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A CHARACTERIZATION OF COMPACTA WHICH ADMIT ACYCLIC UV^{n-1}-RESOLUTIONS

Dedicated to Professor A. Okuyama on his sixtieth birthday

By

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Abstract. Let R be a commutative ring with identity. The main result of the paper is the following:

THEOREM. *Let $f: Z \to X$ be a UV^{n-1}-mapping from a compactum Z of dimension $\leq n$ onto a compactum X. If $H^n(f^{-1}(x); R) = 0$ for all $x \in X$, then $a-dim_R X \leq n$.*

As its consequence, we have a characterization of compacta X of $a-dim_R X \leq n$.

THEOREM. *A compactum X admits a UV^{n-1}-mapping $f: Z \to X$ from a compactum Z of dimension $\leq n$ onto X such that $H^n(f^{-1}(x); R) = 0$ for all $x \in X$ if and only if $a-dim_R X \leq n$.*

1. Introduction

This paper is to devoted to investigation of compacta that admit acyclic UV^{n-1}-resolutions of compacta of dimension $\leq n$.

Let X be a compactum and G be an abelian group. The notation "$c-\dim_G X \leq n$" means that every mapping $f: A \to K(G,n)$ of a closed subset A of X to an Eilenberg-MacLane space $K(G,n)$ can be extended over X and should be read as *cohomological dimension of X with respect to G is at most n*. The existence and the construction of resolutions have played an important role in cohomological dimension theory. The first, a beautiful result, was given by Edwards and Walsh [Wa]:

EDWARDS-WALSH THEOREM. *A compactum X is of $c-\dim_{\mathbf{Z}} X \leq n$ if and only if there exists a compactum Z of dimension $\leq n$ and a cell-like mapping $f: Z \to X$.*

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A similar characterization of cohomological dimension modulo p was given by Dranishnikov [Dr_1]:

DRANISHNIKOV THEOREM. *A compactum X is of $c-\dim_{Z_p} \leq n$ if and only if there exists a compactum Z of dimension $\leq n$ and a UV^{n-1}-mapping $f: Z \to X$ such that $H^n(f^{-1}(x);Z_p)=0$ for all $x \in X$.*

Both results have been extended to non-compact metrizable spaces and non-metrizable compact spaces. See Rubin-Shapiro [R-S], Mardešić-Rubin [M-R] for the integral case, and Koyama-Yokoi [K-Y] for the modulo p case. However, there is no similar characterization of rational cohomological dimension (see [K-Y]). Thereby the author and Yokoi [K-Y] introduced the notation "approximable dimension" as a generalization of cohomological dimension.

DEFINITION. Let G be an abelian group, n be a natural number and ε be a positive number. A mapping $\psi: Q \to P$ between compact polyhedra is (G,n,ε)-*approximable* if there exists a triangulation T of P such that for any triangulation M of Q there is a mapping $\psi': |M^{(n)}| \to |T^{(n)}|$ satisfying the following conditions:
(i) $d(\psi', \psi|_{|M^{(n)}|}) \leq \varepsilon$,
(ii) for any mapping $\alpha: |T^{(n)}| \to K(G,n)$, there exists an extension $\beta: Q \to K(G,n)$ of $\alpha \circ \psi'$.

DEFINITION. A compactum X has *approximable dimension with respect to a coefficient group G of at most n (abbreviated, $a-dim_G X \leq n$)* provided that for every mapping $f: X \to P$ of X to a compact polyhedron P and a positive number ε, there exists a compact polyhedron Q and mapping $\varphi: X \to Q, \psi: Q \to P$ such that
(i) $d(f, \psi \circ \varphi) \leq \varepsilon$,
(ii) ψ is (G,n,ε)-approximable.

An advantage of approximable dimension is the following resolution theorem [K-Y]:

THEOREM A. *Let G be an abelian group. If a compactum X is of $a-\dim_G X \leq n$, then there exists a compactum Z of dimension $\leq n$ and a UV^{n-1}mapping $f: Z \to X$ such that $H^n(f^{-1}(x);G)=0$ for all $x \in X$.*

In this paper we will show that $a-\dim_G X$ is the necessary condition for the existence of acyclic UV^{n-1}-resolutions if G is a commutative ring with identity.

Combining with Theorem A, we give a characterization of compacta which admit acyclic UV^{n-1}-resolutions from n-dimensional compacta. Commutative rings with identity are, for example, $\boldsymbol{Z}, \boldsymbol{Z}_p, \mathrm{Q}, \boldsymbol{Z}_{(p)}, \boldsymbol{R}, \ldots$ *etc*. Thus, our characterization may cover rather wider class of coefficient groups.

The position of approximable dimension is, in fact, between cohomological dimension and the covering dimension. We list the fundamental properties of approximable dimension [K-Y]:

PROPOSITION. *For a compactum X and an abelian group G, we have the following inequalities:*

$$c-\dim_G X \leq a-\dim_G X \leq a-\dim_{\mathbf{Z}} X \leq \dim X.$$

On the other hand, if $G = \boldsymbol{Z}$ *or* $\boldsymbol{Z}_p$, *then* $a-\dim_G X = c-\dim_G X$.

Therefore our characterization may be considered as an extension of Edwards-Walsh and Dranishnikov Theorems. We note that any compactum is a UV^{n-1}-image of a n-dimensional compactum (see [Dr_1]). Hence UV^{n-1}-mappings of compacta of dimension $\leq n$ do not have a part in dimension theory unless other conditions are stated.

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2. The results

THEOREM B. Suppose R is a commutative ring with identity. If a compactum X admits a UV^{n-1}-mapping $f: Z \to X$ from a n-dimensional compactum Z onto X such that $H^n(f^{-1}(x); R) = 0$ for all $x \in X$, then $a-\dim_R X \leq n$.

PROOF. Assume that Z is a closed subset of $\boldsymbol{R}^N$ for sufficiently large N. Let us take a mapping $g: X \to P$ of X to a compact polyhedron P and a positive number $\varepsilon > 0$. We choose a triangulation T of P such that mesh $[T] < \varepsilon$. Then

$$[\,|T^{(n)}|, K(R,n)] = H^n(|T^{(n)}|) \otimes R \cong \underbrace{R \oplus \ldots \oplus R}_{s} \oplus R/m_1R \oplus \ldots \oplus R/m_tR.$$

Thus, $[\,|T^{(n)}|, K(R,n)]$ is the direct sum of $s+t$ commutative rings with identity. For each $i = 1, \ldots s+t$, let $\alpha_i : |T^{(n)}| \to K(R,n)$ be a mapping that represents the identity of the i-th direct summand of the above formula.

Consider the mapping $g \circ f: Z \to P$. Since dim $X = n$, there exists a mapping $h: Z \to |T^{(n)}|$ such that $d(h, g \circ f) < \varepsilon$. Moreover, h can be extended over some neighborhood of Z in $\boldsymbol{R}^N$ keeping the distance condition. Thus, we may assume that (by the same symbol) $h: O \to |T^{(n)}|$ is a mapping of a neighborhood O of Z in $\boldsymbol{R}^N$ to $|T^{(n)}|$ such that

(1) $d(h|_z, g \circ f) < \varepsilon$.

For an arbitrary point $x \in X$, because $0 = H^n(f^{-1}(x); R) = [f^{-1}(x), K(R,n)], \alpha_i \circ h|_{f\text{-}1(x)} \simeq 0$ for all $i = 1, \cdots s+t$. Hence we can have an open neighborhood 0_x of $f^{-1}(x)$ in O such that

(2) $diam[h(O_x)] \leq \varepsilon$,

(3) $\alpha_i \circ h|_{O_x} \simeq 0$ for all $i = 1, \cdots s+t$.

Let consider the open collection $\{O_x \mid x \in X\}$ in O which covers Z. Then, because f is UV^{n-1}, we can find an open collection $\mathcal{U}$ in O which covers Z and satisfies the following conditions:

(4) $f^{-1}(f(U \cap Z)) = U \cap Z$ for each $U \in \mathcal{U}$,

(5) for a simplicial pair (K, L) such that $L \supset K^{(0)}$ and a mapping $\zeta: |L| \to O$ such that for each simplex s of K, $\zeta(|s \cap L|) \subset U_s$ for some $U_s \in \mathcal{U}$, there exists an extension $\bar{\zeta}: |K^{(n)} \cup L| \to O$ of ζ such that for each simplex s of K, $\zeta: (|s \cap K^{(n)}|) \subset O_{x(s)}$ for some $x \in X$.

We call the collection $\mathcal{U}$ a n-*refinement* of $\{O_x \mid x \in X\}$ (see [Dy] §8).Next, by (4) we take a finite open cover $\mathcal{V}$ of X such that

(6) $f^{-1}(\mathcal{V}) \leq^* \mathcal{U}$, where $\leq^*$ means a star-refinement,

(7) $\mathcal{V} \leq^* g^{-1}(\{st(v,T) \mid v \in T^{(0)}\})$.

Then there exists a subpolyhedron N of the nerve $N(\mathcal{V})$ of $\mathcal{V}$ and a surjective mapping $\varphi: X \to N$ such that

(8) $\varphi^{-1}(st(V,N)) \subset V$ for each vertex $V \in N$.

Here N is the subcomplex of $N(\mathcal{V})$. Thereby each vertex V of N can be considered as a member of the covering $\mathcal{V}$. For each vertex V of N, by (7) and (8), there exists a vertex $\psi(V) \in T^{(0)}$ such that

(9) $\varphi^{-1}(st(V,N)) \subset g^{-1}(st(\psi(V),T))$.

Clearly, (9) defines the mapping $\psi: N \to T$, and we can easily see that

(10) $d(\psi \circ \varphi, g) \leq 2\varepsilon$.

Next, we will show that ψ is $(R,n,3\varepsilon)$-approximable. In the definition of (G,n,ε)-approximability we considered an arbitrary triangulation of Q. However, it is sufficient to consider an arbitrary subdivision K of N. For each vertex w of K,

(11) $w \in st(V(w),N)$ for some vertex $V(w) \in N^{(0)}$.

Then we choose an arbitrary point $\mu(w) \in Z$ such that

(12) $\mu(w) \in f^{-1}(V(w)) \subset Z$.

For any simplex $s = \langle w_0, w_1, \ldots w_k \rangle$ of K, the vertices $V(w_0), V(w_1), \ldots V(w_k)$ span a simplex of N. Hence

(13) $\cap_{i=0}^{k} V(w_i) \neq \phi$.

Hence (13), (12) and (6) imply that

(14) $\mu(w_0), \mu(w_1), \ldots, \mu(w_k) \in \cup_{i=0}^{k} f^{-1}(V(w_i)) \subset st(f^{-1}(V(w_0)), f^{-1}(\mathcal{V})) \subset U_s$ for some $U_s \in \boldsymbol{\mathcal{U}}$.

Then the condition (5) implies that there exists a mapping $\xi : |K^{(n)}| \to O$ such that

(15) for each simplex s of K, $\xi(|s \cap K^{(n)}|) \subset O_{x(s)}$ for some $x(s) \in X$.

Here we note that, following the above notation, $V_s \subset O_{x(s)}$ for each simplex s of K.

Now we will show that the composition $h \circ \xi : |K^{(n)}| \to |T^{(n)}|$ satisfies the required conditions.

CLAIM 1. $d(h \circ \xi, \psi|_{|K^{(n)}|}) \leq 3\varepsilon$.

PROOF OF CLAIM 1. For an arbitrary point $z \in |K^{(n)}|$, let take the carrier $s = \langle w_0, w_1, \ldots, w_k \rangle, k \leq n,$ of z. Then

$$\varphi^{-1}(z) \subset \bigcap_{i=0}^{k} V(w_i) \subset g^{-1}(st(\psi(V(w_0)), T)),$$

$$\psi(z) \in st(\psi(V(w_0)), T).$$

Hence we have that for any point $u \in \varphi^{-1}(z)$,

$$d(\psi(z), g(u)) \leq 2\varepsilon.$$

On the other hand, by (14) and the construction,

$$\mu(w_0), \mu(w_1), \ldots, \mu(w_k) \in \bigcup_{i=o}^{k} f^{-1}(V(w_i)) \subset st(f^{-1}(V(w_0), f^{-1}(\mathcal{V}))) \subset U_s,$$

$$\xi(z) \in \xi(\langle w_0, w_1, \ldots, w_k \rangle) \subset O_{x(s)},$$

and $U_s \subset O_{x(s)}$. Hence, because $f^{-1}(\varphi^{-1}(z)) \subset U_s \cap X$, the conditions (1) and (2) yield the inequality:

$$d(h \circ \xi(z), g(u)) \leq d(h \circ \xi(z), h(\tilde{u})) \leq diam[h(O_{x(s)})] \leq \varepsilon,$$

where $\tilde{u} \in f^{-1}(u) \subset Z$. It follows that

$$d(h \circ \xi(z), \psi(z)) \leq d(h \circ \xi(z), g(u)) + d(g(u), \psi(z)) \leq 3\varepsilon,$$

where u is an arbitrary point of $\varphi^{-1}(z)$. Thus, we have Claim 1.

CLAIM 2. *For any mapping* $\alpha : |T^{(n)}| \to K(R,n)$,*there exists an extension* $\beta : N = |K| \to K(R,n)$ *of* $\alpha \circ h \circ \xi$.

PROOF OF CLAIM 2. First, we note that $\alpha \circ h|_{O_x} \simeq 0$ for all $x \in X$. For, the homotopy class $[\alpha]$ can be represented as a linear combination:

$$[\alpha] = r_1[\alpha_1] + \cdots + r_s[\alpha_s] + (r_{s+1} + m_1 R)[\alpha_{s+1}] + \cdots + (r_{s+t} + m_t R)[\alpha_{s+t}],$$

where $r_1, \cdots, r_{s+t} \in R$ and $r_{s+i} + m_i R, i = 1, \cdots, t$, is the coset of r_{s+i} modulo $m_i R$. For each $r \in R$, the homomorphism $\theta_r : R \to R$ given by $\theta_r(x) = rx$ induces a mapping $\bar{\theta}_r : K(R,n) \to K(R,n)$ such that $\bar{\theta}_{r*} = \theta_r : \pi_n(K(R,n)) \to \pi_n(K(R,n))$. Then

$$[\bar{\theta}_r \circ \alpha_i] = \begin{cases} r[\alpha_i] & \text{for } i = 1, \ldots, s \\ (r + m_{i-s}R)[\alpha_i] & \text{for } i = s+1, \ldots, s+t. \end{cases}$$

Hence the condition (3) implies that $\alpha \circ h|_{O_x} \simeq 0$ for all $x \in X$.

Now it suffices to show that the mapping $\alpha \circ h \circ \xi$ can be extended over the $(n+1)$-skeleton $|K^{(n+1)}|$. For each $(n+1)$-simplex s of K, by (5), $\xi(|\partial s|) \subset O_{x(s)}$ for some $x(s) \in X$. Hence by the above discussion, $\alpha \circ h \circ \xi|_{|\partial s|} \simeq 0$. Thereby there exists an extension $\beta_s : |s| \to K(R,n)$ of $\alpha \circ h \circ \xi|_{|\partial s|}$. Then let us define the mapping $\beta : |K^{(n+1)}| \to K(R,n)$ by

$$\beta|_{|K^{(n)}|} = \alpha \circ h \circ \xi$$

$$\beta|_{|s|} = \beta_s \text{ for each} (n+1)-\text{simplex } s \textit{ of } K.$$

Thus, we have a desired extension and thereby Claim 2.

By Claims 1 and 2, ψ is $(R, n, 3\varepsilon)$-approximable. It follows that $a-\dim_R X \le n$.

By Theorems A and B, we have the characterization of $a-\dim_R X \le n$ in terms of acyclic UV^{n-1}-resolutions. The Proposition says that it may be considered as an extension of both Edwards-Walsh and Dranishnikov theorems.

THEOREM C. *Let R be a commutative ring with identity. Then a compactum X is of* $a-\dim_R X \le n$ *if and only if there exists a compactum Z of dimension* $\le n$ *and a* UV^{n-1}*-mapping* $f : Z \to X$ *such that* $H^n(f^{-1}(x); R) = 0$ *for all* $x \in X$.

THEOREM D. *Let G be a finitely generated abelian group. Then for every compactum X we have the equality* $a-\dim_G X = c-\dim_G X$.

Proof. By the Proposition it suffices to show the inequality $a-\dim_G X \le c-\dim_G X$. In the case $G/TorG \ne 0, c-\dim_G X = c-\dim_Z X$. Hence the Proposition induces the inequality.

Thus, we may assume that $G = TorG$. Then by Dranishnikov Theorem (see [Dr3] for more detail), there exists a compactum Z of dimension $\leq n = c - \dim_G X$ and a UV^{n-1}-mapping $f : Z \to X$ such that $H^n(f^{-1}(x); G) = 0$ for all $x \in X$. Thereby Theorem B implies the inequality $a - \dim_G X \leq n = c - \dim_G X$.

REMARK. It may be possible that Theorem D is an easy consequence of Proposition. However, we do not know whether the equality $a - \dim_{G \oplus H} X = \max\{a - \dim_G X, a - \dim_H X\}$ holds. If it does, it easily implies Theorem D.

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