

ON PROPER HELICES AND EXTRINSIC SPHERES IN PSEUDO-RIEMANNIAN GEOMETRY

By

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Abstract. In this paper, we define the notion of a proper helix of order d in a pseudo-Riemannian manifold and investigate those curves in a totally umbilical pseudo-Riemannian submanifold.

Introduction.

In Riemannian geometry, properties of regular curves are well described by the Frenet formula. In [8], K. Sakamoto called a regular curve which has constant curvatures of osculating order d a *helix of order d* . Note that a helix of order one (resp. two) is a geodesic (resp. circle). The research of geodesics, circles and helices (of order three) in Riemannian submanifold theory, has been done by K. Nomizu and K. Yano ([5]), H. Nakagawa ([2]), K. Sakamoto ([7]) and other geometers. Furthermore, K. Sakamoto also has investigated helices of general order in the theory (cf. [8]). For regular curves in a pseudo-Riemannian manifold, we can not necessarily define a formula corresponding to the Frenet formula. Especially, we call a regular curve with a formula corresponding to the Frenet formula a *proper curve*. Furthermore, we call a proper curve which has constant curvatures of osculating order d a *proper helix of order d* . N. Abe, Y. Nakanishi and S. Yamaguchi defined general circles and helices (of order three) in a pseudo-Riemannian manifold. They investigated those curves in a pseudo-Riemannian submanifold (cf. [1], [3], [4]). We shall investigate proper helices of general order in a totally umbilical pseudo-Riemannian submanifold.

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§1. Notations and Basic Equations.

In this paper, the differentiability of all geometric objects will be C^∞ . Let M be a pseudo-Riemannian submanifold in pseudo-Riemannian manifold \bar{M} isometrically immersed by f and denote by g (resp. \bar{g}) the pseudo-Riemannian metric of M (resp. \bar{M}). For all local formulas and calculations, we may assume f as an imbedding and thus we shall often identify $p \in M$ with $f(p) \in \bar{M}$. The tangent space $T_p M$ at p is identified with a subspace $f_*(T_p M)$ of the tangent space $T_p \bar{M}$. We put $\|X\| := \sqrt{|\bar{g}(X, X)|}$ for $X \in T_p \bar{M}$. We denote the tangent bundle of M by TM and the normal bundle by $T^\perp M$. Let $\bar{\nabla}$ and ∇ be the Levi-Civita connections of \bar{M} and M , respectively. Then the Gauss formula is given by

$$(1.1) \quad \bar{\nabla}_X Y = \nabla_X Y + B(X, Y),$$

where X and Y are tangent vector fields of M and B is the second fundamental form of M . The Weingarten formula is given by

$$(1.2) \quad \bar{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi,$$

where X (resp. ξ) is a tangent (resp. normal) vector field of M and A (resp. ∇^\perp) is the shape operator (resp. the normal connection) of M . Clearly A is related to B as

$$g(A_\xi X, Y) = \bar{g}(B(X, Y), \xi).$$

The mean curvature vector field H of M is defined by

$$H := \frac{1}{n} \sum_{i=1}^n g(e_i, e_i) B(e_i, e_i),$$

where $n = \dim M$ and $\{e_1, \dots, e_n\}$ is an orthonormal frame of M . If the second fundamental form B satisfies

$$B(X, Y) = g(X, Y)H$$

for every tangent vector fields X, Y of M , then M is called a totally umbilical submanifold. The mean curvature vector field H is said to be parallel if $\nabla_X^\perp H = 0$ for every tangent vector field X of M . A totally umbilical submanifold with the parallel mean curvature vector field is called an *extrinsic sphere*. If the second fundamental form B vanishes identically, then M is called a totally geodesic submanifold of \bar{M} .

Next we shall define the notion of a proper helix of order d in a pseudo-Riemannian manifold N . Let $\sigma : I \rightarrow N$ be a non-null curve in N parametrized by the arclength s , where I is an open interval of the real line \mathbf{R} . We denote the

tangent vector field of σ by v_0 . We assume that σ satisfies the following Frenet formula:

$$\begin{cases} \nabla_{v_0} v_0 = \lambda_1 v_1 \\ \nabla_{v_0} v_1 + \varepsilon_0 \varepsilon_1 \lambda_1 v_0 = \lambda_2 v_2 \\ \nabla_{v_0} v_2 + \varepsilon_1 \varepsilon_2 \lambda_2 v_1 = \lambda_3 v_3 \\ \vdots \\ \nabla_{v_0} v_{d-2} + \varepsilon_{d-3} \varepsilon_{d-2} \lambda_{d-2} v_{d-3} = \lambda_{d-1} v_{d-1} \\ \nabla_{v_0} v_{d-1} + \varepsilon_{d-2} \varepsilon_{d-1} \lambda_{d-1} v_{d-2} = 0, \end{cases}$$

where

$$\begin{cases} \lambda_1 := \|\nabla_{v_0} v_0\| > 0, \\ \lambda_i := \|\nabla_{v_0} v_{i-1} + \varepsilon_{i-2} \varepsilon_{i-1} \lambda_{i-1} v_{i-2}\| > 0, \quad (2 \leq i \leq d-1) \\ \varepsilon_j := g(v_j, v_j) (= \pm 1) \quad (0 \leq j \leq d-1) \quad \text{on } I. \end{cases}$$

We call such a curve a *proper curve of order d* , λ_i the i -th curvature and v_0, \dots, v_{d-1} the Frenet frame field. Furthermore, if $\lambda_i (1 \leq i \leq d-1)$ are constant along σ , then we call this curve a *proper helix of order d* .

§2. Proper helices in a totally umbilical pseudo-Riemannian submanifold.

Let M be a totally umbilical pseudo-Riemannian submanifold in a pseudo-Riemannian manifold \bar{M} isometrically immersed by f and σ a proper helix of order d in M . We denote a curve $f \circ \sigma$ in \bar{M} by $\bar{\sigma}$. Assume that $\bar{\sigma}$ is a proper helix of order \bar{d} . Let $\lambda_1, \dots, \lambda_{d-1}$ (resp. $\bar{\lambda}_1, \dots, \bar{\lambda}_{\bar{d}-1}$) be the curvatures of σ (resp. $\bar{\sigma}$) and v_0, \dots, v_{d-1} (resp. $\bar{v}_0, \dots, \bar{v}_{\bar{d}-1}$) the Frenet frame field of σ (resp. $\bar{\sigma}$). For convenience, let $\lambda_i = 0, v_i = 0, \bar{\lambda}_j = 0$ and $\bar{v}_j = 0 (i \geq d, j \geq \bar{d})$. Set $\varepsilon_i := g(v_i, v_i)$ and $\bar{\varepsilon}_i := \bar{g}(\bar{v}_i, \bar{v}_i) (i \geq 0)$. We define $\nabla_{v_0}^{\perp(i)} H (i \geq 0)$ by $\nabla_{v_0}^{\perp(0)} H := H$ and $\nabla_{v_0}^{\perp(i)} H := \nabla_{v_0}^{\perp(i-1)} H (i \geq 1)$. Also, we define $\beta_{i,j}$ and $\bar{\beta}_{i,j} (i \geq j \geq 1, i+j : \text{even})$ by

$$(2.1) \quad \begin{cases} \beta_{1,1} = \lambda_1, & \bar{\beta}_{1,1} = \bar{\lambda}_1 \\ \beta_{i,i} = \lambda_i \beta_{i-1,i-1}, & \bar{\beta}_{i,i} = \bar{\lambda}_i \bar{\beta}_{i-1,i-1} \quad (i \geq 2) \\ \beta_{2i+1,1} = -\varepsilon_1 \varepsilon_2 \lambda_2 \beta_{2i,2}, & \bar{\beta}_{2i+1,1} = -\bar{\varepsilon}_1 \bar{\varepsilon}_2 \bar{\lambda}_2 \bar{\beta}_{2i,2} \quad (i \geq 1) \\ \beta_{i,j} = -\varepsilon_j \varepsilon_{j+1} \lambda_{j+1} \beta_{i-1,j+1} + \lambda_j \beta_{i-1,j-1} \quad (i > j \geq 2) \\ \bar{\beta}_{i,j} = -\bar{\varepsilon}_j \bar{\varepsilon}_{j+1} \bar{\lambda}_{j+1} \bar{\beta}_{i-1,j+1} + \bar{\lambda}_j \bar{\beta}_{i-1,j-1} \quad (i > j \geq 2). \end{cases}$$

LEMMA 2.1. *The vector fields $v_i (i \geq 0)$ and $\bar{v}_j (j \geq 0)$ along σ are related as follows:*

$$(F_0) \quad \bar{v}_0 = v_0,$$

$$(F_{2i-1}) \quad \sum_{j=1}^i \bar{\beta}_{2i-1,2j-1} \bar{v}_{2j-1} = \sum_{j=1}^i \beta_{2i-1,2j-1} v_{2j-1} + \varepsilon_0 \nabla_{v_0}^\perp (2i-2) H \quad (i \geq 1),$$

$$(F_{2i}) \quad \sum_{j=1}^i \bar{\beta}_{2i,2j} \bar{v}_{2j} = \sum_{j=1}^i \beta_{2i,2j} v_{2j} + \varepsilon_0 \nabla_{v_0}^\perp (2i-1) H \quad (i \geq 1).$$

PROOF. By using (1.1), the Frenet formulas and the assumption that M is totally umbilic, we get

$$\bar{\lambda}_1 \bar{v}_1 = \bar{\nabla}_{v_0} v_0 = \nabla_{v_0} v_0 + \varepsilon_0 H = \lambda_1 v_1 + \varepsilon_0 H.$$

Thus we obtain (F_1) . Operating $\bar{\nabla}_{v_0}$ to (F_1) , we get

$$\bar{\beta}_{1,1} (-\varepsilon_0 \bar{\varepsilon}_1 \bar{\lambda}_1 v_0 + \bar{\lambda}_2 \bar{v}_2) = \beta_{1,1} (-\varepsilon_0 \varepsilon_1 \lambda_1 v_0 + \lambda_2 v_2) - \varepsilon_0 \bar{g}(H, H) v_0 + \varepsilon_0 \nabla_{v_0}^\perp H,$$

where we use (1.1), (1.2), the Frenet formulas and the assumption that M is totally umbilic. By noticing $\{v_0\}^\perp$ -component of this equality, we see that

$$\bar{\lambda}_2 \bar{\beta}_{1,1} \bar{v}_2 = \lambda_2 \beta_{1,1} v_2 + \varepsilon_0 \nabla_{v_0}^\perp H,$$

which implies (F_2) by (2.1). Assume that (F_{2k}) holds. Operating $\bar{\nabla}_{v_0}$ to (F_{2k}) , we have

$$\begin{aligned} \sum_{j=1}^k \bar{\beta}_{2k,2j} \bar{\nabla}_{v_0} \bar{v}_{2j} &= \sum_{j=1}^k \beta_{2k,2j} \nabla_{v_0} v_{2j} - \varepsilon_0 \bar{g}(\nabla_{v_0}^\perp (2k-1) H, H) v_0 \\ &\quad + \varepsilon_0 \nabla_{v_0}^\perp (2k) H, \end{aligned}$$

where we use (1.1), (1.2) and the assumption that M is totally umbilic. Furthermore, by using the Frenet formulas and (2.1), we have

$$\sum_{j=1}^{k+1} \bar{\beta}_{2k+1,2j-1} \bar{v}_{2j-1} = \sum_{j=1}^{k+1} \beta_{2k+1,2j-1} v_{2j-1} - \varepsilon_0 \bar{g}(\nabla_{v_0}^\perp (2k-1) H, H) v_0 + \varepsilon_0 \nabla_{v_0}^\perp (2k) H.$$

Therefore, by noticing $\text{Span}\{v_0\}^\perp$ -component of this equality, we obtain (F_{2k+1}) . Similarly, by operating $\bar{\nabla}_{v_0}$ to (F_{2k+1}) and using the Frenet formulas and (2.1), we also have

$$\begin{aligned} -\varepsilon_0 \bar{\varepsilon}_1 \bar{\lambda}_1 \bar{\beta}_{2k+1,1} v_0 + \sum_{j=1}^{k+1} \bar{\beta}_{2k+2,2j} \bar{v}_{2j} &= -\varepsilon_0 \{\varepsilon_1 \lambda_1 \beta_{2k+1,1} + \bar{g}(\nabla_{v_0}^\perp (2k) H, H)\} v_0 \\ &\quad + \sum_{j=1}^{k+1} \beta_{2k+2,2j} v_{2j} + \nabla_{v_0}^\perp (2k+1) H. \end{aligned}$$

Thus, by noticing $\text{Span}\{v_0\}^\perp$ -component of this equality, we also have (F_{2k+2}) . Therefore, by the induction, we see that (F_i) holds for every $i \geq 0$. \square

Now we define column vectors $b_i (i \geq 3)$ and matrices $B_i (i \geq 1)$ by

$$b_{2j-1} := \begin{pmatrix} \beta_{2j-1,1} \\ \beta_{2j-1,3} \\ \vdots \\ \beta_{2j-1,2j-3} \end{pmatrix}, \quad b_{2j} := \begin{pmatrix} \beta_{2j,2} \\ \beta_{2j,4} \\ \vdots \\ \beta_{2j,2j-2} \end{pmatrix} \quad (j \geq 2)$$

and

$$B_{2j-1} := \begin{pmatrix} \beta_{1,1} & 0 & \cdots & 0 \\ \beta_{3,1} & \beta_{3,3} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \beta_{2j-1,1} & \beta_{2j-1,3} & \cdots & \beta_{2j-1,2j-1} \end{pmatrix},$$

$$B_{2j} := \begin{pmatrix} \beta_{2,2} & 0 & \cdots & 0 \\ \beta_{4,2} & \beta_{4,4} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \beta_{2j,2} & \beta_{2j,4} & \cdots & \beta_{2j,2j} \end{pmatrix} \quad (j \geq 1).$$

Also, we define formal column vectors $V_i (i \geq 1)$ and $H_i (i \geq 0)$ whose components are vector fields along σ by

$$V_{2j-1} := \begin{pmatrix} v_1 \\ v_3 \\ \vdots \\ v_{2j-1} \end{pmatrix}, \quad V_{2j} := \begin{pmatrix} v_2 \\ v_4 \\ \vdots \\ v_{2j} \end{pmatrix} \quad (j \geq 1)$$

and

$$H_{2j} := \begin{pmatrix} H \\ \nabla_{v_0}^\perp (2) H \\ \vdots \\ \nabla_{v_0}^\perp (2j) H \end{pmatrix}, \quad H_{2j+1} := \begin{pmatrix} \nabla_{v_0}^\perp H \\ \nabla_{v_0}^\perp (3) H \\ \vdots \\ \nabla_{v_0}^\perp (2j+1) H \end{pmatrix} \quad (j \geq 0).$$

Similarly, we define $\bar{b}_i (i \geq 3), \bar{B}_i (i \geq 1)$ and $\bar{V}_i (i \geq 1)$ in terms of $\bar{\beta}_{i,j}$ and \bar{v}_i instead of $\beta_{i,j}$ and v_i . Note that $B_i (i \leq d-1), \bar{B}_i (i \leq \bar{d}-1)$ are nonsingular by (2.1). By using these notations, (F_i) is expressed as follows:

$$(2.2) \quad {}^t \bar{\mathbf{b}}_i \bar{\mathbf{V}}_{i-2} + \bar{\beta}_{i,d} \bar{v}_i = {}^t \mathbf{b}_i \mathbf{V}_{i-2} + \beta_{i,d} v_i + \varepsilon_0 \nabla_{\mathbf{v}_0}^{\perp (i-1)} H. \quad (i \geq 3).$$

Moreover, the systems $(F_1), (F_3), \dots, (F_{2i-1}) (i \geq 1)$ and $(F_2), (F_4), \dots, (F_{2i}) (i \geq 1)$ are expressed as

$$\begin{aligned} \bar{\mathbf{B}}_{2i-1} \bar{\mathbf{V}}_{2i-1} &= \mathbf{B}_{2i-1} \mathbf{V}_{2i-1} + \varepsilon_0 \mathbf{H}_{2i-2}, \\ \bar{\mathbf{B}}_{2i} \bar{\mathbf{V}}_{2i} &= \mathbf{B}_{2i} \mathbf{V}_{2i} + \varepsilon_0 \mathbf{H}_{2i-1}, \end{aligned}$$

respectively. Thus we have

$$(2.3) \quad \bar{\mathbf{B}}_i \bar{\mathbf{V}}_i = \mathbf{B}_i \mathbf{V}_i + \varepsilon_0 \mathbf{H}_{i-1}, \quad (i \geq 1).$$

From (2.2) and (2.3), we have

$$\begin{aligned} (MF_i) \quad -\bar{\beta}_{i,d} \bar{v}_i + \beta_{i,d} v_i &= ({}^t \bar{\mathbf{b}}_i \bar{\mathbf{B}}_{i-2}^{-1} \mathbf{B}_{i-2} - {}^t \mathbf{b}_i) \mathbf{V}_{i-2} \\ &\quad + \varepsilon_0 ({}^t \bar{\mathbf{b}}_i \bar{\mathbf{B}}_{i-2}^{-1} \mathbf{H}_{i-3} - \nabla_{\mathbf{v}_0}^{\perp (i-1)} H) \quad (3 \leq i \leq \bar{d} + 1). \end{aligned}$$

LEMMA 2.2. *The inequality $d \leq \bar{d} \leq d + r$ holds, where r is the codimension of M in \bar{M} .*

PROOF. Suppose $d > \bar{d}$. Then we have $v_{\bar{d}} \neq 0$ and $\bar{v}_{\bar{d}} = 0$. Hence, it follows from $(MF_{\bar{d}})$ that

$$\begin{aligned} \beta_{\bar{d},\bar{d}} v_{\bar{d}} &= ({}^t \bar{\mathbf{b}}_{\bar{d}} \bar{\mathbf{B}}_{\bar{d}-2}^{-1} \mathbf{B}_{\bar{d}-2} - {}^t \mathbf{b}_{\bar{d}}) \mathbf{V}_{\bar{d}-2} \\ &\quad + \varepsilon_0 ({}^t \bar{\mathbf{b}}_{\bar{d}} \bar{\mathbf{B}}_{\bar{d}-2}^{-1} \mathbf{H}_{\bar{d}-3} - \nabla_{\mathbf{v}_0}^{\perp (\bar{d}-1)} H). \end{aligned}$$

Since $v_{\bar{d}}$ is linearly independent of $v_i (i \leq \bar{d} - 2)$ and $\nabla_{\mathbf{v}_0}^{\perp (i)} H (i \leq \bar{d} - 1)$, we have $\beta_{\bar{d},\bar{d}} v_{\bar{d}} = 0$. From (2.1) and $d > \bar{d}$, $\beta_{\bar{d},\bar{d}} = \lambda_1 \lambda_2 \cdots \lambda_{\bar{d}} \neq 0$ is deduced. Therefore, we have $v_{\bar{d}} = 0$. This contradicts $d > \bar{d}$. Thus we have $d \leq \bar{d}$. The remaining part is trivial. □

LEMMA 2.3. (i) *If $\bar{d} = d (\geq 3)$, then $\nabla_{\mathbf{v}_0}^{\perp (d-1)} H = {}^t \mathbf{b}_d \mathbf{B}_{d-2}^{-1} \mathbf{H}_{d-3}$ holds.*

(ii) *If $\bar{d} = d + 1 (d \geq 2)$, then $\nabla_{\mathbf{v}_0}^{\perp (d)} H = {}^t \mathbf{b}_{d+1} \mathbf{B}_{d-1}^{-1} \mathbf{H}_{d-2}$ holds.*

PROOF. (i) By the assumption, $v_d = 0$ and $\bar{v}_d = 0$ holds. Substituting these to (MF_d) , we have

$$({}^t \bar{\mathbf{b}}_d \bar{\mathbf{B}}_{d-2}^{-1} \mathbf{B}_{d-2} - {}^t \mathbf{b}_d) \mathbf{V}_{d-2} + \varepsilon_0 ({}^t \bar{\mathbf{b}}_d \bar{\mathbf{B}}_{d-2}^{-1} \mathbf{H}_{d-3} - \nabla_{\mathbf{v}_0}^{\perp (d-1)} H) = 0.$$

By noticing the tangential component and the normal component of this equality, we have

$${}^t \bar{\mathbf{b}}_d \bar{\mathbf{B}}_{d-2}^{-1} = {}^t \mathbf{b}_d \mathbf{B}_{d-2}^{-1}$$

and

$$\nabla_{\mathbf{v}_0}^{\perp (d-1)} H = {}^t \bar{\mathbf{b}}_d \bar{\mathbf{B}}_{d-2}^{-1} \mathbf{H}_{d-3}.$$

These imply

$$\nabla_{\mathbf{v}_0}^{\perp (d-1)} H = {}^t \mathbf{b}_d \mathbf{B}_{d-2}^{-1} \mathbf{H}_{d-3}.$$

(ii) By the assumption, $v_{d+1} = 0$ and $\bar{v}_{d+1} = 0$ holds. Substituting these to (MF_{d+1}) , we have

$$({}^t \bar{\mathbf{b}}_{d+1} \bar{\mathbf{B}}_{d-1}^{-1} \mathbf{B}_{d-1} - {}^t \mathbf{b}_{d+1}) \mathbf{V}_{d-1} + \varepsilon_0 ({}^t \bar{\mathbf{b}}_{d+1} \bar{\mathbf{B}}_{d-1}^{-1} \mathbf{H}_{d-2} - \nabla_{\mathbf{v}_0}^{\perp (d)} H) = 0.$$

By noticing the tangential component and the normal component of this equality, we have

$${}^t \bar{\mathbf{b}}_{d+1} \bar{\mathbf{B}}_{d-1}^{-1} = {}^t \mathbf{b}_{d+1} \mathbf{B}_{d-1}^{-1}$$

and

$$\nabla_{\mathbf{v}_0}^{\perp (d)} H = {}^t \bar{\mathbf{b}}_{d+1} \bar{\mathbf{B}}_{d-1}^{-1} \mathbf{H}_{d-2}.$$

These imply

$$\nabla_{\mathbf{v}_0}^{\perp (d)} H = {}^t \mathbf{b}_{d+1} \mathbf{B}_{d-1}^{-1} \mathbf{H}_{d-2}. \quad \square$$

Since ${}^t \mathbf{b}_{2i+1} \mathbf{B}_{2i-1}^{-1} (1 \leq 2i-1 \leq d-1)$ is the solution of the equation $(x_1, \dots, x_i) \mathbf{B}_{2i-1} = {}^t \mathbf{b}_{2i+1}$, by Cramér formula, we have

$$(2.4) \quad {}^t \mathbf{b}_{2i+1} \mathbf{B}_{2i-1}^{-1} = \frac{1}{|\mathbf{B}_{2i-1}|} (P_{2i+1,1}(\lambda_1, \dots, \lambda_{2i}), \dots, P_{2i+1,i}(\lambda_1, \dots, \lambda_{2i})),$$

where $P_{2i+1,j}(\lambda_1, \dots, \lambda_{2i}) (1 \leq j \leq i)$ is the determinant replaced the j -th row of $|\mathbf{B}_{2i-1}|$ by ${}^t \mathbf{b}_{2i+1}$. Similarly, we have

$$(2.5) \quad {}^t \mathbf{b}_{2i} \mathbf{B}_{2i-2}^{-1} = \frac{1}{|\mathbf{B}_{2i-2}|} (P_{2i,1}(\lambda_1, \dots, \lambda_{2i-1}), \dots, P_{2i,i-1}(\lambda_1, \dots, \lambda_{2i-1}))$$

(2 \leq 2i-2 \leq d-1),

where $P_{2i,j}(\lambda_1, \dots, \lambda_{2i-1}) (1 \leq j \leq i-1)$ is the determinant replaced the j -th row of $|\mathbf{B}_{2i-2}|$ by ${}^t \mathbf{b}_{2i}$. Then we have the following lemma.

LEMMA 2.4. (i) *The polynomial $P_{2i+1,j}(\lambda_1, \dots, \lambda_{2i}) (1 \leq j \leq i)$ is a homogeneous*

polynomial of degree $(i^2 + 2i - 2j + 2)$ and $P_{2i,j}(\lambda_1, \dots, \lambda_{2i-1}) (1 \leq j \leq i-1)$ is a homogeneous polynomial of degree $(i^2 + i - 2j)$.

(ii) The polynomial $P_{2i+1,1}(\lambda_1, \dots, \lambda_{2i})$ is expressed as follows:

$$P_{2i+1,1}(\lambda_1, \dots, \lambda_{2i}) = -\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{2i} \lambda_2 \lambda_4 \cdots \lambda_{2i} |\mathbf{B}_{2i}|.$$

PROOF. (i) By (2.1), we see that $\beta_{i,j}$ is a homogeneous polynomial of degree i with variables $\lambda_1, \dots, \lambda_i$. Hence the conclusion is directly deduced from the definitions of $P_{2i+1,j}(\lambda_1, \dots, \lambda_{2i})$ and $P_{2i,j}(\lambda_1, \dots, \lambda_{2i-1})$.

(ii) Define $\hat{\beta}_{j,k} (j > k \geq 1, j+k : \text{even})$ by

$$\hat{\beta}_{j,k} = \begin{cases} 0 & (j > k = 1) \\ \lambda_k \beta_{j-1, k-1} & (j > k > 1). \end{cases}$$

Then, from (2.1), we have

$$(b_{j,k}) \quad \beta_{j,k} = -\varepsilon_k \varepsilon_{k+1} \lambda_{k+1} \beta_{j-1, k+1} + \hat{\beta}_{j,k} \quad (j > k > 2).$$

Also, we define a matrix C_j of type $(2, j)$ and a matrix D_j of type $(j, 2) (j \geq 1)$ by

$$C_j := \begin{pmatrix} \beta_{2j+3,1} & \beta_{2j+3,3} & \dots & \beta_{2j+3,2j-1} \\ \beta_{2j+5,1} & \beta_{2j+5,3} & \dots & \beta_{2j+5,2j-1} \end{pmatrix}$$

and

$$D_j := \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \beta_{2j+1,2j+1} & 0 \end{pmatrix}.$$

Furthermore, we define matrices A_j and $\hat{A}_j (j \geq 1)$ by

$$A_1 := (\beta_{3,1}), \quad A_2 := \begin{pmatrix} \beta_{3,1} & \beta_{3,3} \\ \beta_{5,1} & \beta_{5,3} \end{pmatrix},$$

$$A_j := \begin{pmatrix} A_{j-2} & D_{j-2} \\ C_{j-2} & \begin{pmatrix} \beta_{2j-1,2j-3} & \beta_{2j-1,2j-1} \\ \beta_{2j+1,2j-3} & \beta_{2j+1,2j-1} \end{pmatrix} \end{pmatrix} \quad (j \geq 3)$$

and

$$\hat{A}_1 := (\hat{\beta}_{3,1}), \quad \hat{A}_2 := \begin{pmatrix} \beta_{3,1} & \beta_{3,3} \\ \beta_{5,1} & \beta_{5,3} \end{pmatrix},$$

$$\hat{A}_j := \begin{pmatrix} A_{j-2} & D_{j-2} \\ C_{j-2} & \begin{pmatrix} \beta_{2j-1,2j-3} & \beta_{2j-1,2j-1} \\ \beta_{2j+1,2j-3} & \beta_{2j+1,2j-1} \end{pmatrix} \end{pmatrix} \quad (j \geq 3).$$

From the definition of $P_{2j+1,1}(\lambda_1, \dots, \lambda_{2j})$, we have

$$\begin{aligned} P_{2j+1,1}(\lambda_1, \dots, \lambda_{2j}) &= (-1)^{j-1} |A_j| \\ &= (-1)^{j-1} \begin{vmatrix} A_{j-2} & D_{j-2} \\ C_{j-2} & \begin{pmatrix} \beta_{2j-1,2j-3} & \beta_{2j-1,2j-1} \\ \beta_{2j+1,2j-3} & \beta_{2j+1,2j-1} \end{pmatrix} \end{vmatrix}. \end{aligned}$$

Substituting $(b_{2j+1,2j-1})$ to this equality and using the linearity of the determinant for the final column, we have

$$\begin{aligned} (2.6) \quad & P_{2j+1,1}(\lambda_1, \dots, \lambda_{2j}) \\ &= (-1)^{j-1} \left\{ \begin{vmatrix} A_{j-2} & D_{j-2} & 0 \\ C_{j-2} & \begin{pmatrix} \beta_{2j-1,2j-3} & 0 \\ \beta_{2j+1,2j-3} & -\varepsilon_{2j-1}\varepsilon_{2j}\lambda_{2j}\beta_{2j,2j} \end{pmatrix} \end{vmatrix} + |\hat{A}_j| \right\} \\ &= (-1)^{j-1} \{ -\varepsilon_{2j-1}\varepsilon_{2j}\lambda_{2j}\beta_{2j,2j}|A_{j-1}| + |\hat{A}_j| \} \\ &= \varepsilon_{2j-1}\varepsilon_{2j}\lambda_{2j}\beta_{2j,2j}P_{2j-1,1}(\lambda_1, \dots, \lambda_{2j-2}) + (-1)^{j-1}|\hat{A}_j| \quad (j \geq 2). \end{aligned}$$

Next we shall show $|\hat{A}_j| = 0 (j \geq 1)$. Clearly we have $|\hat{A}_j| = |\hat{\beta}_{3,1}| = 0$. Assume that $|\hat{A}_j| = 0$ for every $j \leq k$. Substituting $(b_{2k+1,2k-1}), (b_{2k+3,2k-1}), \beta_{2k+1,2k+1} = \lambda_{2k+1}\beta_{2k,2k}$ and $\hat{\beta}_{2k+3,2k+1} = \lambda_{2k+1}\beta_{2k+2,2k}$ to

$$|\hat{A}_{k+1}| = \begin{vmatrix} A_{k-1} & D_{k-1} \\ C_{k-1} & \begin{pmatrix} \beta_{2k+1,2k-1} & \beta_{2k+1,2k+1} \\ \beta_{2k+3,2k-1} & \beta_{2k+3,2k+1} \end{pmatrix} \end{vmatrix}$$

and adding $\frac{\varepsilon_{2k-1}\varepsilon_{2k}\lambda_{2k}}{\lambda_{2k+1}}$ multiple of the final column to the k -th column, we obtain

$$|\hat{A}_{k+1}| = \begin{vmatrix} A_{k-1} & D_{k-1} \\ C_{k-1} & \begin{pmatrix} \hat{\beta}_{2k+1,2k-1} & \beta_{2k+1,2k+1} \\ \hat{\beta}_{2k+3,2k-1} & \hat{\beta}_{2k+3,2k+1} \end{pmatrix} \end{vmatrix}.$$

Expanding this determinant with respect to the final column and using the assumption of the induction, we obtain

$$\begin{aligned} |\hat{A}_{k+1}| &= -\beta_{2k+1,2k+1} \begin{vmatrix} A_{k-2} & D_{k-2} \\ \begin{pmatrix} \beta_{2k-1,1} & \dots & \beta_{2k-1,2k-5} \\ \beta_{2k+3,1} & \dots & \beta_{2k+3,2k-5} \end{pmatrix} & \begin{pmatrix} \beta_{2k-1,2k-3} & \beta_{2k-1,2k-1} \\ \beta_{2k+3,2k-3} & \hat{\beta}_{2k+3,2k-1} \end{pmatrix} \end{vmatrix} \\ &\quad + \hat{\beta}_{2k+3,2k+1} |\hat{A}_k| \end{aligned}$$

$$= -\beta_{2k+1,2k+1} \left| \begin{array}{ccc} & A_{k-2} & \\ \beta_{2k-1,1} & \cdots & \beta_{2k-1,2k-5} \\ \beta_{2k+3,1} & \cdots & \beta_{2k+3,2k-5} \end{array} \right| \left| \begin{array}{cc} D_{k-2} & \\ \beta_{2k-1,2k-3} & \beta_{2k-1,2k-1} \\ \beta_{2k+3,2k-3} & \hat{\beta}_{2k+3,2k-1} \end{array} \right|.$$

By repeating the same process, we can obtain

$$\begin{aligned} |\hat{A}_{k+1}| &= (-1)^{k-2} \beta_{7,7} \beta_{9,9} \cdots \beta_{2k+1,2k+1} \left| \begin{array}{cc} A_1 & D_1 \\ \beta_{5,1} & \beta_{5,3} \quad \beta_{5,5} \\ \beta_{2k+3,1} & \beta_{2k+3,3} \quad \hat{\beta}_{2k+3,5} \end{array} \right| \\ &= (-1)^{k-2} \beta_{7,7} \beta_{9,9} \cdots \beta_{2k+1,2k+1} \left| \begin{array}{ccc} \beta_{3,1} & \beta_{3,3} & 0 \\ \beta_{5,1} & \beta_{5,3} & \beta_{5,5} \\ \beta_{2k+3,1} & \beta_{2k+3,3} & \hat{\beta}_{2k+3,5} \end{array} \right| \\ &= (-1)^{k-1} \beta_{5,5} \beta_{7,7} \cdots \beta_{2k+1,2k+1} \left| \begin{array}{cc} \beta_{3,1} & \beta_{3,3} \\ \beta_{2k+3,1} & \hat{\beta}_{2k+3,3} \end{array} \right| \\ &= (-1)^k \beta_{3,3} \beta_{5,5} \cdots \beta_{2k+1,2k+1} |\hat{\beta}_{2k+3,1}| \\ &= 0. \end{aligned}$$

Thus, by the induction, we can conclude $|\hat{A}_j|=0$ every $j \geq 1$. Substituting $|\hat{A}_j|=0$ to (2.6), we have

$$P_{2j+1,1}(\lambda_1, \dots, \lambda_{2j}) = \varepsilon_{2j-1} \varepsilon_{2j} \lambda_{2j} \beta_{2j,2j} P_{2j-1,1}(\lambda_1, \dots, \lambda_{2j-2}) \quad (j \geq 2).$$

After all we can obtain

$$\begin{aligned} &P_{2i+1,1}(\lambda_1, \dots, \lambda_{2i}) \\ &= \varepsilon_3 \varepsilon_4 \cdots \varepsilon_{2i} \lambda_4 \lambda_6 \cdots \lambda_{2i} \beta_{4,4} \beta_{6,6} \cdots \beta_{2i,2i} P_{3,1}(\lambda_1, \lambda_2) \\ &= -\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{2i} \lambda_2 \lambda_4 \cdots \lambda_{2i} \beta_{2,2} \beta_{4,4} \cdots \beta_{2i,2i} \\ &= -\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{2i} \lambda_2 \lambda_4 \cdots \lambda_{2i} |B_{2i}|. \end{aligned} \quad \square$$

Also, we have the following lemma.

LEMMA 2.5. (i) The normal vector field $\nabla_{\mathbf{v}_0}^{\perp (2i)} H (i \geq 1)$ along σ is written as

$$(H_{2i}) \quad \nabla_{\mathbf{v}_0}^{\perp (2i)} H = \sum_{j=1}^{i-1} Q_{2i,2j-1}(\lambda_1, \dots, \lambda_{2i-2}) \nabla_{\mathbf{v}_{2j-1}}^{\perp} H + \lambda_1 \lambda_2 \cdots \lambda_{2i-1} \nabla_{\mathbf{v}_{2j-1}}^{\perp} H + N_{2i}(\lambda_1, \dots, \lambda_{2i-2}),$$

where $Q_{2i,2j-1}(\lambda_1, \dots, \lambda_{2i-1}) (1 \leq j \leq i-1)$ is a homogeneous polynomial of degree

$(2i-1)$ and $N_{2i}(\lambda_1, \dots, \lambda_{2i-2})$ is a normal vector field-valued polynomial of degree at most $(2i-2)$,

(ii) The normal vector field $\nabla_{v_0}^\perp{}^{(2i+1)} H (i \geq 1)$ along σ is written as

$$\begin{aligned} (H_{2i+1}) \nabla_{v_0}^\perp{}^{(2i+1)} H &= \sum_{j=0}^{i-1} Q_{2i+1,2j}(\lambda_1, \dots, \lambda_{2i-1}) \nabla_{v_{2j}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2i} \nabla_{v_{2i}}^\perp H \\ &\quad + N_{2i+1}(\lambda_1, \dots, \lambda_{2i-1}), \end{aligned}$$

where $Q_{2i+1,2j}(\lambda_1, \dots, \lambda_{2i}) (0 \leq j \leq i-1)$ is a homogeneous polynomial of degree $2i$ and $N_{2i+1}(\lambda_1, \dots, \lambda_{2i-1})$ is a normal vector field-valued polynomial of degree at most $(2i-1)$.

PROOF. Define a normal bundle-valued $(0,j)$ -tensor field T_j on M by $T_1 := \nabla^\perp H$ and $T_k(X_1, \dots, X_k) := (\bar{\nabla}_{X_1} T_{k-1})(X_2, \dots, X_k) (k \geq 2)$ for $X_1, \dots, X_k \in TM$, where $\bar{\nabla}$ is the connection induced from ∇ and ∇^\perp . We shall show (H_3) . By using the definition of T_j and the Frenet formula, $\nabla_{v_0}^\perp{}^{(3)} H$ is rewritten in terms of T_j as follows:

$$\begin{aligned} \nabla_{v_0}^\perp{}^{(3)} H &= \nabla_{v_0}^\perp{}^{(2)} (T_1(v_0)) = \nabla_{v_0}^\perp (T_2(v_0, v_0) + \lambda_1 T_1(v_1)) \\ &= T_3(v_0, v_0, v_0) + \lambda_1 T_2(v_1, v_0) + 2\lambda_1 T_2(v_0, v_1) \\ &\quad - \varepsilon_0 \varepsilon_1 \lambda_1^2 \nabla_{v_0}^\perp H + \lambda_1 \lambda_2 \nabla_{v_0}^\perp H \\ &= Q_{3,0}(\lambda_1) \nabla_{v_0}^\perp H + \lambda_1 \lambda_2 \nabla_{v_0}^\perp H + N_3(\lambda_1), \end{aligned}$$

where we set $Q_{3,0}(\lambda_1) := -\varepsilon_0 \varepsilon_1 \lambda_1^2$ and $N_3(\lambda_1) := T_3(v_0, v_0, v_0) + \lambda_1 T_2(v_1, v_0) + 2\lambda_1 T_2(v_0, v_1)$. Thus (H_3) is shown. Similarly, $(H_i) (i \geq 4)$ is also shown. \square

By using these lemmas, we can prove the following theorem.

THEOREM 2.6. *Let M be a totally umbilical pseudo-Riemannian submanifold in \bar{M} isometrically immersed by f . Assume that for every proper helix σ of order d in M , $\bar{\sigma} := f \circ \sigma$ is a proper helix of order d in \bar{M} , where d is a positive integer. Then*

- (i) *if d is odd, then M is totally geodesic,*
- (ii) *if d is even, then M is an extrinsic sphere.*

PROOF. Assume that $d \geq 3$. Fix $p \in M$. For any orthonormal system X_0, X_1, \dots, X_{d-1} of $T_p M$ and any positive numbers $\lambda_1, \dots, \lambda_{d-1}$, there exists a proper helix σ of order d through p with the curvatures $\lambda_1, \dots, \lambda_{d-1}$ whose Frenet frame field v_0, v_1, \dots, v_{d-1} coincide with X_0, X_1, \dots, X_{d-1} at p . Since $\bar{\sigma} := f \circ \sigma$ is a proper

helix of order d in M , by Lemma 2.3, we have

$$(2.7) \quad \nabla_{\mathbf{v}_0}^\perp{}^{(d-1)} H = {}^t \mathbf{b}_d \mathbf{B}_{d-2}^{-1} \mathbf{H}_{d-3}.$$

(i) Let $d = 2i + 1$. It follows from (2.4) and Lemma 2.5 that

$$\begin{aligned} \nabla_{\mathbf{v}_0}^\perp{}^{(d-1)} H &= \nabla_{\mathbf{v}_0}^\perp{}^{(2i)} H \\ &= \sum_{k=1}^{i-1} Q_{2i,2k-1}(\lambda_1, \dots, \lambda_{2i-2}) \nabla_{\mathbf{v}_{2k-1}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2i-1} \nabla_{\mathbf{v}_{2i-1}}^\perp H \\ &\quad + N_{2i}(\lambda_1, \dots, \lambda_{2i-2}) \end{aligned}$$

and

$$\begin{aligned} {}^t \mathbf{b}_d \mathbf{B}_{d-2}^{-1} \mathbf{H}_{d-3} &= \frac{1}{|\mathbf{B}_{d-2}|} \sum_{j=0}^{i-1} P_{d,j+1}(\lambda_1, \dots, \lambda_{d-1}) \nabla_{\mathbf{v}_0}^\perp{}^{(2j)} H \\ &= \frac{1}{|\mathbf{B}_{d-2}|} \left\{ P_{d,1}(\lambda_1, \dots, \lambda_{d-1}) H + \sum_{j=1}^{i-1} P_{d,j+1}(\lambda_1, \dots, \lambda_{d-1}) \right. \\ &\quad \left. \left\{ \sum_{k=1}^{j-1} Q_{2j,2k-1}(\lambda_1, \dots, \lambda_{2j-2}) \nabla_{\mathbf{v}_{2k-1}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2j-1} \nabla_{\mathbf{v}_{2j-1}}^\perp H + N_{2j}(\lambda_1, \dots, \lambda_{2j-2}) \right\} \right\}. \end{aligned}$$

Substituting these equalities to (2.7) and noticing the point p , we have

$$(2.8) \quad \begin{aligned} &|\mathbf{B}_{d-2}| \left\{ \sum_{k=1}^{i-1} Q_{2i,2k-1}(\lambda_1, \dots, \lambda_{2i-2}) \nabla_{\mathbf{x}_{2k-1}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2i-1} \nabla_{\mathbf{x}_{2i-1}}^\perp H + N_{2i}(\lambda_1, \dots, \lambda_{2i-2}) \right\} \\ &= P_{d,1}(\lambda_1, \dots, \lambda_{d-1}) H + \sum_{j=1}^{i-1} P_{d,j+1}(\lambda_1, \dots, \lambda_{d-1}) \\ &\quad \left\{ \sum_{k=1}^{j-1} Q_{2j,2k-1}(\lambda_1, \dots, \lambda_{2j-2}) \nabla_{\mathbf{x}_{2k-1}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2j-1} \nabla_{\mathbf{x}_{2j-1}}^\perp H + N_{2j}(\lambda_1, \dots, \lambda_{2j-2}) \right\} \end{aligned}$$

Since the degrees of $|\mathbf{B}_{d-2}|, Q_{2j,2k-1}(\lambda_1, \dots, \lambda_{2j-1}) (j > k \geq 1)$ and $P_{d,j}(\lambda_1, \dots, \lambda_{d-1}) (j \geq 1)$ are $i^2, (2j-1)$, and $(i^2 + 2i - 2j + 2)$, respectively, the left-hand side of (2.8) is a polynomial of degree $(i^2 + 2i - 1)$, the first term $P_{d,1}(\lambda_1, \dots, \lambda_{d-1}) H$ of the right-hand side is of degree $(i^2 + 2i)$ and other terms of the right-hand side are of degree at most $(i^2 + 2i - 1)$. Hence, since (2.8) holds for every positive numbers $\lambda_1, \dots, \lambda_{2i-1}$, we obtain $P_{d,1}(\lambda_1, \dots, \lambda_{d-1}) H = 0$. From Lemma 2.4-(ii), $P_{d,1}(\lambda_1, \dots, \lambda_{d-1}) \neq 0$ holds. Therefore, we see that $H = 0$ at p . By the arbitrariness of $p \in M$, we see that $H \equiv 0$, that is, M is totally geodesic. In case of $d = 1$, it is directly deduced from Lemma 2.1 that so is M .

(ii) Let $d = 2i$. It follows from (2.5), (2.7) and Lemma 2.5 that

$$\begin{aligned}
 & \left| \mathbf{B}_{d-2} \left\{ \sum_{k=0}^{i-2} Q_{2i-1,2k}(\lambda_1, \dots, \lambda_{2i-3}) \nabla_{X_{2k}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2i-2} \nabla_{X_{2i-2}}^\perp H + N_{2i-1}(\lambda_1, \dots, \lambda_{2i-3}) \right\} \right. \\
 (2.9) &= P_{d,1}(\lambda_1, \dots, \lambda_{d-1}) \nabla_{X_0}^\perp H + \sum_{j=1}^{i-2} P_{d,j+1}(\lambda_1, \dots, \lambda_{d-1}) \\
 & \left. \left\{ \sum_{k=0}^{j-1} Q_{2j+1,2k}(\lambda_1, \dots, \lambda_{2j-1}) \nabla_{X_{2k}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2j} \nabla_{X_{2j}}^\perp H + N_{2j+1}(\lambda_1, \dots, \lambda_{2j-1}) \right\}. \right.
 \end{aligned}$$

Since the degrees of $|\mathbf{B}_{d-2}|, Q_{2j+1,2k-1}(\lambda_1, \dots, \lambda_{2j-1}) (j > k \geq 0)$ and $P_{d,j}(\lambda_1, \dots, \lambda_{d-1}) (j \geq 1)$ are $(i^2 - i), 2j$ and $(i^2 + i - 2j)$, respectively, both sides of (2.9) are polynomials of degree $(i^2 + i - 2)$. Hence, since (2.9) holds for every positive numbers $\lambda_1, \dots, \lambda_{2i-2}$, terms of degree $(i^2 + i - 2)$ of the both sides are mutually equal, that is,

$$\begin{aligned}
 & \left| \mathbf{B}_{d-2} \left\{ \sum_{k=0}^{i-2} Q_{2i-1,2k}(\lambda_1, \dots, \lambda_{2i-3}) \nabla_{X_{2k}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2i-2} \nabla_{X_{2i-2}}^\perp H \right\} \right. \\
 &= P_{d,1}(\lambda_1, \dots, \lambda_{d-1}) \nabla_{X_0}^\perp H + \sum_{j=1}^{i-2} P_{d,j+1}(\lambda_1, \dots, \lambda_{d-1}) \\
 & \left. \left\{ \sum_{k=0}^{j-1} Q_{2j+1,2k}(\lambda_1, \dots, \lambda_{2j-1}) \nabla_{X_{2k}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2j} \nabla_{X_{2j}}^\perp H \right\}. \right.
 \end{aligned}$$

Furthermore, since this equality holds for every orthonormal system $X_0, X_2, \dots, X_{2i-2}$ of $T_p M$, we see that $|\mathbf{B}_{d-2}| \lambda_1 \lambda_2 \cdots \lambda_{2i-2} \nabla_{X_{2i-2}}^\perp H = 0$, that is $\nabla_{X_{2i-2}}^\perp H = 0$. By the arbitrariness of X_{2i-2} , we see that $\nabla^\perp H = 0$ at p . Furthermore, from the arbitrariness of $p \in M, \nabla^\perp H \equiv 0$ is deduced. Thus M is an extrinsic sphere. In case of $d = 2$, it is directly deduced from Lemma 2.1 that so is M . \square

In the case where M and \bar{M} are Riemannian manifolds, this theorem is written as follows.

COROLLARY 2.7. *Let M be a totally umbilical submanifold in a Riemannian manifold \bar{M} isometrically immersed by f . Assume that for every helix σ of order d in M , $\bar{\sigma} (= f \circ \sigma)$ is a helix of order d in \bar{M} , where d is a positive integer. Then*

- (i) *if d is odd, then M is totally geodesic,*
- (ii) *if d is even, then M is an extrinsic sphere.*

Also, we can prove the following theorem.

THEOREM 2.8. *Let M be a totally umbilical pseudo-Riemannian submanifold in \bar{M} isometrically immersed by f . Assume that for every proper helix σ of order d in M , $\bar{\sigma} (= f \circ \sigma)$ is a proper helix of order $d + 1$ in \bar{M} , where d is a positive*

integer. Then d is odd and M is an extrinsic sphere.

PROOF. Assume that $d \geq 2$. Fix $p \in M$. For any orthonormal system X_0, X_1, \dots, X_{d-1} of $T_p M$ and any positive numbers $\lambda_1, \dots, \lambda_{d-1}$, there exists a proper helix σ of order d through p with the curvatures $\lambda_1, \dots, \lambda_{d-1}$ whose Frenet frame field v_0, v_1, \dots, v_{d-1} coincide with X_0, X_1, \dots, X_{d-1} at p . Since $\bar{\sigma} (= f \circ \sigma)$ is a proper helix of order $d + 1$ in M , by Lemma 2.3, we have

$$(2.10) \quad \nabla_{v_0}^\perp H = {}^t \mathbf{b}_{d+1} \mathbf{B}_{d-1}^{-1} H_{d-2}.$$

Suppose that d is even. Let $d = 2i$. It follows from (2.4), (2.10) and Lemma 2.5 that

$$(2.11) \quad \begin{aligned} & \left\{ \mathbf{B}_{d-1} \left[\sum_{k=1}^{i-1} Q_{2i,2k-1}(\lambda_1, \dots, \lambda_{2i-2}) \nabla_{X_{2k-1}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2i-1} \nabla_{X_{2i-1}}^\perp H + N_{2i}(\lambda_1, \dots, \lambda_{2i-2}) \right] \right\} \\ & = P_{d+1,1}(\lambda_1, \dots, \lambda_d) H + \sum_{j=1}^{i-1} P_{d+1,j+1}(\lambda_1, \dots, \lambda_d) \\ & \quad \left\{ \sum_{k=1}^{j-1} Q_{2j,2k-1}(\lambda_1, \dots, \lambda_{2j-2}) \nabla_{X_{2k-1}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2j-1} \nabla_{X_{2j-1}}^\perp H + N_{2j}(\lambda_1, \dots, \lambda_{2j-2}) \right\}. \end{aligned}$$

Since (2.11) holds for every positive numbers $\lambda_1, \dots, \lambda_{2i-1}$, by noticing the term of the highest degree, we have $P_{d+1,1}(\lambda_1, \dots, \lambda_d) H = 0$. From Lemma 2.4-(ii), $P_{d+1,1}(\lambda_1, \dots, \lambda_d) \neq 0$ holds. Therefore, we obtain $H = 0$ at p . By the arbitrariness of $p \in M$, we see that $H \equiv 0$, that is, M is totally geodesic. This implies $\bar{d} = d$. Thus a contradiction results. Therefore, d is odd. Let $d = 2i + 1$. It follows from (2.5), (2.10) and Lemma 2.5 that

$$(2.12) \quad \begin{aligned} & \left\{ \mathbf{B}_{d-1} \left[\sum_{k=0}^{i-1} Q_{2i+1,2k}(\lambda_1, \dots, \lambda_{2i-1}) \nabla_{X_{2k}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2i} \nabla_{X_{2i}}^\perp H + N_{2i+1}(\lambda_1, \dots, \lambda_{2i-1}) \right] \right\} \\ & = P_{d+1,1}(\lambda_1, \dots, \lambda_d) \nabla_{X_0}^\perp H + \sum_{j=1}^{i-1} P_{d+1,j+1}(\lambda_1, \dots, \lambda_d) \\ & \quad \left\{ \sum_{k=0}^{j-1} Q_{2j+1,2k}(\lambda_1, \dots, \lambda_{2j-1}) \nabla_{X_{2k}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2j} \nabla_{X_{2j}}^\perp H + N_{2j+1}(\lambda_1, \dots, \lambda_{2j-1}) \right\}. \end{aligned}$$

Since (2.12) holds for every positive numbers $\lambda_1, \dots, \lambda_{2i-1}$, by noticing terms of the highest degree, we have

$$\begin{aligned} & \left\{ \mathbf{B}_{d-1} \left[\sum_{k=0}^{i-1} Q_{2i+1,2k}(\lambda_1, \dots, \lambda_{2i-1}) \nabla_{X_{2k}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2i} \nabla_{X_{2i}}^\perp H \right] \right\} \\ & = P_{d+1,1}(\lambda_1, \dots, \lambda_d) \nabla_{X_0}^\perp H + \sum_{j=1}^{i-1} P_{d+1,j+1}(\lambda_1, \dots, \lambda_d) \\ & \quad \left\{ \sum_{k=0}^{j-1} Q_{2j+1,2k}(\lambda_1, \dots, \lambda_{2j-1}) \nabla_{X_{2k}}^\perp H + \lambda_1 \lambda_2 \cdots \lambda_{2j} \nabla_{X_{2j}}^\perp H \right\}. \end{aligned}$$

Furthermore, since this equality holds for every orthonormal system X_0, X_2, \dots, X_{2i} of $T_p M$, we see that $|B_{d-1}| \lambda_1 \lambda_2 \cdots \lambda_{2i} \nabla_{X_{2i}}^\perp H = 0$, that is, $\nabla_{X_{2i}}^\perp H = 0$. By the arbitrariness of X_{2i} , we see that $\nabla^\perp H = 0$ at p . Furthermore, from the arbitrariness of $p \in M$, $\nabla^\perp H \equiv 0$ is deduced. Thus M is an extrinsic sphere. In case of $d = 1$, it is directly deduced from Lemma 2.1 that so is M . \square

In the case where M and \bar{M} are Riemannian manifolds, this theorem is written as follows.

COROLLARY 2.9. *Let M be a totally umbilical submanifold in a Riemannian manifold \bar{M} isometrically immersed by f . Assume that for every helix σ of order d in M , $\bar{\sigma} := f \circ \sigma$ is a helix of order $d + 1$ in \bar{M} , where d is a positive integer. Then d is odd and M is an extrinsic sphere.*

§3. Proper helices in an extrinsic sphere.

Let M be an extrinsic sphere in a pseudo-Riemannian manifold \bar{M} isometrically immersed by f and σ a proper helix of order d in M . We put $\bar{\sigma} := f \circ \sigma$. Assume that $\bar{\sigma}$ is a proper curve of order \bar{d} . Let $\lambda_1, \dots, \lambda_{d-1}$ (resp. $\lambda_1, \dots, \lambda_{\bar{d}-1}$) be the curvatures of σ (resp. $\bar{\sigma}$), v_0, \dots, v_{d-1} (resp. $\bar{v}_0, \dots, \bar{v}_{\bar{d}-1}$) the Frenet frame field of σ (resp. $\bar{\sigma}$). For convenience, let $\lambda_i = 0, v_i = 0, \bar{\lambda}_j = 0$ and $\bar{v}_j = 0$ ($i \geq d, j \geq \bar{d}$). Set $\varepsilon_i := g(v_i, v_i)$ and $\bar{\varepsilon}_i := \bar{g}(\bar{v}_i, \bar{v}_i)$ ($i \geq 0$). Also, we define $\beta_{i,j}$ and $\bar{\beta}_{i,j}$ ($i \geq j \geq 1, i + j$: even) as (2.1).

LEMMA 3.1. *The curve $\bar{\sigma}$ is a proper helix in \bar{M} and the vector fields v_i ($i \geq 0$) and \bar{v}_j ($j \geq 0$) along σ are related as follows:*

$$\begin{aligned} (F'_0) \quad & \bar{v}_0 = v_0, \\ (F'_1) \quad & \bar{\beta}_{1,1} \bar{v}_1 = \beta_{1,1} v_1 + \varepsilon_0 H, \\ (F'_{2i-1}) \quad & \sum_{j=1}^i \bar{\beta}_{2i-1,2j-1} \bar{v}_{2j-1} = \sum_{j=1}^i \beta_{2i-1,2j-1} v_{2j-1} \quad (i \geq 2), \\ (F'_{2i}) \quad & \sum_{j=1}^i \bar{\beta}_{2i,2j} \bar{v}_{2j} = \sum_{j=1}^i \beta_{2i,2j} v_{2j} \quad (i \geq 1). \end{aligned}$$

PROOF. From (1.1), the Frenet formulas and the assumption that M is totally umbilic, we get

$$\bar{\lambda}_1 \bar{v}_1 = \bar{\nabla}_{v_0} v_0 = \nabla_{v_0} v_0 + \varepsilon_0 H = \lambda_1 v_1 + \varepsilon_0 H.$$

Thus we obtain (F'_1) . Furthermore, from this equality, we get

$$\bar{\varepsilon}_1 \bar{\lambda}_1^2 = \varepsilon_1 \lambda_1^2 + \bar{g}(H, H).$$

Since M is an extrinsic sphere, $\bar{g}(H, H)$ is constant. Therefore, $\bar{\lambda}_1$ is constant. Operating $\bar{\nabla}_{v_0}$ to (F_1') , we get

$$\bar{\beta}_{1,1}(-\varepsilon_0 \bar{\varepsilon}_1 \bar{\lambda}_1 v_0 + \bar{\lambda}_2 \bar{v}_2) = \beta_{1,1}(-\varepsilon_0 \varepsilon_1 \lambda_1 v_0 + \lambda_2 v_2) - \varepsilon_0 \bar{g}(H, H) v_0,$$

where we use (1.1), (1.2), the Frenet formulas and the assumption that M is an extrinsic sphere. By noticing $\text{Span}\{v_0\}^\perp$ -component of this equality, we see that

$$\bar{\lambda}_2 \bar{\beta}_{1,1} \bar{v}_2 = \lambda_2 \beta_{1,1} v_2,$$

which implies (F_2') by (2.1). Furthermore, from this equality, we get

$$\bar{\varepsilon}_2 \bar{\lambda}_2^2 \bar{\beta}_{1,1}^2 = \varepsilon_2 \lambda_2^2 \beta_{1,1}^2,$$

which implies that $\bar{\lambda}_2$ is constant. Assume that (F'_{2k}) holds and $\bar{\lambda}_i (1 \leq i \leq 2k)$ are constant. Since $\beta_{2k,2j}$ (resp. $\bar{\beta}_{2k,2j}$) ($1 \leq j \leq k$) are polynomials with variables $\lambda_1, \dots, \lambda_{2k}$ (resp. $\bar{\lambda}_1, \dots, \bar{\lambda}_{2k}$), these are constant along σ . Hence, operating $\bar{\nabla}_{v_0}$ to (F'_{2k}) , we have

$$\sum_{j=1}^k \bar{\beta}_{2k,2j} \bar{\nabla}_{v_0} \bar{v}_{2j} = \sum_{j=1}^k \beta_{2k,2j} \nabla_{v_0} v_{2j},$$

where we use (1.1) and the assumption that M is an extrinsic sphere. Applying the Frenet formulas and (2.1) to this equality, we obtain (F'_{2k+1}) . Furthermore, from (F'_{2k+1}) , we get

$$\sum_{j=1}^{k+1} \bar{\varepsilon}_{2j-1} \bar{\beta}_{2k+1,2j-1}^2 = \sum_{j=1}^{k+1} \varepsilon_{2j-1} \beta_{2k+1,2j-1}^2,$$

that is,

$$(3.1) \quad \bar{\varepsilon}_{2k+1} \bar{\beta}_{2k+1,2k+1}^2 = \sum_{j=1}^{k+1} \varepsilon_{2j-1} \beta_{2k+1,2j-1}^2 - \sum_{j=1}^k \bar{\varepsilon}_{2j-1} \bar{\beta}_{2k+1,2j-1}^2.$$

Since $\beta_{2k+1,2j-1}$ ($1 \leq j \leq k+1$) are polynomials with variables $\lambda_1, \dots, \lambda_{2k+1}$ and $\bar{\beta}_{2k+1,2j-1}$ ($1 \leq i \leq k$) are polynomials with variables $\bar{\lambda}_1, \dots, \bar{\lambda}_{2k}$, these are constant along σ , that is, the right-hand side of (3.1) is constant along σ . Also, the left-hand side of (3.1) is equal to $\bar{\varepsilon}_{2k+1} \bar{\lambda}_1^2 \bar{\lambda}_2^2 \dots \bar{\lambda}_{2k+1}^2$. Therefore, we see that $\bar{\lambda}_{2k+1}$ is constant. Since $\beta_{2k+1,2j-1}$ (resp. $\bar{\beta}_{2k+1,2j-1}$) ($1 \leq j \leq k+1$) are polynomials with variables $\lambda_1, \dots, \lambda_{2k+1}$ (resp. $\bar{\lambda}_1, \dots, \bar{\lambda}_{2k+1}$), these are constant along σ . Hence, operating $\bar{\nabla}_{v_0}$ to (F'_{2k+1}) , we have

$$\sum_{j=1}^{k+1} \bar{\beta}_{2k+1,2j-1} \bar{\nabla}_{v_0} \bar{v}_{2j-1} = \sum_{j=1}^{k+1} \beta_{2k+1,2j-1} \nabla_{v_0} v_{2j-1},$$

where we use (1.1) and the assumption that M is an extrinsic sphere. Applying the Frenet formulas and (2.1) to this equality, we obtain (F'_{2k+2}) . Furthermore, from (F'_{2k+2}) , we get

$$\sum_{j=1}^{k+1} \bar{\epsilon}_{2j} \bar{\beta}_{2k+2,2j}^2 = \sum_{j=1}^{k+1} \epsilon_{2j} \beta_{2k+2,2j}^2,$$

that is,

$$(3.2) \quad \bar{\epsilon}_{2k+2} \bar{\beta}_{2k+2,2k+2}^2 = \sum_{j=1}^{k+1} \epsilon_{2j} \beta_{2k+2,2j}^2 - \sum_{j=1}^k \bar{\epsilon}_{2j} \bar{\beta}_{2k+2,2j}^2.$$

Since $\beta_{2k+2,2j} (1 \leq j \leq k+1)$ are polynomials with variables $\lambda_1, \dots, \lambda_{2k+2}$ and $\bar{\beta}_{2k+2,2j} (1 \leq i \leq k)$ are polynomials with variables $\bar{\lambda}_1, \dots, \bar{\lambda}_{2k+1}$, these are constant along σ , that is, the right-hand side of (3.2) is constant along σ . Also, the left-hand side of (3.2) is equal to $\bar{\epsilon}_{2k+2} \bar{\lambda}_1^2 \bar{\lambda}_2^2 \dots \bar{\lambda}_{2k+2}^2$. Therefore, we see that $\bar{\lambda}_{2k+2}$ is constant. Thus, by the induction, we see that $(F'_i) (i \geq 0)$ hold and $\lambda_j (j \geq 1)$ are constant (i.e., $\bar{\sigma}$ is a proper helix). \square

By using this lemma, we can prove the following theorem.

THEOREM 3.2. *Let M be an extrinsic sphere in a pseudo-Riemannian manifold M isometrically immersed by f and σ a proper helix of order d in M such that $\bar{\sigma} (= f \circ \sigma)$ is a proper curve in \bar{M} , where d is a positive integer. Then*

- (i) *if d is odd, then $\bar{\sigma}$ is a proper helix of order d or $d + 1$,*
- (ii) *if d is even, then $\bar{\sigma}$ is a proper helix of order d .*

PROOF. Let $v_i (0 \leq i \leq d-1)$ (resp. $v_i (0 \leq i \leq \bar{d}-1)$) the Frenet frame field of σ (resp. $\bar{\sigma}$) and, for convenience, $v_i = 0 (i \geq d)$ and $\bar{v}_i = 0 (i \geq \bar{d})$. According to Lemma 3.1, $\bar{\sigma}$ is a proper helix, $\bar{v}_{2i} \in \text{Span} \{v_0, v_2, \dots, v_{2i}\} (i \geq 0)$ and $\bar{v}_{2i+1} \in \text{Span} \{v_1, v_3, \dots, v_{2i+1}, H\} (i \geq 0)$. The conclusion is directly deduced from these facts. \square

In the case where M and \bar{M} are Riemannian manifolds, this theorem is written as follows.

COROLLARY 3.3. *Let M be an extrinsic sphere in a Riemannian manifold \bar{M} isometrically immersed by f and σ a helix of order d in M , where d is a positive integer. Then*

- (i) if d is odd, then $f \circ \sigma$ is a helix of order d or $d + 1$,
(ii) if d is even, then $f \circ \sigma$ is a helix of order d .

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