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CONSTANT SCALAR CURVATURES ON WARPED PRODUCT MANIFOLDS

By

"Dedicated to Professor Tsunero Takahashi on his 60th birthday"

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1. Introduction

In a recent study [D.D.], F. Dobarro and E. L. Dozo have studied from the viewpoint of partial differential equations and variational methods, the problem of showing when a Riemannian metric of constant scalar curvature can be produced on a product manfild $B\times F$ by a warped product construction applied to the two Riemannian manifolds (B,g_B) and (F,g_F), especially in the case when the fibre (F,g_F) is of constant curvature. Particularly, in Theorem 3.6 of [D.D.], the uniqueness of the warping function is considered. In [D.D.], the eigenvalue problem for the elliptic operator $Lu=-\frac{4n}{n+1}\Delta u+Ru$ of a warped product $B\times_f F$ of Riemannian manifolds B and F, where Δ is the Laplacian on B and R is the scalar curvature on B, is studied. Basically, the fact that the operator $L-\lambda I:C^{2,\alpha}(M)\rightarrow C^{\alpha}(M)$ is an isomorphism for some λ, is employed.

For Riemannian manifolds, warped products have been useful in producing examples of spectral behavior, examples of manifolds of negative curvature (cf. [B.O.], [D.G.], [D.D.], [Eb], [Ej], [K.K.P.], [M.M.]), and also in studying L_2-cohomology (cf. [Z.]).

Perhaps even more interestingly on physical grounds than purely Riemannian constructions employing warped products, many of the known exact solutions of the Einstein field equations of General Relativity are warped product metrics of the form $B\times_f F$ where (B,g_B) is a Lorentzian manifold and (F,g_F) is a Riemannian manifold. A most notable class of examples are the Robertson-

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Walker space-times of cosmology theory as well as the Schwarzschild space-time. So, in Lorentzian geometry, the warped product is also widely used for studying space times with various applications (cf. [A1], [D.D.V.], [D.V.V.], [G.], [M.], etc.)

In this paper, then, we consider the problem of achieving constant scalar curvature for two different classes of pseudo-Riemannian manifolds. The first class consisting of the case B is an interval (a,b) with negative definite metric $-dt^2$ and $-\infty \leq a < b \leq +\infty$ and F is a Riemannian manifold of dimension $n > 1$. Here $n = 3$ turns out to be a special case because of a term in the general curvature formula which is multiplied by a factor of $(n-3)$. In this first case, the problem may be studied directly from an associated second order linear equation $(n=3)$, or from an autonomous differential equation, when $n \neq 3$. Correspondingly, when $n=3$ and one is seeking to produce constant positive scalar curvature on the warped product, it may be done for any constant scalar curvature value chosen for the fiber (F, g_F) (cf. Remark 3.6-(1)). Whereas for $n \neq 3$, nonnegative scalar curvature may be needed for the fiber to produce constant positive scalar curvature on the warped product (cf. Remark 3.8-(2)).

The second class studied consists of taking (B, g_B) to be a compact Riemannian manifold and (F, g_F) to be a pseudo-Riemannian manifold. In our study of this case, we apply the method of upper and lower solutions and also variational considerations.

Although we will assume throughout this paper that all data (M, metric g, and curvature, etc.) are smooth, this is merely for convenience. Our arguments go through with little or no change if one makes minimal smoothness hypotheses, such as assuming that the give data is Holder continuous.

2. Preliminaries on a warped product manifold

In this section, we briefly recall some results on warped product manifolds. Complete details may be found in [B.E.], [B.O.], or [O.].

On a (semi)Riemannian product manifold $B \times F$, let π and σ be the projections of $B \times F$ onto B and F, respectively, and let $f > 0$ be a smooth function on B.

DEFINITION 2.1. The warped product manifold $M = B \times_f F$ is the product manifold $M = B \times F$ furnished with metric tensor

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F),$$

where g_B and g_F are metric tensors of B and F, respectively. In other words, if

v is tangent to M at (p,q), then

$$g(v,v) = g_B(d\pi(v), d\pi(v)) + f^2(p) g_F(d\sigma(v), d\sigma(v)).$$

Here B is called the base of M and F the fiber. We denote the metric g by $\langle\, , \rangle$. In view of Remark 2.2-(1) and Lemma 2.3, we may also denote the metric g_B by $\langle\, , \rangle$. The metric g_F will be denoted by $(\, ,)$.

REMARK 2.2. Some well known elementary properties of the warped product manifold $M = B \times_f F$ are as follows.

(1) For each $q \in F$, the map $\pi|_{\sigma^{-1}(q)=B\times q}$ is an isometry onto B.

(2) For each $p \in B$, the map $\sigma|_{\pi^{-1}(q)=p\times F}$ is a positive homothetic map onto F with homothetic factor $1/f(p)$.

(3) For each $(p,q) \in M$, the horizontal leaf $B \times q$ and the vertical fiber $p \times F$ are orthogonal at (p,q).

(4) The horizontal leaf $\sigma^{-1}(q) = B \times q$ is a totally geodesic submanifold of M and the vertical fiber $\pi^{-1}(p) = p \times F$ is a totally umbilic submanifold of M.

(5) If ϕ is an isometry of F, then $1 \times \phi$ is an isometry of M. And if ψ is an isometry of B such that $f = f \circ \psi$, then $\psi \times 1$ is an isometry of M.

Recall that vectors tangent to leaves are called horizontal and vectors tangent to fibers are called vertical. From now on, we will often use a natural identification $T_{(p,q)}(B \times_f F) \cong T_{(p,q)}(B \times F) \cong T_p B \times T_q F$. The decomposition of vectors into horizontal and vertical parts plays a role in our proofs. If X is a vector field on B, we define $\overline{X}$ at (p,q) by setting $\overline{X}(p,q) = (X_p, 0_q)$. Then $\overline{X}$ is π-related to X and σ-related to the zero vector field on F. Similarly, if Y is a vector field on F, $\overline{Y}$ is defined by $\overline{Y}(p,q) = (0_p, Y_q)$.

LEMMA 2.3. If h is a smooth function on B, then the gradient of the lift $h \circ \pi$ of h to M is the lift to M of gradient of h on B.

PROOF. See Lemma 7.34 in [O.].

In view of Lemma 2.3, we simplify the notation by writing h for $h \circ \pi$ and $grad(h)$ for $grad(h \circ \pi)$. For a covariant tensor A on B, its lift $\overline{A}$ to M is just its pullback $\pi^*(A)$ under the projection $\pi : M \to B$. That is, if A is a (1,s)-tensor, and if $v_1, \ldots, v_s \in T_{(p,q)}M$, then $\overline{A}(v_1, \ldots, v_s) = A(d\pi(v_1)), \ldots, d\pi(v_s)) \in T_p(B)$. Hence if v_k is vertical, then $\overline{A} = 0$ on B. For example, if f is a smooth function on B, the lift to M of the Hessian of f is also denoted by H^f. This agrees with the Hessian of the lift $f\circ\pi$ generally only on horizontal vectors. For detailed computations, see Lemma 5.1 in [B.E.P.].

Now we recall the formula for the Ricci curvature tensor Ric of the warped product manifold $M=B\times_f F$. We write Ric^B for the pullback by π of the Ricci curvature of B and similarly for Ric^F.

LEMMA 2.4. On a warped product manifold $M=B\times_f F$ with $n=\dim F>1$, let X, Y be horizontal and V, W vertical. Then

(1) $Ric(X,Y)=Ric^B(X,Y)-\dfrac{n}{f}H^f(X,Y)$

(2) $Ric(X,V)=0$

(3) $Ric(V,W)=Ric^F(V,W)-\langle V,W\rangle f^{\#}$,

where $f^{\#}=\dfrac{\Delta f}{f}+(n-1)\dfrac{\langle \mathrm{grad}f,\mathrm{grad}f\rangle}{f^2}$, and $\Delta f=C(H^f)=\mathrm{trace}(H^f)$ is the Laplacian on B.

PROOF. See Corollary 7.43 in [O.].

On the given warped product manifold $M=B\times_f F$, we also write S^B for the pullback by π of the scalar curvature S^B of B and similarly for S^F. From now on, we denote $grad(f)$ by ∇f.

COROLLARY 2.5. If S is the scalar curvature of $M=B\times_f F$ with $n=\dim F>1$, then

$$S=S^B+\frac{S^F}{f^2}-2n\frac{\Delta f}{f}-n(n-1)\frac{\langle\nabla f,\nabla f\rangle}{f^2}, \tag{2.1}$$

where Δ is the Laplacian on B.

PROOF. For each $(p,q)\in M=B\times_f F$, let $\{e_i\}$ be an orthonormal basis for T_pB. Then by the natural isomorphism $\{\overline{e_i}=(e_i,0)\}$ is an orthonormal set in $T_{(p,q)}M$. We can choose $\{d_j\}$ on T_qF such that $\{\overline{e_i},\overline{d_j}\}$ forms an orthonormal basis for $T_{(p,q)}M$. Then

$$1=\langle\bar d_j,\bar d_j\rangle=f(p)^2(d_j,d_j)=(f(p)d_j,f(p)d_j),$$

which implies that $\{f(p)d_j\}$ forms an orthonormal basis for T_qF.

By Lemma 2.4 (1) and (3), for each i and j,

$$Ric(\overline{e_i},\overline{e_i}) = Ric^B(\overline{e_i},\overline{e_i}) - \Sigma_i \frac{n}{f} H^f(\overline{e_i},\overline{e_i}),$$

and

$$Ric(\overline{d_j},\overline{d_j}) = Ric^F(\overline{d_j},\overline{d_j}) - f^2(d_j,d_j)\left(\frac{\Delta f}{f} + (n-1)\frac{\langle \nabla f, \nabla f\rangle}{f^2}\right)$$

Hence for $\varepsilon_\alpha = g(e_\alpha, e_\alpha)$,

$$\begin{aligned} S(p,q) &= \Sigma_\alpha \varepsilon_\alpha R_{\alpha\alpha} \\ &= \Sigma_i \varepsilon_i Ric(\overline{e_i},\overline{e_i}) + \Sigma_j \varepsilon_j Ric(\overline{d_j},\overline{d_j}) \\ &= S^B(p,q) + \frac{S^F}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{\langle \nabla f, \nabla f\rangle}{f^2}, \end{aligned}$$

which is a nonlinear partial differential equation on $B \times q$ for each $q \in F$.

Now we may pose the following question: if $S_F(q) \equiv c$ (constant) on F, can we find a warping function $f > 0$ on B such that the warped metric g has constant scalar curvature $S(p,q) = k$ on $M = B \times_f F$? If $S(p,q) \equiv k$ for all $(p,q) \in M$, then equation (2.1) is the pullback by π of the following equation:

$$k = S_B(p) + \frac{c}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{\langle \nabla f, \nabla f\rangle}{f^2},$$

or equivalently,

$$\Delta f + \frac{1}{2n}(k - S_B)f - \frac{c}{2nf} + \frac{n-1}{2}\frac{\langle \nabla f, \nabla f\rangle}{f^2} = 0. \tag{2.2}$$

3. Generalized Robertson-Walker space-times

In this section, we restrict our results to the case that $B = (a,b)$ is an open connected subset of R^1_1 with the negative definite metric $-dt^2$ and $-\infty \leq a < b \leq +\infty$. Recalling that $\Delta f = -f''(t)$ and $\langle \nabla f, \nabla f\rangle = -(f'(t))^2$, and making the change of variable $f(t) = \sqrt{v(t)}$, we have the following equation from equation (2.2),

$$v''(t) + \frac{(n-3)}{4}\frac{|v'(t)|^2}{v(t)} - \frac{k}{n}v(t) + \frac{c}{n} = 0 \tag{3.1}$$

where we assume that F is a Riemannian manifold with constant scalar curvature c and $\dim F = n > 1$ (cf. equation (2.16) in [B.E., p. 78]).

Now we consider the following problem:

Problem I: Given a fiber F with constant scalar curvature c, can we find a warping function $f > 0$ on $B = (a,b)$ such that for any real number k, the warped metric g admits k as the constant scalar curvature on $M = (a,b) \times_f F$?

We consider several cases according to the dimension of F and the value of the given c.

THEOREM 3.1. If $\dim F = n = 3$, i.e., M is a generalized Robertson-Walker space-time, then for any real number k the following warping function $v(t)$ produces constant scalar curvature k on (M, g):

i) $k > 0$, $v(t) = c_1 \exp\left(\sqrt{\frac{k}{3}}t\right) + c_2 \exp\left(-\sqrt{\frac{k}{3}}t\right) + \frac{c}{k}$,

ii) $k = 0$, $v(t) = -\frac{c}{6}t^2 + c_1 t + c_2$,

iii) $k < 0$, $v(t) = c_1 \sin\left(\sqrt{-\frac{k}{3}}t\right) + c_2 \cos\left(\sqrt{-\frac{k}{3}}t\right) + \frac{c}{k}$,

where c_1 and c_2 are suitable constants chosen (if possible) so that $v(t)$ is positive on $B = (a, b)$.

PROOF. If $n = 3$, then we have a simple differential equation,

$$v''(t) - \frac{k}{3}v(t) + \frac{c}{3} = 0$$

Putting $h(t) = \frac{k}{3}v(t) - \frac{c}{3}$, it follows that $h''(t) - \frac{k}{3}h(t) = 0$. Hence, according to sign of k, the above solutions follow directly from elementary methods for ordinary differential equations.

REMARK 3.2. The difficulty in applying Theorem 3.1 is simply to insure that c_1, c_2 may be chosen, depending on c, k, and the interval $B = (a, b)$ such that $v(t)$ is positive for all $t \in (a,b)$. The strongest statement that may be made independent of choice of (a,b) is the following.

COROLLARY 3.3. For $\dim F = 3$ and (a,b) arbitrary,
i) for $k > 0$, Problem I may be solved affirmatively for all c,
ii) for $k = 0$, Problem I may be solved affirmatively for all $c \le 0$,
iii) for $k < 0$, Problem I may be solved affirmatively for all $c < 0$.

REMARK 3.4. (1) If $k=0, c>0$ and $B=(a,b)=(-\infty,+\infty)$, then no values of c_1, c_2 may be chosen which will produce a warping function positive on all of $(-\infty,+\infty)$. Similarly, if $k<0, c\ge 0$ and $B=(a,b)=(-\infty,+\infty)$, then no values of c_1, c_2 will produce $\upsilon(t)>0$ on all of $(-\infty,+\infty)$.

(2) By Remark 2.58 in [B.E.] and Corollary 5.6 in [P.], if $B=(a,b)$ is a finite interval and $\dim F = n = 3$, then all nonspacelike geodesics are incomplete. But if $B=(-\infty,+\infty)$, then there exists $\upsilon(t)$ so that all non-spacelike geodesics are complete. For Theorem 5.5 in [P.] implies that all timelike geodesics are future (resp. past) complete on $(a,b)\times_{\upsilon(t)}F$ if and only if $\int_{t_0}^{+\infty}\left(\frac{\upsilon}{1+\upsilon}\right)^{1/2}dt=+\infty$ (resp. $\int_{-\infty}^{t_0}\left(\frac{\upsilon}{1+\upsilon}\right)^{1/2}dt=+\infty$) and Remark 2.58 in [B.E.] implies that all null geodesics are future (resp. past) complete if and only if $\int_{t_0}^{+\infty}\upsilon^{1/2}dt=+\infty$ $\left(\text{resp. } \int_{-\infty}^{t_0}\upsilon^{1/2}dt=+\infty\right)$ (cf. Theorem 4.1, Remark 4.2 in [B.E.P.]).

THEOREM 3.5 If $\dim F = n \neq 3$ and $c = 0$, then for any real number k the warping function $\upsilon(t)$ produces constant scalar curvature k on (M, g):

i) $$k>0, \upsilon(t)=\left(c_1\exp\left(\sqrt{\frac{(n+1)k}{4n}}t\right)+c_2\exp\left(-\sqrt{\frac{(n+1)k}{4n}}t\right)\right)^{4/n+1},$$

ii) $k=0, \upsilon(t)=(c_1t+c_2)^{4/n+1}$,

iii) $$k<0, \upsilon(t)=\left(c_1\cos\left(\sqrt{\frac{-(n+1)k}{4n}}t\right)+c_2\left(\sin\left(\sqrt{\frac{-(n+1)k}{4n}}t\right)\right)\right)^{4/n+1},$$

where c_1 and c_2 are suitable constants chosen (if possible) so that $\upsilon(t)$ is positive.

PROOF. In this case, equation (3.1) is changed into the simpler form,

$$\frac{\upsilon''(t)}{\upsilon(t)}+\frac{(n-3)}{4}\frac{\upsilon'(t)^2}{\upsilon(t)^2}-\frac{k}{n}=0.$$

Putting $\upsilon(t)=\omega(t)^{\frac{4}{n+1}}$, then $\omega(t)$ satisfies the equations,

$$\upsilon'(t)=\frac{4}{n+1}\omega(t)^{3-n/n+1}\omega'(t)$$

and

$$v''(t) = \frac{4(3-n)}{(n+1)^2}\omega^{(4/_{n+1})-2}\omega'(t)^2 + \frac{4}{n+1}\omega^{(4/_{n+1})-1}\omega''(t).$$

Hence $\omega''(t) = \dfrac{n+1}{4n}k\omega(t)$ and our solutions follow.

REMARK 3.6 (1) If $k > 0$ and $(a\ , b)$ is arbitrary, taking $c_1 = c_2 = 1$ in Theorem 3.5 provides an affirmative solution to Problem I.

(2) If $k = 0$ and $B = (-\infty, +\infty)$, only a constant warping function $v(t)$ with $c_1 = 0, c_2 > 0$ will satisfy $v(t) > 0$ on all of B.

(3) If $k < 0$ and $B = (-\infty, +\infty)$, then iii) reveals that Problem I may not be solved on all of B. In the case that B is a finite interval, evidently iii) reveals that a positive warping function $v(t)$ may be constructed, but all nonspacelike geodesics will necessarily be incomplete.

THEOREM 3.7. If $\dim F = n \neq 1,3$ and $c \neq 0$, then for any real number k the warping function $v(t)$ produces constant scalar curvature k on $(M\ , g)$:

i) $k > 0, v(t) = \left[c_1 \exp\left(\sqrt{\dfrac{k}{n(n+1)}}t\right) + \dfrac{n+1}{n-1}\dfrac{c}{4kc_1}\exp\left(-\sqrt{\dfrac{k}{n(n+1)}}t\right)\right]^2,$

ii) $k = 0, v(t) = \dfrac{-c}{n(n-1)}t^2 + c_1 t - \dfrac{n(n-1)}{4c}c_1^2,$

iii) $k < 0, v(t) = \dfrac{n+1}{n-1}\dfrac{c}{k}[\tan^2(\pm\sqrt{\dfrac{-k}{n(n+1)}}t + c_1) + 1]^{-1},$

where c_1 is a suitable constant chosen (if possible) so that $v(t)$ is positive.

PROOF. Suppose $v(t)$ is a solution of equation (3.1). If $v(t)$ is a constant, then $v(t) = \dfrac{c}{k}$, which is defined only when $ck > 0$. If $v(t)$ is nonconstant, putting $v(t) = \omega(t)^{4/_{n+1}}$, then $\omega(t)$ satisfies the equations,

$$v'(t) = \frac{4}{n+1}\omega(t)^{3-n/_{n+1}}\omega'(t)$$

and

$$v''(t) = \frac{4(3-n)}{(n+1)^2}\omega^{(4/_{n+1})-2}\omega'(t)^2 + \frac{4}{n+1}\omega^{(4/_{n+1})-1}\omega''(t).$$

Hence

$$\omega''(t) - \frac{n+1}{4n}k\omega(t) + \frac{n+1}{4n}c\omega^{1-(4/_{n+1})} = 0.$$

Putting $\dfrac{d\omega(t)}{dt} = y$ and $\dfrac{dy}{dt} = \omega''(t)$,

$$\frac{d\omega}{dy} = \frac{y}{\frac{k(n+1)}{4n}\omega - \frac{c(n+1)}{4n}\omega^{1-(4/_{n+1})}}.$$

$$y^2 = \frac{n+1}{4n}\omega^2\left(k - \frac{n+1}{n-1}c\omega^{(-4/_{n+1})}\right)$$

and

$$\frac{d\omega}{\omega\sqrt{k - \frac{n+1}{n-1}c\omega^{(-4/_{n+1})}}} = \pm\sqrt{\frac{n+1}{4n}}dt.$$

Here we have three following cases:

$$\int \frac{d\omega}{\omega\sqrt{k - \frac{n+1}{n-1}c\omega^{(-4/_{n+1})}}} = -\frac{n+1}{4\sqrt{k}}\log\frac{\sqrt{k - \frac{n+1}{n-1}c\omega^{(-4/_{n+1})}} - \sqrt{k}}{\sqrt{k - \frac{n+1}{n-1}c\omega^{(-4/_{n+1})}} + \sqrt{k}}, \quad k > 0,$$

$$= \sqrt{-\frac{(n+1)(n-1)}{4c}\omega^{(2/_{n+1})}}, \quad k = 0$$

$$= -\frac{n+1}{2\sqrt{-k}}\tan^{-1}\left(\frac{\sqrt{k - \frac{n+1}{n-1}c\omega^{(-4/_{n+1})}}}{\sqrt{-k}}\right), \quad k < 0$$

Hence our results follow easily. For example, if $k > 0$, then

$$\sqrt{k - \frac{n+1}{n-1}c\omega^{(-4/_{n+1})}} = \frac{\sqrt{k}\left(1 + \tilde{c}_1\exp\left(-\sqrt{\frac{4k}{n(n+1)}}t\right)\right)}{1 - \tilde{c}_1\exp\left(-\sqrt{\frac{4k}{n(n+1)}}t\right)}$$

for some constant $\tilde{c}_1$. Thus

$$\upsilon(t) = \omega(t)^{(4/_{n+1})} = \frac{n+1}{n-1}\frac{c}{(-4k\tilde{c}_1)}\left[\exp\left(\sqrt{\frac{k}{n(n+1)}}t\right) - \tilde{c}_1\exp\left(-\sqrt{\frac{k}{n(n+1)}}t\right)\right]^2,$$

which implies the first case, replacing $\tilde{c}_1 = \dfrac{n+1}{n-1}\dfrac{c}{(-4kc_1^2)}$

REMARK 3.8. (1) If $k \le 0$ and $B = (-\infty, +\infty)$, then ii) and iii) of Theorem 3.7

reveal that no warping function $v(t)$ may be found which is positive on all of B.
(2) If $k>0$, then i) of Theorem 3.7 reveals that Problem I may be solved affirmatively for any B provided that $c \geq 0$.

4. Warped products with semi-Riemannian fiber

In this section, we treat the case that, for $M = B \times_f F, B$ is a compact Riemannian manifold and F is a (semi) Riemannian manifold. We denote the volume element of g_B by dV, the gradient by ∇, and the associated Laplacian by Δ. The mean value $\bar{h}$ of a function h on B is, of course,

$$\bar{h} = \frac{1}{\mathrm{vol}(B)} \int_B h dV.$$

We let $H_{s,p}(B)$ denote the Sobolev space of functions on B whose derivatives through order s are in $L_p(B)$. The norm on $H_{s,p}(B)$ will be denoted by $\| \ \|_{s,p}$. The usual norm $L_2(B)$ inner product will be written $\| \ \|$.

By equation (2.2), on B, assuming that M has a constant scalar curvature k,

$$\Delta f + \frac{1}{2n}(k - S_B)f - \frac{c}{2nf} + \frac{n-1}{2}\frac{\langle \nabla f, \nabla f \rangle}{f} = 0. \tag{4.1}$$

By the change of variables $f = v^{\frac{2}{n+1}}$,

$$\nabla f = \frac{2}{n+1} v^{(2/n+1)-1} \nabla v,$$

and

$$\Delta f = \frac{2}{n+1}\left(\frac{2}{n+1} - 1\right) v^{(2/n+1)-2} |\nabla v|^2 + \frac{2}{n+1} v^{(2/n+1)-1} \Delta v.$$

Hence equation (4.1) is changed into

$$\Delta v + \frac{n+1}{4n}(k - S(p))v - \frac{n+1}{4n} c v^{(1-4/n+1)} = 0. \tag{4.2}$$

In Theorem 2.1 of [D.D.], F. Dobarro and E. L. Dozo obtained the same equations (2.1) and (4.2) by using conformal deformations since if $g = g_B + f^2 g_F = f^2(f^{-2} g_B + g_F)$, then g is conformal to $f^{-2} g_B + g_F$ on $B \times_f F$ and $f^{-2} g_B$ is conformal to g_B.

THEOREM 4.1. Let v_+, v_- satisfy the following equations

$$\Delta v_+ + \frac{n+1}{4n}(k - S(p))v_+ - \frac{n+1}{4n} c v_+^{(1-4/n+1)} \leq 0$$

and

$$\Delta v_- + \frac{n+1}{4n}(k - S(p))v_- - \frac{n+1}{4n} c v_-^{(1-4/_{n+1})} \geq 0$$

on B with $0 < v_- \leq v_+$. Then there exists $v > 0$ on B such that $0 < v_- \leq v \leq v_+$ and v is a solution of equation (4.2).

PROOF. We briefly outline the proof. Let

$$N = \text{l.u.b}_{p \in B, 0 < v_- \leq v \leq v_+} \left[-\frac{n+1}{4n}(k - S(p)) + \frac{n+1}{4n} c\left(1 - \frac{4}{n+1}\right) v^{(-4/_{n+1})} \right]$$

and, if necesssary, add a positive constant to N to insure that $N > 0$. Set $v_0 = v_+$, and then define the sequence $\{v_j\}$ recurrently as the unique solution on B of

$$\Delta v_{j+1} - N v_{j+1} = -\frac{n+1}{4n}(k - S(p))v_j + \frac{n+1}{4n} c v_j^{(1-4/_{n+1})} - N v_j .$$

One uses the maximum principle to show that $0 < v_- \leq v_{j+1} \leq v_j \leq \cdots \leq v_+$. A standard argument shows that the sequence $\{v_j\}$ converges to a positive solution v of equation (4.2) with $0 < v_- \leq v \leq v_+$. (For details, see [C.H., pp. 370–371], [K.W. 1, 2, 3], [K.K.], or [K.]).

Here v_+ is called an upper solution of equation (4.2) and v_- a lower solution of equation (4.2).

THEOREM 4.2. If equation (4.2) has a solution for c, then equation (4.2) has a solution of Nc, where N is any positive constant.

PROOF. If v is a solution of equation (4.2) for c, then $v_1 = N^{((n+1)/_4)} v$ is also a solution of equation (4.2) for Nc.

THEOREM 4.3. If there exists a solution of equation (4.2) and $c \leq 0$, then

$$\int (k - S(p))dV \leq 0, i.e., \int_B S(p)dV \geq k\,\text{vol}(B).$$

PROOF. Multiply both sides of equation (4.2) by v^{-1} and integrate.

Even though there always exists a metric on a compact Riemannian manifold such that $\int_B S(p)dV < 0$, we do not assume negative total scalar curvature. Recall also that there are topological obstructions for zero scalar curvature and positive scalar curvature ([K.W.1]).

Now we turn to consideration of the following problem:

Problem II. Does B admit a warping function $f = v^{(2/n+1)} > 0$ such that the associated warped metic g has constant scalar curvature on $M = B \times_f F$, given constant scalar curvature c on (F, g_F)?

As in section 3, we consider several cases according to the value of c.

THEOREM 4.4. If $c = 0$, then the Problem II admits a solution.

PROOF. Letting $L(v) = -\Delta v + \frac{n+1}{4n} S(p)v$, we consider the first eigenvalue of the differential operator L on the Sobolev space $H_{1,2}(B)$, i.e.,

$$\begin{aligned}\lambda_1 &= \min_{v \neq 0 \in H_{1,2}(B)} \frac{\int_B vL(v)dV}{\int_B v^2 dV} \\ &= \min_{v \neq 0 \in H_{1,2}(B)} \frac{\int_B |\nabla v|^2 dV + \frac{n+1}{4n}\int_B S(p)v^2 dV}{\int_B v^2 dV}\end{aligned}$$

Put $\lambda_1 = \frac{n+1}{4n}\bar{k}$. Then $L(v) = \lambda_1 v = \frac{n+1}{4n}\bar{k}v,$, where v is a positive eigenfunction. (Recall that the eigenfunction is never zero and smooth, so we can assume that $v > 0$ ([K.W.1])). Hence

$$\Delta v + \frac{n+1}{4n}(\bar{k} - S(p))v = 0,$$

which implies that the warped metric g has the constant $\bar{k}$ as the scalar curvature of g.

REMARK 4.5. Note that if v is an eigenfunction, then rv is also an eigenfunction for any real positive number r. Therefore, in case that $c = 0$, there are infinitely many warped metrics all of which have constant scalar curvature $\bar{k}$.

THEOREM 4.6 If $c < 0$ and equation (4.2) has a solution for k_1, then for any $k \le k_1$, there exists a solution of equation (4.2) for k.

PROOF. Let $v > 0$ be a solution of equation (4.2) for k_1, i.e.,

$$\begin{aligned}0 &= \Delta v + \frac{n+1}{4n}(k_1 - S(p))v - \frac{n+1}{4n}cv^{(1-4/n+1)} \\ &= \Delta v + \frac{n+1}{4n}(k - S(p))v - \frac{n+1}{4n}cv^{(1-4/n+1)} + \frac{n+1}{4n}(k_1 - k)v.\end{aligned}$$

Since $k_1 - k \geq 0$ and $v > 0$,

$$\Delta v + \frac{n+1}{4n}(k - S(p))v - \frac{n+1}{4n} c v^{1-\frac{4}{n+1}} \leq 0 \ ,$$

which implies that v is an upper solution of equation (4.2) for k. Since $c < 0$ and the exponent $(1 - 4/_{n+1})$ is less than 1, a sufficiently small positive constant less than v is a lower solution of equation (4.2) for k. Hence Theorem 4.1 implies that there exists a solution of equation (4.2) for k.

THEOREM 4.7. If $c < 0$, then there exists a constant k_0 such that we can solve equation (4.2) for $k < k_0$, but not for $k > k_0$.

PROOF. If $k_1 < \min_{p \in B} S(p)$, then a large positive constant is an upper solution and a small positive constant is a lower solution of equation (4.2) for k_1. Hence by Theorem 4.1 there exists a solution v of equation (4.2) with $v > 0$. Theorem 4.6 implies that if $k < k_1$, then we can solve equation (4.2) for k. Define k_0 by $k_0 = l.u.b\{k|$ equation (4.2) admits a solution for that value of $k\}$.

Now we observe that if $c < 0$, then the above k_0 is finite. For suppose $k_0 = \infty$. Choose $k > \max_{p \in B} S(p)$ and let v be a corresponding solution tø equation (4.2). By choice of c and k,

$$\frac{n+1}{4n} \int (k - S(p)) v dV - \frac{n+1}{4n} c \int v^{(1 - 4/_{n+1})} dV > 0 \, ,$$

which is a contradiction to v satisfying equation (4.2).

REMARK 4.8. (1) According to Theorem 3.2 in [D.D.], k_0 should be the first eigenvalue of $-\frac{4n}{n+1}\Delta + S(p)$.

(2) We can obtain another result using the variational method. For $k_1 < \min_{p \in B} S(p)$, if we define the functional $J(v)$ by

$$J(v) = \frac{1}{2}\left[\int_B |\nabla v|^2 dV - \frac{n+1}{4n}\int_B (k_1 - S(p)) v^2 dV\right]$$

on. $\left\{v \in H_{1,2}(B) | v \geq 0, \int_B v^{(2 - 4/_{n+1})} dV = 1\right\}$ Clearly $\inf_v J(v) \geq 0$. We can see that there exist a minimizing sequence $\{v_j\}$ and a function v_0 such that $v_j \to v_0$ (strongly in $L_2(B)$, weakly in $H_{1,2}(B)$, and almost everywhere pointwise) and $J(v_j) \to \bar{c}$, whose value may not be equal to the originally given value c. Then v_0 is a solution of equation (4.2) for k_1 and $\bar{c}$ (since this method is similar to the proof of Theorem 4.10 below, we omit details).

In case that c is positive, little is known about the existence of a positive solution of the equation (4.2). But using the variational method, we have some partial results, i.e., if $S(p)$ does not change "too much," then we can solve the equation (4.2) for some k and some c.

We consider the functional

$$J(v)=\frac{1}{2}\left[\int_B|\nabla v|^2dV-\frac{n+1}{4n}\int_B(k-S(p))v^2dV\right]\left[\int_B v^{(2-4/n+1)}dV\right]^{-2/(2-4/n+1)}$$
$$=\frac{1}{2}\left[\int_B|\nabla v|^2dV-\frac{n+1}{4n}\int_B(k-S(p))v^2dV\right]$$

on the set $D=\left\{v\in H_{1,2}(B)\,\middle|\, v\geq 0,\int_B v^{(2-4/n+1)}dV=1\right\}$. It is well known that if $\int_B\varphi dV=0$, then $\|\varphi\|\leq\lambda_1^{-1}\|\nabla\varphi\|$, where λ_1 is the first positive eigenvalue of Δ on B.

LEMMA 4.9. If the maximum value of $|k-S(p)|$ is less than $\frac{4n}{n+1}\lambda_1^2$ for $n>2$, where λ_1 is the first positive eigenvalue of Δ on B, then the above given functional has an infinimum on the set D.

PROOF. Consider $v-\bar{v}$ for all $v\in H_{1,2}(B)$, $v\not\equiv 0, v\geq 0$. Then $\|v-\bar{v}\|\leq\lambda_1^{-1}\|\nabla v\|$. Since $\|v-\bar{v}\|^2=\|v\|^2-\frac{1}{\mathrm{vol}(B)}(\int_B vdV)^2$, on the set D,

$$\begin{aligned}J(v)&=\frac{1}{2}\left[\int_B|\nabla v|^2dV-\frac{n+1}{4n}\int_B(k-S(p))v^2dV\right]\\
&\geq\lambda_1^2\|v\|^2-\frac{\lambda_1^2}{\mathrm{vol}(B)}\left(\int_B vdV\right)^2-\frac{n+1}{4n}M_1\|v\|^2\\
&(\leftarrow M_1=max_{p\in B}|k-S(p)|)\\
&=\left(\lambda_1^2-\frac{n+1}{4n}M_1\right)\|v\|^2-\frac{\lambda_1^2}{\mathrm{vol}(B)}\|v\|_1^2,\\
&\qquad\left(\leftarrow\int_B vdV=\|v\|_1\quad since\ \ v\geq 0\right)\\
&\geq-\frac{\lambda_1^2}{\mathrm{vol}(B)}\|v\|_1^2.\end{aligned}$$

Since B is compact, $\|v\|_1^2\leq C_1\|v\|^2_{(2-4/n+1)}=C_1$ for some constant C_1. Hence

$$J(\upsilon) \geq -\frac{\lambda_1^2}{\mathrm{vol}(B)} C_1.$$

Let $\bar{c} = \inf_\upsilon J(\upsilon)$. Even if the maximum and minimum values of $S(p)$ are close together, $\bar{c}$ may admit positive, zero, or negative values depending on the choice of k. We now consider the case $\bar{c} < 0$ (i.e., c is positive since $-\frac{n+1}{4n} c = 2\bar{c}$, cf., the proof of the following Theorem 4.10) because the cases $\bar{c} \geq 0$ (i.e., c is negative or zero) have already been treated.

THEOREM 4.10. If $S(p)$ is of small variation in the sense of Lemma 4.9 and $\inf_\upsilon J(\upsilon) < 0$, then Problem II admits a solution for such k and all $c > 0$.

PROOF. By Lemma 4.9, there exists a minimizing sequence $\{\upsilon_i\}$ in $D \subset H_{1,2}(B)$ such that $J(\upsilon_i) \to \bar{c}$. For the proof, we follow several steps.

Step 1. $\{\upsilon_i\}$ is bounded in $H_{1,2}(B)$.

Since $J(\upsilon_i) \to \bar{c}$, we may assume that for all i,

$$J(\upsilon_i) = \frac{1}{2}\left[\int_B |\nabla \upsilon_i|^2 dV - \frac{n+1}{4n}\int_B (k - S(p))\upsilon_i^2 dV\right] \leq C_2$$

for some positive C_2. Since $\|\nabla \upsilon_i\|_2^2 \geq \lambda_1^2 \|\upsilon_i\|_2^2 - \frac{\lambda_1^2}{\mathrm{vol}(B)}\|\upsilon_i\|_1^2$,

$$\left(\lambda_1^2 - \frac{n+1}{4n} M_1\right)\|\upsilon_i\|_2^2 - \frac{\lambda_1^2}{\mathrm{vol}(B)}\|\upsilon_i\|_1^2 \leq C_2$$

and

$$\left(\lambda_1^2 - \frac{n+1}{4n} M_1\right)\|\upsilon_i\|_2^2 \leq C_2 + \frac{\lambda_1^2}{\mathrm{vol}(B)}\|\upsilon_i\|_1^2 \leq C_1 + C_2.$$

Hence $\|\upsilon_i\|^2 \leq C_3$ for some positive C_3 since $\lambda_1^2 - \frac{n+1}{4n} M_1 > 0$.

Now since $J(\upsilon_i) = \frac{1}{2}\left[\int_B |\nabla \upsilon_i|^2 dV - \frac{n+1}{4n}\int_B (k - S(p))\upsilon_i^2 dV\right] \leq C_2$.

$$\begin{aligned}\|\nabla \upsilon_i\|_2^2 &\leq \frac{n+1}{4n}\int_B (k - S(p))\upsilon_i^2 dV + 2C_2 \\ &\leq \frac{n+1}{4n} M_1 \|\upsilon_i\|_2^2 + 2C_2 \\ &\leq \frac{n+1}{4n} M_1 C_3 + 2C_2,\end{aligned}$$

which implies that $\{\upsilon_i\}$ is bounded in $H_{1,2}(B)$.

Step 2. By Kondrokov's theorem for compact manifolds (cf. [Au]), the imbedding $H_{1,2}(B) \to L_2(B)$ is compact. A bounded closed set in $H_{1,2}(B)$ is weakly compact, so there exist a subsequence $\{v_j\}$ of $\{v_i\}$ and a function $v_0 \in H_{1,2}(B)$ such that

i) $v_j \to v_0$ strongly in $L_2(B)$, so $v_j \to v_0$ strongly in $L_{(2-4/n+1)}(B)$,

ii) $v_j \to v_0$ weakly in $H_{1,2}(B)$,

iii) $v_j \to v_0$ almost everywhere pointwise.

By i) and iii), $\int_B v_0^{(2-4/n+1)} dV = 1$ and $v_0 \geq 0$. And by ii), $\|v_0\|_{1,2} \leq \liminf_{j\to\infty} \|v_j\|_{1,2}$. Since $v_j \to v_0$ strongly in $L_2(B)$, we can see that $J(v_0) \leq \bar{c}$.. The minimum property implies that $J(v_0) = \bar{c}$.

Step 3. Set $\varphi = v_0 + t\psi$, where t is a small real number and $\psi \in H_{1,2}(B)$. An asymptotic expansion gives

$$\begin{aligned} J(\varphi) &= \Big[\frac{1}{2}\Big\{\int_B |\nabla v_0|^2 dV - \frac{n+1}{4n}\int_B (k - S(p))v_0^2 dV\Big\} + t\Big\{\int_B \nabla v_0 \nabla \psi dV \\ &\quad - \frac{n+1}{4n}\int_B (k - S(p))v_0 \psi dV\Big\} + 0(t^2)\Big]\Big[\Big\{\int_B v_0^{(2-4/n+1)} dV\Big\}^{-2/(2-4/n+1)} \\ &\quad -2t\int_B v_0^{1-(4/n+1)} \psi dV + 0(t^2)\Big] \\ &= \bar{c} + t\Big[\int_B \nabla v_0 \nabla \psi dV - \frac{n+1}{4n}\int_B (k - S(p))v_0 \psi dV \\ &\quad -2\bar{c}\int_B v_0^{1-(4/n+1)} \psi dV\Big] + 0(t^2). \end{aligned}$$

From $\frac{dJ(\varphi)}{dt}|_{t=0} = 0$, v_0 satisfies that for all $\psi \in H_{1,2}(B)$,

$$\int_B \nabla v_0 \nabla \psi dV - \frac{n+1}{4n}\int_B (k - S(p))v_0 \psi dV - 2\bar{c}\int_B v_0^{1-(4/n+1)} \psi dV = 0.$$

Thus we have

$$\Delta v_0 + \frac{n+1}{4n}(k - S(p))v_0 + 2\bar{c} v_0^{1-(4/n+1)} = 0,$$

where $2\bar{c} = -\frac{n+1}{4n}c$. By the elliptic regularity theory, v_0 is a C^2 solution. And then by the maximum principle for C^2 solutions, v_0 can not attain zero, cf. Proposition 3.75 in [Au] since $\Delta v_0 \geq \frac{n+1}{4n}(S(p) - k)v_0$. Hence $v_0 > 0$, which is our desired smooth solution.

REMARK 4.11. From Theorem 4.7, in case c is negative, there is a discriminant value k_0 for which $k < k_0$ is necessary in order to solve equation (4.2). However, in case c is nonnegative, we do not know if there exists such a discriminant value for k.

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