_3 C_1 r_\Sigma^{d-n+2} \\ &= C_5 r_\Sigma^{-\{l-2+4d/(n-2)\}(n-2)/4+d-n+2} \\ &= C_5 r_\Sigma^{-(l+2)(n-2)/4} \\ &= C_5 r_\Sigma^{-(l+2)/(q-1)}, \end{aligned}$$

where $C_5 := (q-1)q^{-q/(q-1)} C_4^{-1/(q-1)} C_3 C_1$. In particular, we have $u \geq C_5 r_\Sigma^{-(l+2)/(q-1)}$ near Σ. Remark that the assumption $l \geq 0$ is not used here.

In the case when $l \geq n-(n+2)d/(n-2)$, put $u_+ := u^\theta G_\Sigma^{1-\theta}$, where θ is a positive number chosen to satisfy

$$\theta < \frac{n-2-d}{l-2+\dfrac{4d}{n-2}} (\leq 1).$$

By direct computation, we get

$$\begin{aligned} L_g u_+ &= \theta f u^{\theta+q-1} G_\Sigma^{1-\theta} + a_n \theta(1-\theta) u^{\theta-2} G_\Sigma^{-\theta-1} |G_\Sigma \nabla_g u - u \nabla_g G_\Sigma|^2 \\ &\geq \theta f u^{\theta+q-1} G_\Sigma^{1-\theta} \end{aligned}$$

and hence

$$u_+^{-q} L_g u_+ \geq \theta f \left(\frac{u}{G_\Sigma}\right)^{(1-\theta)(q-1)}.$$

Since we assume

$$-C_6 r_\Sigma^l \leq f \leq -C_7 r_\Sigma^l \quad \text{near } \Sigma$$

for some positive constants C_6 and C_7, by the proof of [9, Theorem IV], there is a positive constant C_8 such that

$$u \leq C_8 r_\Sigma^{-(l+2)/(q-1)}.$$

Therefore we get

$$\begin{aligned} \theta f \left(\frac{u}{G_\Sigma}\right)^{(1-\theta)(q-1)} &\geq \theta(-C_6 r_\Sigma^l) \left(\frac{C_8 r_\Sigma^{-(l+2)/(q-1)}}{C_1 r_\Sigma^{d-n+2}}\right)^{(1-\theta)(q-1)} \\ &= -C_9 r_\Sigma^{l_\theta} \quad \text{near } \Sigma, \end{aligned}$$

where $C_9 := \theta C_6(C_1^{-1}C_8)^{(1-\theta)(q-1)}$ and

$$l_\theta := 2 - \frac{4d}{n-2} + \theta\left(l - 2 + \frac{4d}{n-2}\right).$$

Note here that, by the assumption on θ, $2 - 4d/(n-2) < l_\theta < n - (n+2)d/(n-2)$ holds. Let f_0 be a nonpositive smooth function on M such that $f_0 \le \theta f(u/G_\Sigma)^{(1-\theta)(q-1)}$ on M and $f_0 \equiv -C_9 r_\Sigma^{l_\theta}$ near Σ. Then u_+ is a supersolution of the equation (f_0, M). Clearly $u_+(x)/G_\Sigma(x) = (u(x)/G_\Sigma(x))^\theta \to +\infty$ as $x \to \Sigma$. Hence we can use the estimate given in the case when $2 - 4d/(n-2) < l < n - (n+2)d/(n-2)$ with $l = l_\theta$ and $u_+ = u^\theta G_\Sigma^{1-\theta}$, and get

$$u^\theta G_\Sigma^{1-\theta} \ge C_5 r_\Sigma^{-(l_\theta+2)/(q-1)}.$$

Therefore we have

$$\begin{aligned} u^\theta &\ge C_5 r_\Sigma^{-(l_\theta+2)/(q-1)} G_\Sigma^{\theta-1} \\ &\ge C_5 r_\Sigma^{-(l_\theta+2)/(q-1)} (C_2 r_\Sigma^{d-n+2})^{\theta-1} \\ &\ge C_{10} r_\Sigma^{-\theta(l+2)/(q-1)}, \end{aligned}$$

where $C_{10} := C_5 C_2^{\theta-1}$, from which it follows that $u \ge C_{10}^{1/\theta} r_\Sigma^{-(l+2)/(q-1)}$ near Σ.

Now, in both cases, because $u > 0$ on M, we get $u \ge C r_\Sigma^{-(l+2)/(q-1)}$ on M. Since l is nonnegative, the metric $u^{q-1}g$ is complete and, by Corollary 2.3, we conclude $u \equiv U$. q.e.d.

Corollary 3.2. *Let (M, g) and f be as in Theorem 2. Suppose f satisfies $f \sim -r_\Sigma^l$ near Σ for a nonnegative number $l > 2 - 4d/(n-2)$, and suppose U is the maximal solution of the equation (f, M) and, for any $\gamma > 0$, u_γ is a solution of the equation (f, M) satisfying $u_\gamma \le \gamma G_\Sigma$ and $\lim_{x\to\Sigma}(u_\gamma(x)/G_\Sigma(x)) = \gamma$. Then it holds that $\lim_{\gamma\to+\infty} u_\gamma \equiv U$ on M.*

Proof. By [9, Lemma 2.2], $\{u_\gamma\}_{\gamma>0}$ is monotonically increasing and bounded above by U. Therefore, if we set $u := \lim_{\gamma\to+\infty} u_\gamma$, then u is a smooth solution of the equation (f, M). It is clear that u satisfies $\lim_{x\to\Sigma}(u(x)/r_\Sigma(x)^{d-n+2}) = +\infty$. Hence, by Theorem 2, we have $u \equiv U$, namely, $\lim_{\gamma\to+\infty} u_\gamma \equiv U$ on M. q.e.d.

4. A structure theorem in the case Σ is a point

The aim of this section is to prove Theorem 3 which describes the structure of the scalar curvature equation $(f, \overline{M}\backslash\{p\})$ in the case when $f \sim -r_p^l$ near p.

When Σ is a point, we can prove a Harnack type inequality for the solutions of the equation (f, M) by the same way as Cheng-Ni [5, Proposition 5.2]. This and Corollary 3.2 imply Theorem 3.

LEMMA 4.1. *Let* $(\overline{M}, \bar{g})$ *be a compact Riemannian manifold* ($n = \dim \overline{M} \geq 3$) *with* $S_{\bar{g}} > 0$, p *a point in* $\overline{M}$, *and* $(M, g) := (\overline{M}\backslash\{p\}, \bar{g}|_{\overline{M}\backslash\{p\}})$. *Suppose* f *satisfies* $f \sim -r_p^l$ *near* p *for a number* $l > 2$, *and suppose* u *is a solution of the equation* (f, M). *Then there is a positive constant* C_{11} *which is independent of both* u *and* r *and satisfies*

$$\max_{\partial B_r(p)} u \leq C_{11} \min_{\partial B_r(p)} u$$

for any positive number r *small enough.*

PROOF. Set $d(x) := (S_g + |f|u^{q-1})/a_n$. Then $(\Delta_g - d(x))u = 0$. Since we assume $f \sim -r_p^l$, by the proof of [9, Theorem IV], we have $u^{q-1} \leq C_{12} r_p^{-(l+2)}$ and hence $0 \leq d(x) \leq C_{13} r_p^{-2}$, where C_{12} and C_{13} are positive constants independent of u. Now, apply the Harnack inequality [8, Theorem 8.20] to $(\Delta_g - d(x))u = 0$ in a domain $\Omega = \overline{M}\backslash B_{4R}(p)$ for $R > 0$ small enough. Then, for any $y \in \partial B_{8R}(p)$, since $B_{4R}(y) \subset \Omega$, we get the estimate

$$\sup_{B_R(y)} u \leq C_{14} \inf_{B_R(y)} u$$

for some positive constant C_{14} depending only on n, g and C_{13}, and independent of y and R. We can cover $\partial B_{8R}(p)$ by a finite number (independent of R) of $B_R(y)$'s, and our assertion is proved. q.e.d.

PROOF OF THEOREM 3. We may assume $S_{\bar{g}} > 0$ without loss of generality.

The existence follows from [9, Theorems IV and V] (see also [10, Theorem 2]). In particular, for any $\gamma \in (0, +\infty)$, we also know the uniqueness of solutions u_γ of the equation (f, M) satisfying $\lim_{x\to p}(u_\gamma(x)/G_p(x)) = \gamma$.

Suppose u is a solution of the equation (f, M). If u/r_p^{2-n} is bounded, then it is clear that

$$\Phi_u(x) := \int_M G(x, y)|f(y)|u(y)^q\, dy$$

is a positive smooth function on M and satisfies $L_g \Phi_u = |f|u^q$. Note that $|f|u^q \leq C r_p^{l-n-2}$ for a positive constant C. Then we get, by standard calculation, that

$$\Phi_u \leq \begin{cases} Cr_p^{l-n} & \text{if } 2 < l < n \\ C\log(r_p^{-1}) & \text{if } l = n \\ C & \text{if } l > n \end{cases}$$

near p for a positive constant C. In particular, it holds that $\Phi_u(x)/G_p(x) \to 0$ as $x \to p$. Set $\varphi := u + \Phi_u$. Then φ is a solution of the equation $(0, M)$, and hence there is a positive number γ such that $\varphi \equiv \gamma G_p$ (see [9, Section 4]). Since $u(x)/G_p(x) = \gamma - \Phi_u(x)/G_p(x) \to \gamma$ as $x \to p$, we have $u \equiv u_\gamma$.

On the other hand, if u/r_p^{2-n} is unbounded, then $\limsup_{x\to p}(u(x)/r_p(x)^{2-n}) = +\infty$. Hence, there is a sequence $\{x_i\}_{i\in N}$ of points in M such that $\lim_{i\to+\infty} x_i = p$ and $u(x_i)/r_p(x_i)^{2-n} \geq C_{15} i$ for any i, where $C_{15} := C_2 C_{11}$, C_2 is a positive constant satisfying $G_p \leq C_2 r_p^{2-n}$, and C_{11} is the constant given in Lemma 4.1. By Lemma 4.1, it holds that $u(x)/G_p(x) \geq i$ on $\partial B_{r_p(x_i)}(p)$ for any i. Hence $u \geq u_i$ on $\partial B_{r_p(x_i)}(p)$, where u_i is the unique solution of the equation (f, M) satisfying $u_i \leq iG_p$ on M and $\lim_{x\to p}(u_i(x)/G_p(x)) = i$. Therefore, by [9, Lemma 2.2], $u \geq u_i$ in $M \backslash B_{r_p(x_i)}(p)$ for any i, from which it follows that $u \geq \lim_{i\to+\infty} u_i$ on M. Now, by Corollary 3.2, $\lim_{i\to+\infty} u_i \equiv U$. Since U is the maximal solution, we get $u \equiv U(\equiv u_\infty)$. This completes the proof. q.e.d.

As a consequence of Theorem 3, we have the following symmetry argument.

COROLLARY 4.2. *Let (M, g) and f be as in Theorem 3. If f is invariant under the action of some subgroup Γ of* $\mathrm{Isom}(M, g)$, *then any solution u of the equation (f, M) is also Γ-invariant.*

PROOF. Since any two solutions of the equation (f, M) with the same asymptotic behavior coincide with each other, the assertion above is clear. q.e.d.

The most typical example of this corollary is as follows.

EXAMPLE 4.3. Let $\bar{g}$ be a (non-standard) rotationally symmetric metric on $\mathbf{S}^n$ with $\lambda_1(L_{\bar{g}}) > 0$, p a central point of the symmetry. Let $(M, g) := (\mathbf{S}^n \backslash \{p\}, \bar{g}|_{S^n\backslash\{p\}})$. If a nonpositive smooth function f on M satisfies $f \sim -r_p^l$ near p for a number $l > 2$, and is rotationally symmetric, then any solution u of the equation (f, M) is also rotationally symmetric.

5. Solutions with mixed singular behavior

In this section, we construct examples of solutions which are not only maximal but also asymptotic to no solutions of the equation $(0, M)$. First, we prepare the following

LEMMA 5.1. *Let (M, g) be a Riemannian manifold ($n = \dim M \geq 3$), and f a nonpositive smooth function on M.*

(1) *If u_{1+} and u_{2+} are supersolutions of the equation (f, M), then $u_{1+} + u_{2+}$ is also a supersolution of the equation (f, M).*

(2) *If u_{1-} and u_{2-} are subsolutions of the equation (f, M), then $u_{1-} + u_{2-}$ is a subsolution of the equation $(2^{1-q}f, M)$.*

PROOF. Note that the following inequality holds.

$$2^{1-q} \leq \frac{s^q + t^q}{(s+t)^q} < 1 \quad \text{for } s, t > 0.$$

(1) Set $u_+ := u_{1+} + u_{2+}$. Then we get

$$L_g u_+ = L_g u_{1+} + L_g u_{2+} \geq f u_{1+}^q + f u_{2+}^q = f\,\frac{u_{1+}^q + u_{2+}^q}{(u_{1+} + u_{2+})^q}\, u_+^q \geq f u_+^q,$$

namely, u_+ is a supersolution of the equation (f, M).

(2) Set $u_- := u_{1-} + u_{2-}$. Then we get

$$L_g u_- = L_g u_{1-} + L_g u_{2-} \leq f u_{1-}^q + f u_{2-}^q = f\,\frac{u_{1-}^q + u_{2-}^q}{(u_{1-} + u_{2-})^q}\, u_-^q \leq 2^{1-q} f u_-^q,$$

namely, u_- is a subsolution of the equation $(2^{1-q}f, M)$. q.e.d.

Now, we observe the case when Σ consists of a finite number of points.

PROPOSITION 5.2. *Let $(\overline{M}, \bar{g})$ be a compact Riemannian manifold ($n = \dim \overline{M} \geq 3$) with $\lambda_1(L_{\bar{g}}) > 0$, Σ a set $\{p_1, \ldots, p_k\}$ of a finite number of points in $\overline{M}$, and $(M, g) := (\overline{M} \backslash \Sigma, \bar{g}|_{\overline{M} \backslash \Sigma})$. Set $G_\Sigma(x) := \Sigma_{i=1}^k G(p_i, x)$, where G is the Green function of $L_{\bar{g}}$. Let f be a nonpositive smooth function on M. If f satisfies $-C r_\Sigma^l \leq f < 0$ near Σ for a positive constant C and a number $l > 2$, then, for any $\gamma = (\gamma_1, \ldots, \gamma_k) \in (0, +\infty]^k$, the equation (f, M) possesses a solution u_γ such that $u_\gamma(x)/G_\Sigma(x) \to \gamma_i$ as $x \to p_i$ for any $i = 1, \ldots, k$ and the metric $u_\gamma^{q-1} g$ is complete. Namely, there are complete conformal metrics on M with scalar curvature f which are parametrized by $(0, +\infty]^k$.*

PROOF. When $\gamma \in (0, +\infty)^k$ or $\gamma = (+\infty, \dots, +\infty)$, the existence follows from [9, Theorems IV and V].

When some γ_i's are finite and the others are $+\infty$, we may assume $\gamma_i < +\infty$ for $i \le k'$ and $\gamma_i = +\infty$ for $i \ge k' + 1$ without loss of generality. Set $\Sigma_1 := \{p_1, \dots, p_{k'}\}$ and $\Sigma_2 := \{p_{k'+1}, \dots, p_k\}$.

Let $f_{1\pm}$ be nonpositive smooth functions on $\overline{M}\backslash\Sigma_1$ such that $f_{1+} \ge f \ge f_{1-}$ on M and $f_{1\pm} \equiv f$ near Σ_1, and u_{1+} (resp. u_{1-}) the solution of the equation $(f_{1+}, \overline{M}\backslash\Sigma_1)$ (resp. $(2^{q-1}f_{1-}, \overline{M}\backslash\Sigma_1)$) satisfying $\lim_{x\to p_i}(u_{1\pm}(x)/G_{\Sigma_1}(x)) = \gamma_i$ for any $i = 1, \dots, k'$. Then, by the same way as in the proof of [9, Lemma 2.2], we have $u_{1+} \ge u_{1-}$.

Let $f_{2\pm}$ be nonpositive smooth functions on $\overline{M}\backslash\Sigma_2$ such that $f_{2+} \ge f \ge f_{2-}$ on M and $f_{2\pm} \equiv f$ near Σ_2, and U_{2+} (resp. U_{2-}) the maximal solution of the equation $(f_{2+}, \overline{M}\backslash\Sigma_2)$ (resp. $(2^{q-1}f_{2-}, \overline{M}\backslash\Sigma_2)$). Then, by [9, Proposition 2.3 (1)], we have $U_{2+} \ge U_{2-}$.

Note here that both u_{1+} and U_{2+} are supersolutions of the equation (f, M), and that both u_{1-} and U_{2-} are subsolutions of the equation $(2^{q-1}f, M)$. Set $u_\pm := u_{1\pm} + U_{2\pm}$. Then clearly $u_+ \ge u_- > 0$ and , by Lemma 5.1, u_+ (resp. u_-) is also a supersolution (resp. subsolution) of the equation (f, M). Therefore the equation (f, M) possesses a solution u satisfying $u_+ \ge u \ge u_-$. Now, since

$$\begin{aligned}\frac{u_\pm(x)}{G_\Sigma(x)} &= \frac{u_{1\pm}(x)}{G_{\Sigma_1}(x)} \times \frac{G_{\Sigma_1}(x)}{G_{\Sigma_1}(x) + G_{\Sigma_2}(x)} + \frac{U_{2\pm}(x)}{G_\Sigma(x)} \\ &\to \gamma_i \times 1 + 0 = \gamma_i \quad \text{as } x \to p_i\end{aligned}$$

for any $i = 1, \dots, k'$, and

$$\begin{aligned}\frac{u_-(x)}{G_\Sigma(x)} &= \frac{u_{1-}(x)}{G_\Sigma(x)} + \frac{U_{2-}(x)}{G_{\Sigma_2}(x)} \times \frac{G_{\Sigma_2}(x)}{G_{\Sigma_1}(x) + G_{\Sigma_2}(x)} \\ &\to 0 + \infty \times 1 = +\infty \quad \text{as } x \to p_i\end{aligned}$$

for any $i = k' + 1, \dots, k$, we have $u(x)/G_\Sigma(x) \to \gamma_i$ as $x \to p_i$ for any $i = 1, \dots, k$.
q.e.d.

Since any solution of the equation $(0, M)$ coincides with $\Sigma_{i=1}^k \gamma_i G(p_i, x)$ for some $(\gamma_1, \dots, \gamma_k) \in [0, +\infty)^k \backslash \{(0, \dots, 0)\}$, if at least one of γ_i's is $+\infty$, then the solution u_γ given in the proof above does not behave asymptotically to any solution of the equation $(0, M)$.

Next, we consider solutions of the equation (f, M) which do not give complete metrics.

EXAMPLE 5.3. Let $(\overline{M}, \bar{g})$ be a compact Riemannian manifold ($n = \dim \overline{M} \geq 3$) with $\lambda_1(L_{\bar{g}}) > 0$, Σ a compact submanifold ($(n-2)/2 < d = \dim \Sigma < n-2$), and $(M, g) := (\overline{M} \backslash \Sigma, \bar{g}|_{\overline{M} \backslash \Sigma})$. Let f be a nonpositive smooth function on M satisfying $f \sim -1$ near Σ. Let Σ' be a compact submanifold of $\Sigma((n-2)/2 < d' = \dim \Sigma' < d)$. Then the equation (f, M) possesses a solution u such that

$$\begin{cases} \dfrac{u(x)}{G_{\Sigma'}(x)} \to +\infty & \text{as } x \to \Sigma' \\ \dfrac{u(x)}{G_{\Sigma}}(x) \to 1 & \text{as } x \to \Sigma \backslash B_{\varepsilon}(\Sigma') \text{ for any } \varepsilon > 0, \end{cases}$$

where G_{Σ} and $G_{\Sigma'}$ are as in the proof of Theorem 2.

Indeed, let $f_{\pm}$ be a nonpositive smooth function on $\overline{M}$ such that $f_{+} \geq f \geq f_{-}$ on M and $f_{\pm} < 0$ near Σ. Let u_{1+} (resp. u_{1-}) be a solution of the equation (f_{+}, M) (resp. $(2^{q-1} f_{-}, M)$) satisfying $\lim_{x \to \Sigma}(u(x)/G_{\Sigma}(x)) = 1$ which is given e.g. by combining Delanoë [6, Theorem 5] and the proof of [9, Theorem V], and U_{2+} (resp. U_{2-}) the maximal solution of the equation $(f_{+}, \overline{M} \backslash \Sigma')$ (resp. $(2^{q-1} f_{-}, \overline{M} \backslash \Sigma')$). Then, by the same consideration as in the proof of Proposition 5.2, we get a solution u of the equation (f, M) with the desired property.

In particular, if u is asymptotic to a solution φ of $(0, M)$, then $\varphi(x)/G_{\Sigma'}(x) \to +\infty$ as $x \to \Sigma'$. Therefore, by the maximal principle, $\varphi \geq \gamma G_{\Sigma'}$ for any $\gamma > 0$, namely $\varphi \equiv +\infty$. Hence there are no such φ, namely, u does not behave asymptotically to any solution of the equation $(0, M)$. This solution is, of course, of essentially different type from the solutions constructed by Finn-McOwen [7, Section 6], since each of them is asymptotic to a solution $\gamma G_{\Sigma} + \gamma' G_{\Sigma'}$ of $(0, M)$ for some γ and γ'.

Repeating the same process as the proof above, we can construct solutions with more complicated behavior. Consequently, for generic $(M, g) = (\overline{M} \backslash \Sigma, \bar{g}|_{\overline{M} \backslash \Sigma})$ and f, the space of solutions of the equation (f, M) has more complicated structure than that of $(f, \overline{M} \backslash \{p\})$.

6. The case f has a compact support

In the last section of this paper, we consider the structure of the space of solutions of the equation (f, M) satisfying $\lim_{x \to \Sigma} u(x) = +\infty$ in the case when f has a compact support. The following result are partially observed in [9, Section 4]. Here, we state it precisely with an outline of the proof.

THEOREM 6.1. *Let $(\overline{M}, \bar{g})$ be a compact Riemannian manifold* $(n = \dim \overline{M} \geq 3)$ *with* $\lambda_1(L_{\bar{g}}) > 0$, Σ *a compact submanifold* $(d = \dim \Sigma \leq n-2)$, *and* $(M, g) := (\overline{M} \backslash \Sigma, \bar{g}|_{\overline{M} \backslash \Sigma})$. *Let f be a nonpositive smooth function on M. If the support of f is compact, then the space of solutions of the equation (f, M) satisfying* $\lim_{x \to \Sigma} u(x) = +\infty$ *is parametrized naturally by that of* $(0, M)$.

PROOF. Denote the space of solutions of the equation (f, M) (resp. $(0, M)$) satisfying $\lim_{x \to \Sigma} u(x) = +\infty$ by $\mathscr{M}_f$ (resp. $\mathscr{M}_0$). For any $\varphi \in \mathscr{M}_0$, if there exists a solution $u \in \mathscr{M}_f$ satisfying $\lim_{x \to \Sigma}(u(x)/\varphi(x)) = 1$, then write $\mathscr{F}_f(\varphi) = u$. Now, we show that the map $\mathscr{F}_f : \mathscr{M}_0 \to \mathscr{M}_f$ is well-defined and bijective.

Since $f \equiv 0$ near Σ, it is clear that, for any $\varphi \in \mathscr{M}_0$,

$$\Phi_\varphi(x) := \int_{\overline{M}} G(x,y)|f(y)|\varphi(y)^q dy$$

is a positive smooth function on $\overline{M}$ and satisfies $L_{\bar{g}} \Phi_\varphi = |f| \varphi^q$. Set $\beta := (2 \sup_M (\Phi_\varphi / \varphi))^{-1}$ and define functions $u_\pm$ on M by

$$u_{\gamma+} := \gamma\varphi, \quad u_{\gamma-} := \gamma(\varphi - \beta\Phi_\varphi).$$

Then $u_{\gamma+} \geq u_{\gamma-} > 0$ and it can be easily checked that, for any $0 < \gamma \leq \beta^{1/(q-1)}$, $u_{\gamma+}$ (resp. $u_{\gamma-}$) is a supersolution (resp. subsolution) of the equation (f, M) (see the proof of [10, Theorem 2′]). Hence, by the method of supersolutions and subsolutions, the equation (f, M) possesses a solution u_γ satisfying $u_{\gamma+} \geq u_\gamma \geq u_{\gamma-}$. In particular, since $\varphi(x) \to +\infty$ as $x \to \Sigma$ and Φ_φ is bounded, u_γ satisfies $\lim_{x \to \Sigma}(u_\gamma(x)/\varphi(x)) = \gamma$. Now, by the same way as in the proof of [9, Theorem V] (see also the proof of Lemma 3.1 of this paper), we can show that γ can take an arbitrarily positive value. Set $u := u_1$. Then u is a solution of the equation (f, M) satisfying $\lim_{x \to \Sigma}(u(x)/\varphi(x)) = 1$. In particular, $u \in \mathscr{M}_f$.

For any $\varphi \in \mathscr{M}_0$, set $\hat{g} := \varphi^{q-1} g$. Then $u = \mathscr{F}_f(\varphi)$ satisfies $-a_n \Delta_{\hat{g}}(u/\varphi) = f(u/\varphi)^q$ on M. Therefore, the uniqueness of $\mathscr{F}_f(\varphi)$ is established by the same method as in Cheng-Ni [5, Theorem 3.1], and hence the map $\mathscr{F}_f$ is well-defined. Moreover, if $\mathscr{F}_f(\tilde{\varphi}) = \mathscr{F}_f(\varphi)$ for some $\tilde{\varphi} \in \mathscr{M}_0$, then we get $\Delta_{\hat{g}}(\tilde{\varphi}/\varphi) = 0$ on M and $\lim_{x \to \Sigma}(\tilde{\varphi}(x)/\varphi(x)) = 1$. Therefore the injectivity of $\mathscr{F}_f$ also follows from the maximal principle, and we have only to show the surjectivity.

For any $u \in \mathscr{M}_f$, set

$$\Phi_u(x) := \int_{\overline{M}} G(x,y)|f(y)|u(y)^q \, dy.$$

Then Φ_u is a positive smooth function on $\overline{M}$ and satisfies $L_{\bar{g}}\Phi_u = |f|u^q$. Set $\varphi := u + \Phi_u$. Then φ is a solution of the equation $(0, M)$ satisfying $\lim_{x\to\Sigma}(u(x)/\varphi(x)) = 1$, namely, $\varphi \in \mathscr{M}_0$ and $\mathscr{F}_f(\varphi) = u$. q.e.d.

Even if $d = n-1$ or Σ is not a submanifold, the proof above is valid. However both $\mathscr{M}_0$ and $\mathscr{M}_f$ are empty in the case when $d = n-1$ and some other cases.

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