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ISOPARAMETRIC HYPERSURFACES IN A SPACE FORM AND METRIC CONNECTIONS

Dedicated to Professor Yoshihiro Tashiro for his 70th birthday

By

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1. Introduction

As is known, homogeneous hypersurfaces in a space form are isoparametric (for instance, see [2], [4], [5], [6] and [7]), but there are many non-homogeneous isoparametric hypersurfaces ([3] and [8]). On the other hand, T. Tricerri and L. Vanhecke ([9]) have introduced the notion of a homogeneous Riemannian structure on a Riemannian manifold, following the homogeneous results which were given by W. Ambrose and I. M. Singer ([1]).

The main purpose of this paper is to study the existence of a metric connection on hypersurfaces in a space form such that this connection determines the hypersurfaces to be isoparametric, and to investigate some properties of isoparametric hypersurface with the connection. This metric connection makes the shape operator of the hypersurfaces parallel and the torsion tensor of this connection gives rise to a homogeneous Riemannian structure under a certain conditions. In fact, after a brief survey of a hypersurface in a space of constant curvature in section 2, we give the above mentioned connection (see Theorem 3.1), and investigate properties of isoparametric hypersurfaces in a space form in section 3. In section 4 we study hypersurface with a homogeneous Riemannian structure in a space of constant curvature. We show that a connection defined by the homogeneous Riemannian structure and the Levi-Civita connection makes the shape operator parallel, and some properties concerned with isoparametric ones are obtained.

2. Preliminaries

Let $(\hat{M}^{n+1}(c), \hat{g})$ be an $(n+1)$-dimensional space of constant curvature c, that is, a Riemannian manifold with the curvature form $\hat{R}$ defined by

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$$\hat{R}(\hat{X}, \hat{Y}) = [\hat{\nabla}_{\hat{X}}, \hat{\nabla}_{\hat{Y}}] - \hat{\nabla}_{[\hat{X}, \hat{Y}]} = c\hat{X} \wedge \hat{Y},$$

where $\hat{\nabla}$ is its Levi-Civita connection and $\hat{X} \wedge \hat{Y}$ is given by

$$(\hat{X} \wedge \hat{Y})\hat{Z} = \hat{g}(\hat{Y}, \hat{Z})\hat{X} - \hat{g}(\hat{X}, \hat{Z})\hat{Y}$$

for any vector fields $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ on $\hat{M}$. If $\hat{M}^{n+1}(c)$ is simply connected and complete, then it is called a *space form.*

Let M^n be an n-dimensional Riemannian manifold, and $\iota : M^n \to \hat{M}^{n+1}(c)$ be an isometric immersion of M into a space $\hat{M}$ of constant curvature. The Riemannian metric of M is the induced one $g = \iota^*\hat{g}$, and the Levi-Civita connection on M with respect to the metric g is denoted by ∇. Then the Gauss and Weingarten formulas are given by

$$\hat{\nabla}_X Y = \nabla_X Y + g(AX, Y)N$$

and

$$\hat{\nabla}_X N = -AX$$

for any vector fields X and Y on M and normal vector field N to M respectively, where A is the shape operator of ι. The *principal curvatures* of the hypersurface M are the eigenvalues of the shape operator A of ι, and the *type number* $t(p)$ of M at $p \in M$ is defined by the rank of A at p, that is, the number of non-zero principal curvatures of M at p. If the principal curvatures of M are constant, then M is said to be an *isoparametric hypersurface* of $\hat{M}^{n+1}(c)$. Let R be the curvature form with respect to the Levi-Civita connection ∇ on M. Then the equations of Gauss and Codazzi are given by

$$R(X, Y) = cX \wedge Y + AX \wedge AY \tag{2.1}$$

and

$$(\nabla_X A)Y = (\nabla_Y A)X \tag{2.2}$$

for any vector fields X and Y on M respectively. It follows from (2.1) that

$$(\nabla_X R)(Y, Z) = (\nabla_X A)Y \wedge AZ + AY \wedge (\nabla_X A)Z \tag{2.3}$$

for any vector fields X, Y and Z on M. If $\nabla_X R = 0$ identically, then M is said to be *locally symmetric.*

On the other hand, if a Riemannian manifold (M,g) admits the tensor field T of type $(1,2)$ such that

$$g(T_X Y,Z)+g(Y,T_X Z)=0, \tag{2.4}$$

$$(\nabla_X R)(Y,Z)=[T_X,R(Y,Z)]-R(T_X Y,Z)-R(Y,T_X Z), \tag{2.5}$$

$$(\nabla_X T)_Y=[T_X,T_Y]-T_{T_X Y} \tag{2.6}$$

for any vector fields X, Y and Z on M, then T is said to be a *homogeneous Riemannian structure* (see [9]). It is known ([1], [9]) that a connected, complete and simply connected Riemannian manifold M is homogeneous if and only if M admits a homogeneous Riemannian structure.

3. Isoparametric hypersurfaces

In this section we assume that $\hat{M}^{n+1}(c)$ is a space form and M^n is a hypersurface in $\hat{M}^{n+1}(c)$. Then we first prove

THEOREM 3.1. *M is an isoparametric hypersurface in a space form $\hat{M}(c)$ if and only if there exists a metric connection D on M so that the shape operator A is parallel.*

PROOF. Assume that M is isoparametric. Let

$$\mathcal{O}(M)=\{u=(p;e_1,\dots,e_n)\,|\,p\in M,\ g(e_i,e_j)=\delta_{ij}\}$$

be an orthonormal frame bundle over M, where and in the sequal the indices $i,j,k,\dots$ run over the range $1,2,\dots,n$. Then $\mathcal{O}(N)$ is a principal fibre bundle with structure group $\mathcal{O}(n)$. Let

$$A_p e_j=\sum_i A_{ji}(u)e_i. \tag{3.1}$$

Then $A_{ji}(u)$ are differentiable functions on $\mathcal{O}(M)$. If we put

$$\mathscr{B}=\{u\in\mathcal{O}(M)\,|\,A_{ji}(u)=\lambda_j(u)\delta_{ji}\}, \tag{3.2}$$

where λ_j are the principal curvatures of M, then $\mathscr{B}$ is a subbundle of $\mathcal{O}(M)$ and the structure group G of $\mathscr{B}$ is closed since λ_j are constant. It is easily seen from (3.1) and (3.2) that a connection on $\mathscr{B}$ induces a metric connection D on M so that $DA=0$.

Conversely, if there exists a metric connection D on M so that $DA=0$, then it is easily seen that the principal curvatures are constant. ■

From now on, we assume that M is an isoparametric hypersurface in $\hat{M}^{n+1}(c)$ and the shape operator A of M is parallel by the metric connection D, Then it follows from (2.1) that

$$D_X R = 0 \tag{3.3}$$

for any vector field X on M. Let S be the torsion tensor of D. Then we see that $D - (1/2)S$ is a torsion-free connection on M, and it is equal to the Levi-Civita connection ∇ on M if and only if the torsion tensor S satisfies

$$g(S_X Y, Z) + g(Y, S_X Z) = 0 \tag{3.4}$$

for any vector fields X, Y and Z on M.

We also asumme that the connection $D - (1/2)S$ is the Levi-Civita connection, that is,

$$\nabla = D - \tfrac{1}{2}S. \tag{3.5}$$

Since $D_X A = 0$ by Theorem 3.1, it follows from (3.5) that

$$\nabla_X A = \tfrac{1}{2}[A, S_X], \tag{3.6}$$

and from (2.2) and (3.6) that

$$2AS_X Y = S_X AY - S_Y AX. \tag{3.7}$$

Let $c_1, \ldots, c_m (m \leq n)$ be mutually distinct principal curvatures of M, and V_α be the eigenspace of A relative to the eigenvalue $c_\alpha (\alpha = 1, \ldots, m)$. We choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ of M such that $Ae_i = \lambda_i e_i$. Then we have

LEMMA 3.2. *Assume that $D - (1/2)S$ is the Levi-Civita connection ∇ on M. Then we have*

$$[A, S_X] = 0 \tag{3.8}$$

for any vector field X on M.

PROOF. Let $\alpha \neq \beta$. Substituting $X = e_\alpha \in V_\alpha$ and $Y = e_\beta \in V_\beta$ into (3.7), we obtain

$$2AS_{e_\alpha} e_\beta = (c_\alpha + c_\beta) S_{e_\alpha} e_\beta. \tag{3.9}$$

If we multiply (3.9) by $e_q \in V_\alpha$, then we get

$$g(S_{e_\alpha} e_\beta, e_q) = 0 \tag{3.10}$$

because $c_\alpha \neq c_\beta$. For the vector $e_\gamma \in V_\gamma$ $(\gamma \neq \alpha, \beta)$, we also get

$$(c_\alpha + c_\beta - 2c_\gamma)g(S_{e_\alpha}e_\beta, e_\gamma) = 0. \tag{3.11}$$

On the other hand, if we substitute $X = e_\alpha \in V_\alpha$ and $Y = e_\gamma \in V_\gamma$ into (3.7) and multiply it by $e_\beta \in V_\beta$, we find

$$(c_\alpha + c_\gamma - 2c_\beta)g(S_{e_\alpha}e_\gamma, e_\beta) = 0. \tag{3.12}$$

Therefore it follows from (3.4), (3.11) and (3.12) that

$$g(S_{e_\alpha}e_\beta, e_\gamma) = 0. \tag{3.13}$$

Thus, for $e_\alpha \in V_\alpha$ and $e_\beta \in V_\beta$ $(\alpha \neq \beta)$, (3.10) and (3.13) show that

$$S_{e_\alpha}e_\beta = 0. \tag{3.14}$$

Let e_p, $e_q \in V_\alpha$. Then, by putting $X = e_p$ and $Y = e_q$ into (3.7), we easily see that

$$AS_{e_p}e_q = c_\alpha S_{e_p}e_q. \tag{3.15}$$

The relation (3.8) follows from (3.14) and (3.15). ■

REMARK 3.3. As seen in the proof of Lemma 3.2, if all the multiplicities of the principal curvatures of M are not greater than 3, then D is torsion-free, that is, $S = 0$.

From (2.3) and Lemma 3.2, we can state

THEOREM 3.4. *Let M be an isoparametric hypersurface in a space form $\hat{M}^{n+1}(c)$, D a metric connection on M so that the shape operator A is parallel and S the torsion tensor of D. If $D - (1/2)S$ is the Levi-Civita connection on M, then M is locally symmetric.*

In this case, M has at most two distinct principal curvatures.

PROOF. If M has more than two distinct principal curvatures, then we have $[A, S_X] = 0$ by Lemma 3.2. If M is totally umbilical (included totally geodesic), then it is clear that $[A, S_X] = 0$. Therefore it follows from (2.3) and (3.6) that M is locally symmetric. Since we have $\nabla_X A = 0$ from (3.6) and hence

$$R(X, Y)AZ = AR(X, Y)Z,$$

we obtain from the equation (2.1) of Gauss

$$
\begin{aligned}
(3.16)\qquad & c\{g(Y,AZ)X - g(X,AZ)Y\} + g(AY,AZ)AX - g(AX,AZ)AY \\
&= c\{g(Y,Z)AX - g(X,Z)AY\} + g(AY,Z)A^2X - g(AX,Z)A^2Y
\end{aligned}
$$

for any vector fields (X, Y and Z on M. With respect to the orthonormal frame field $\{e_1, \dots, e_n\}$ of M which diagonalize A, if we substitute $X = e_i$, $Y = e_j$ and $Z = e_k$ into (3.16), we have

$$(\lambda_i - \lambda_j)(c + \lambda_i\lambda_j) = 0,$$

where λ_j are principal curvatures of M. This implies that the number of the distinct principal curvatures is at most two. ■

Let K be the curvature tensor of M with respect to the connection D. Then it follows from (3.5) that

$$(3.17)\qquad K(X,Y) = R(X,Y) + \tfrac{1}{2}\{(D_XS)_Y - (D_YS)_X\} - \tfrac{1}{4}[S_X, S_Y] + \tfrac{1}{2}S_{S_XY}$$

for any vector fields X and Y on M. We put

$$T_X = -\tfrac{1}{2}S_X.$$

Then, since S_XY is skew-symmetric for X and Y, we have

$$S_XY = -T_XY + T_YX.$$

The relation (3.4) is equivalent to

$$(3.18)\qquad g(T_XY,Z) + g(Y,T_XZ) = 0.$$

If the torsion tensor S is parallel by D, that is,

$$(3.19)\qquad D_XS = 0,$$

then we see from (3.3) and (3.17) that

$$(3.20)\qquad D_XK = 0.$$

It is easily seen that the conditions (3.18), (3.19) and (3.20) are equivalent to that of (2.4), (2.5) and (2.6) (see [9]). Thus we can state

THEOREM 3.5. *Let M be an isoparametric hypersurface in a space form $\hat{M}^{n+1}(C)$, D a metric connection on M so that the shape operator is parallel, and S the torsion tensor of D. Assume that $D - (1/2)S$ is the Levi-Civita connection* *on M. Then the torsion tensor S defines a homogeneous Riemannian structure on M if and only if it is parallel by D.*

4. Homogeneous Riemannian structures

In this section we assume that the ambient manifold $\hat{M}^{n+1}(c)$ is a space of constant curvature c, $\iota : M^n \to \hat{M}^{n+1}(c)$ is an isometric immersion of an n-dimensional Riemannian manifold M into $\hat{M}^{n+1}(c)$, and M admits a homogeneous Riemannian structure T. Then it follows from (2.1) and (2.5) that

$$(\nabla_X R)(Y,Z) = (T_X A - AT_X)Y \wedge AZ + AY \wedge (T_X A - AT_X)Z \tag{4.1}$$

for any vector fields X, Y and Z on M. Comparing (2.3) with (4.1), we have

$$\begin{aligned}\{A(\nabla_X Y - T_X Y) - \nabla_X AY + T_X AY\} \wedge AZ \\ = \{A(\nabla_X Z - T_X Z) - \nabla_X AZ + T_X AZ\} \wedge AY.\end{aligned} \tag{4.2}$$

Now we prove

THEOREM 4.1. *Let $\iota : M^n \to \hat{M}^{n+1}(c)$ be an isometric immersion of an n-dimensional Riemannian manifold M into an $(n+1)$-dimensional space $\hat{M}^{n+1}(c)$ of constant curvature and ∇ be the Levi-Civita connection on M. Assume that M admits a homogeneous Riemannian structure T and the type number of ι is not equal to* 1 *and* 2. *Then $D = \nabla - T$ is a metric connection on M so that the shape operator A of ι is parallel and M is an isoparametric hypersurface in $\hat{M}^{n+1}(c)$.*

PROOF. At first, we see easily from (2.4) that D is a metric connection. It follows from (4.2) that

$$(AD_X Y - D_X AY) \wedge AZ = (AD_X Z - D_X AZ) \wedge AY. \tag{4.3}$$

Let $\{e_1, \dots, e_n\}$ be the local orthonormal frame field of M such that $Ae_i = \lambda_i e_i$. Substituting $Y = e_i$ and $Z = e_j$ into (4.3), we obtain

$$\lambda_j(AD_X e_i - \lambda_i D_X e_i - X\lambda_j \cdot e_i) \wedge e_j = \lambda_i(AD_X e_j - \lambda_j D_X e_j - X\lambda_j \cdot e_j) \wedge e_i. \tag{4.4}$$

If we multiply (4.4) by e_j, then we get

$$\lambda_j(AD_X e_i - \lambda_i D_X e_i) = X(\lambda_i \lambda_j) \cdot e_i + \lambda_j(\lambda_j - \lambda_i) g(D_X e_i, e_j) e_j \tag{4.5}$$

for $i \neq j$. Multiplying (4.5) by e_i, we find

$$X(\lambda_i \lambda_j) = 0 \tag{4.6}$$

for $i \neq j$, that is, $\lambda_i\lambda_j$ is a constant on M. If we compare (4.5) with (4.6), then we have

$$\lambda_j(AD_X e_i - \lambda_i D_X e_i) = \lambda_j(\lambda_j - \lambda_i)g(D_X e_i, e_j)e_j \tag{4.7}$$

for any $i(\neq j)$.

If the type number of ι is greater than 2, then for any index i there exist two indices $j(\neq i)$ and $k(\neq i)$ such that $j \neq k$, $\lambda_j = 0$ and $\lambda_k \neq 0$. Then from (4.6) we have

$$\lambda_i^2\lambda_j^2\lambda_k^2 = (\lambda_i\lambda_j)(\lambda_j\lambda_k)(\lambda_k\lambda_i) = \text{constant}$$

And also we have

$$\lambda_j\lambda_k = \text{constant} \neq 0$$

So we can conclude that λ_i is constant and M is isoparametric.

If we compare (4.5) with (4.6), then we have

$$AD_X e_i - \lambda_i D_X e_i = (\lambda_j - \lambda_i)g(D_X e_i, e_j)e_j.$$

Changing the index j to k we have also

$$AD_X e_i - \lambda_i D_X e_i = (\lambda_k - \lambda_i)g(D_X e_i, e_k)e_k.$$

Since e_j and e_k are linearly independent, we have

$$AD_X e_i - \lambda_i D_X e_i = 0 \tag{4.8}$$

Then we have

$$AD_X e_i - D_X A e_i = AD_X e_i - \lambda_i D_X e_i - X\lambda_i \cdot e_i = 0.$$

Hence it follows that

$$AD_X Y = D_X A Y \tag{4.9}$$

for any vector fields X and Y on M, which shows that A is parallel with respect to D.

If the type number is equal to zero, then M is isoparametric and it is clear that (4.9) is satisfied.

This completes the proof. ∎

REMARK 4.2. Under the assumptions of Theorem 4.1, if the type number of ι is equal to 2 and the non-zero principal curvatures are equal to each other, then we also see from (4.6) that M is an isoparametric hypersurface in $\hat{M}^{n+1}(c)$.

Since the shape operator A is parallel by the connection $D = \nabla - T$ by Theorem 4.1, we have

$$\nabla_X A = [T_X, A] \tag{4.10}$$

for any vector field X on M. It follows from the equation (2.2) of Codazzi and (4.10) that

$$T_X AY - T_Y AX = A(T_X Y - T_Y X). \tag{4.11}$$

Differentiating (4.10) covariantly with respect to the Levi-Civita connection ∇ on M and using (2.4), (2.6), (4.10) and (4.11), we have

$$(R(X, Y)A)Z = [[T_X, T_Y], A]Z - [T_Z, A](T_X Y - T_Y X). \tag{4.12}$$

Since it is easily seen that $(R(X, Y)A) = R(X, Y)A - AR(X, Y)$ on M, it follows from (2.1) and (4.12) that

$$\begin{aligned} &[[T_X, T_Y], A]Z - [T_Z, A](T_X Y - T_Y X) \\ &\quad = c\{g(Y, AZ)X - g(X, AZ)Y - g(Y, Z)AX + g(X, Z)AY\} \\ &\qquad + g(AY, AZ)AX - g(AX, AZ)AY - g(AY, Z)A^2X + g(AX, Z)A^2Y \end{aligned} \tag{4.13}$$

for any vector fields X, Y and Z on M. Now we state

THEOREM 4.3. *Let $\imath : M^n \to \hat{M}^{n+1}(c)$ be an isometric immersion of an n-dimensional Riemannian manifold M into an $(n+1)$-dimensional space $\hat{M}^{n+1}(c)$ of constant curvature and A be the shape operator of $\imath$. Assume that M admits a homogeneous Riemannian structure T and the type number of $\imath$ is not equal to* 1 *and* 2. *Then the followings are mutually equivalent*:

(1) *M is locally symmetric*;
(2) $[T_X, A] = 0$ *for any vector field X on M*;
(3) *M has at most two distinct principal curvatures.*

PROOF. If M is locally symmetric, then it follows from (2.3) that

$$(A\nabla_X Y - \nabla_X AY) \wedge AZ = (A\nabla_X Z - \nabla_X AZ) \wedge AY \tag{4.14}$$

for any vector fields X, Y and Z on M. As a similar argument as the proof of parallelism of A by D in Theorem 4.1, we can verify from (4.14) that $(\nabla_X A)Y = 0$, and hence (2) follows from (3.10).

Assume that (2) holds on M. Since it is easily seen that

$$[[T_X, T_Y], A] + [[T_Y, A], T_X] + [[A, T_X], T_Y] = 0$$

on M, we see that the left hand side of (4.13) vanishes identically, and hence we have

$$\begin{aligned}(4.15)\qquad &\{g(AY,AZ)AX - g(AX,AZ)AY - g(AY,Z)A^2X + g(AX,Z)A^2Y\\ &+ c\{g(Y,AZ)X - g(X,AZ)Y - g(Y,Z)AX + g(X,Z)AY\} = 0.\end{aligned}$$

Let $\{e_1,\dots,e_n\}$ be the local orthonormal frame field of M such that $Ae_i = \lambda_i e_i$. Then, substituting $Y = Z = e_i$ into (4.15), we have

$$(4.16)\qquad \lambda_i A^2 + (c - \lambda_i^2)A - c\lambda_i I = 0,$$

where I is the identity transformation on M. Applying e_j to (4.16), we obtain

$$(\lambda_i - \lambda_j)(c + \lambda_i\lambda_j) = 0,$$

which shows that M has at most two distinct principal curvatures. Thus (3) is proved.

Now we assume that M has at most two distinct principal curvatures. If M is totally umbilical (included totally geodesic), then M is locally symmetric. Therefore to show (1), we only consider that M has just two principal curvatures λ and μ. Since the type number is not equal to 1 at each point of M, we see from Theorem 4.1 and Remark 4.2 that λ and μ are constant. By arranging the orthonormal frame field if necessary, we put $Ae_p = \lambda e_p$ and $Ae_\alpha = \mu e_\alpha$, where and in the sequal the indices p, q, r and α, β, γ run over the ranges $1,\dots,\ell$ and $\ell+1,\dots,n$ respectively. Since it is easily seen that

$$(\nabla_X A)e_p = (\lambda - \mu)\sum_\beta g(\nabla_X e_p, e_\beta)e_\beta,$$
$$(\nabla_X A)e_\alpha = (\mu - \lambda)\sum_q g(\nabla_X e_\alpha, e_q)e_q$$

for any vector field X on M, then we have

$$(4.17)\qquad g(\nabla_{e_p} e_\alpha, e_q) = 0, \quad g(\nabla_{e_\alpha} e_p, e_\beta) = 0$$

from the equation (2.2) of Codazzi, Using (4.17), we see that

$$(\nabla_{e_p} A)e_q = 0, \quad (\nabla_{e_p} A)e_\beta = 0, \quad (\nabla_{e_\alpha} A)e_q = 0, \quad (\nabla_{e_\alpha} A)e_\beta = 0$$

identically. This shows that $\nabla_X A = 0$ for any vector field X on M, and hence (1) follows from (2.3). ■

Let $V = T_p(M)$, $T_p(M)$ being the tangent space of M at $p \in M$, and let $C(V)$ be the set of all homogeneous Riemannian structures on M. Then it is known ([9]) that $C(V)$ is a Euclidean space and the orthogonal direct sum of the following subspace;

$$C_1(V) = \{T \in C(V) | T_X Y = g(X, Y)\xi - g(\xi, Y)X, \xi \in V\}, \tag{4.18}$$

$$C_2(V) = \{T \in C(V) | \oplus g(T_X Y, Z), 0, \text{trace } T = 0\}, \tag{4.19}$$

$$C_3(V) = \{T \in C(V) | T_X Y + T_Y X = 0\}, \tag{4.20}$$

where. $\oplus$ denotes the cyclic sum for X, Y and Z. These subspaces $C_1(V)$, $C_2(V)$ and $C_3(V)$ are invariant and irreducible under the action of the orthogonal group $O(V)$ on $C(V)$ ([9]). Finally we prove

THEOREM 4.4. *Let $\iota : M^n \to \hat{M}^{n+1}(c)$ be an isometric immersion of an n-dimensional Riemannian manifold M into an $(n+1)$-dimensional space $\hat{M}$ of constant curvature. Assume that the type number of ι is not equal to* 1 *and* 2. *If M admits a homogenous Riemannian structure T belonging to each subspace $C_a (a = 1, 2, 3)$, then M is locally symmetric.*

PROOF. By Theorem 4.1, $D = \nabla - T$ is a metric connection on M and $D_X A = 0$ for any vector field X on M. Let $\{e_1, \dots, e_n\}$ be the orthonormal frame field of M such that $Ae_i = \lambda_i e_i$. Then, putting $X = e_i$ and $Y = e_j$ into (4.11) and multiplying it by e_k, we obtain

$$(\lambda_j - \lambda_k) T_{ijk} = (\lambda_i - \lambda_k) T_{jik}, \tag{4.21}$$

where we have put

$$T_{ijk} = g(T_{e_i} e_j, e_k).$$

Let $T \in C_1$. Then, substituting (4.18) into (4.13), we have

$$\begin{aligned}
&\{\eta(Y)\eta(AZ) - |\xi|^2 g(Y, AZ)\}X - \{\eta(X)\eta(AZ) - |\xi|^2 g(X, AZ)\}Y \\
&\quad + \{\eta(X)\eta(AY) - \eta(Y)\eta(AX)\}Z - \{\eta(Y)\eta(Z) - |\xi|^2 g(Y, Z)\}AX \\
&\quad + \{\eta(X)\eta(Z) - |\xi|^2 g(X, Z)\}AY \\
&= c\{g(Y, AZ)X - g(X, AZ)Y - g(Y, Z)AX + g(X, Z)AY\} \\
&\quad + g(AY, AZ)AX - g(AX, AZ)AY - g(AY, Z)A^2 X + g(AX, Z)A^2 Y,
\end{aligned} \tag{4.22}$$

where η is the dual 1-form of the vector field ξ on M and $|\xi|^2$ is the square length of ξ. Putting $Y = Z = e_i$ into (4.22), we obtain

$$\{\eta(e_i)^2 - |\xi|^2\}(AX - \lambda_i X) - \eta(e_i)\{\eta(AX) - \lambda_i\eta(X)\}e_i = \lambda_i A^2 X + (c - \lambda_i^2)AX - c\lambda_i X. \tag{4.23}$$

If we multiply (4.23) by e_i, then we find

$$\eta(e_i)\{\eta(AX) - \lambda_i\eta(X)\} = 0, \tag{4.24}$$

and hence (4.23) is reduced to

$$\lambda_i A^2 + (c - \lambda_i^2 + |\xi|^2 - \eta(e_i)^2)A - \lambda_i(c + |\xi|^2 - \eta(e_i)^2)I = 0,$$

where I is the identity transformation. Appying e_j to the above equation, we have

$$(\lambda_j - \lambda_i)(c + \lambda_i\lambda_j + |\xi|^2 - \eta(e_i)^2) = 0. \tag{4.25}$$

It follows from (4.24) and (4.25) that M is totally umbilical. In fact, if there are two distinct principal curvatures λ_i and λ_j, we obtain $\eta(e_i)^2 = \eta(e_j)^2$ from (4.25) by exchanging i and j. Comparing this fact with (4.24), we see that $\eta(e_i) = 0$ for any i, that is, $T = 0$, and $T \notin C_1$. Therefore M is locally symmetric.

If the homogeneous Riemannian structure T satisfies $\oplus g(T_X Y, Z) = 0$, or equivalently,

$$T_{ijk} + T_{jki} + T_{kij} = 0, \tag{4.26}$$

then, by comparing (4.21) with (4.26), we have

$$(\lambda_j - \lambda_i)T_{ijk} = (\lambda_k - \lambda_i)T_{kji}. \tag{4.27}$$

Multiplying (4.27) by $\lambda_j - \lambda_i$ and using (4.21), we obtain

$$\{(\lambda_i - \lambda_j)^2 + (\lambda_j - \lambda_k)^2 + (\lambda_k - \lambda_i)^2\}T_{ijk} = 0. \tag{4.28}$$

Moreover, substituting $X = e_i$ and $Y = Z = e_j$ into (4.13), we find

$$\sum_{k,l}\{(\lambda_j - \lambda_l)(T_{jjk} - T_{ikl}) - (\lambda_k - \lambda_l)T_{kji}T_{jkl}\}e_l = (\lambda_j - \lambda_i)(c + \lambda_i\lambda_j)e_i \tag{4.29}$$

for $i \neq j$. Since it is easily seen from (4.21) and (4.27) that

$$(\lambda_j - \lambda_i)T_{iij} = 0 \quad \text{and} \quad (\lambda_k - \lambda_i)T_{kji}T_{jki} = (\lambda_j - \lambda_i)T_{ijk}T_{jki}$$

respectively, then we obtain

$$(\lambda_j - \lambda_i)\Big(c + \lambda_i\lambda_j - 2\sum_k T_{ijk}T_{jik}\Big) = 0 \tag{4.30}$$

by multiplying (4.29) by e_i. We see from (4.28) and (4.30) that M has at most two distinct principal curvatures. Therefore, by Theorem 4.3, M is locally symmetric if $T \in C_2$.

Let $T \in C_3$. Then it follows from (4.20) and (4.21) that

$$(\lambda_i + \lambda_j - 2\lambda_k) T_{ijk} = 0. \tag{4.31}$$

Substituting $X = e_i$ and $Y = Z = e_j$ into (4.13) and using (4.20), (4.21) and (4.31), we have

$$\sum_{k,l} (\lambda_l - \lambda_k) T_{ijk} T_{jkl} e_l = (\lambda_j - \lambda_i)(c + \lambda_i \lambda_j) e_i \tag{4.32}$$

for $i \neq j$. If we multiply (4.32) by e_i, then we obtain

$$(\lambda_j - \lambda_i)\Big(c + \lambda_i \lambda_j + \sum_k T_{ijk}^2\Big) = 0. \tag{4.33}$$

It is easily seen from (4.31) and (4.33) that M has at most two distinct principal curvatures, and hence M is locally symmetric by Theorem 4.3. ■

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