

A NOTE ON THE TITS SYSTEMS OF KAC-MOODY STEINBERG GROUPS

By

Richard MARCUSON

Abstract. Let $G(H)$ be the Chevalley (Steinberg) Kac-Moody group of the Kac-Moody Lie algebra L . If σ is the canonical homomorphism of H onto G , and $\{B_G, N_G\}$ is the Tits system in G , then $\{\sigma^{-1}(B_G), \sigma^{-1}(N_G)\}$ is a Tits system in the Weyl-simple subgroup of H .

More than twenty years have passed since the study of Kac-Moody groups was begun by Moody and Teo [M–T] and Marcuson [Mar]. During this period, interest in the subject has swelled ($[A–M], [Mat], [Mor_1], [Mor_2], [Mor_3], [M–R]$ and $[T_2]$) along with that for the general theory of Kac-Moody Lie algebras [K]. One focus of this research has been to describe the Tits systems, or B - N pairs, within the groups.

For both the Chevalley (adjoint) and Steinberg (nonadjoint) Kac-Moody groups, the methods for constructing Tits systems have closely resembled those used in the classical finite dimensional case. We note here that, for the Steinberg Kac-Moody groups, it is possible to construct a second Tits system within a naturally arising subgroup by elementary means.

For ϕ a field of characteristic 0, let L be a Kac-Moody ϕ -Lie algebra with Weyl group W . Let Δ, Π , and P denote the set of roots, the set of simple roots, and the set of positive roots respectively. We have $\Delta W = \Delta, \Pi W \subseteq \Delta$, and $P W \subseteq \Delta$. If $\alpha \in \Pi W$, we say that α is Weyl-simple [Mar]. Denote by L_α the root space corresponding to the root α .

Let G be the Kac-Moody Chevalley group of L , i.e. the group generated by all $\exp(ad te_\alpha)$,

$$G = \langle \exp(ad te_\alpha) : \alpha \in \Pi W, e_\alpha \in L_\alpha, t \in \phi \rangle.$$

Assume that we have a representation of L on the e -extreme module \mathfrak{M} with dominant highest weight. In particular, e_α acts on \mathfrak{M} . The Kac-Moody Steinberg group of L associated with \mathfrak{M} is then

$$H = \langle \exp(te_\alpha) : \alpha \in PW, e_\alpha \in L_\alpha, t \in \phi \rangle.$$

Let σ be given by $\exp(te_\alpha) \rightarrow \exp(ad te_\alpha)$.

Let H_Π be the Weyl-simple subgroup of H , i.e. the subgroup generated by all $\exp(te_\alpha)$ where α is Weyl-simple. Define B'_H to be $B_H \cap H_\Pi$ and N'_H to be N_H .

THEOREM 1. σ is a canonical group map of H_Π onto G .

PROOF. See Marcuson [Mar].

The notion of a Tits system, or $B-N$ pair [T₁], plays an important role in the simplicity proofs for the classical Chevalley groups [C].

DEFINITION. A group G , subgroups B and N , and a subset S of $N/(B \cap N)$ is a Tits system if

- i. $\langle B \cup N \rangle = G$,
- ii. $B \cap N$ is normal in G ,
- iii. S is a set of involutions which generate $W \equiv N/(B \cap N)$,
- iv. for all $s \in S$ and all $w \in W$, $wBs \subseteq BwB \cup BwsB$, and
- v. for all $s \in S$ $sBs \not\subseteq B$.

In the sequel, we will say that $\{B, N\}$ forms a Tits system when the nature of G and S are clear.

In 1972, Moody and Teo showed by construction that a Tits system $\{B_G, N_G\}$ exists in G . Soon afterward, Marcuson (1975) constructed the Tits system $\{B_H, N_H\}$ generalizing that of Steinberg [S] for the representation of L on \mathfrak{M} . In 1983, Peterson and Kac [P-K] studied the theory of $B-N$ pairs $\{B_{PK}, N_{PK}\}$ in Kac-Moody groups for general integral representations. It follows from their work that $\{B_G, N_G\}$ coincides with $\{B_{PK}, N_{PK}\}$ for adjoint representations and that $\{B'_H, N'_H\}$ coincides with $\{B_{PK}, N_{PK}\}$ for highest weight

representations. To see this, note first that $B_{PK} \subseteq B_G$ and $B_{PK} \subseteq B'_H$. Then $B_{PK} = B_G$, and $B_{PK} = B'_H$. Now, because $B'_H \cap N'_H$ contains the kernel of $\sigma [P-K]$, it follows that $\{\sigma^{-1}(B_G), \sigma^{-1}(N_G)\}$ coincides with $\{B_{PK}, N_{PK}\}$.

In the finite dimensional case, $H_\Pi = H$, and the Tits system in H is the inverse image under σ of the Tits system in G . It is therefore natural to ask whether B_{PK} and N_{PK} form a Tits system simply as the inverse images of B_G and N_G under σ .

THEOREM 2. $B''_H \equiv \sigma^{-1}(B_G)$ and $N''_H \equiv \sigma^{-1}(N_G)$ form a Tits system in H_Π

PROOF. Let $K_H = \text{kernel } \sigma$.

$$\text{i. } H_\Pi = \langle B''_H \cup N''_H \rangle.$$

Let $h \in H_\Pi$, and write $\sigma(h) = g_1 \cdots g_n$ where $g_i \in B_G \cup N_G$ for $i = 1, \dots, n$. Let $h_i \in H_\Pi$ be such that $\sigma(h_i) = g_i$. Then $h_0 = h_1 \cdots h_n \in \langle B''_H \cup N''_H \rangle$, and $h_0^{-1}h = k \in K_H \subseteq B''_H \cup N''_H$. Hence $h = h_0k \in \langle B''_H \cup N''_H \rangle$ so that $H_\Pi \subseteq \langle B''_H \cup N''_H \rangle$. Thus $H_\Pi = \langle B''_H \cup N''_H \rangle$.

$$\text{ii. } B''_H \cap N''_H \text{ is normal in } N''_H.$$

This follows from $B''_H \cap N''_H = \sigma^{-1}(B_G \cap N_G)$.

$$\text{iii. } N''_H / (B''_H \cap N''_H) = W \text{ is generated by a set of involutions.}$$

This is a consequence of the fact that σ induces an isomorphism of $N''_H / (B''_H \cap N''_H)$ onto $N_G / (B_G \cap N_G)$.

$$\text{iv. For all } s \in S, \text{ and all } w \in W,$$

$$wB''_Hs \subseteq B''_HsB''_H \cup B''_HwsB''_H.$$

Let w'_i and n'' be representatives of s and w respectively in $N''_H / (B''_H \cap N''_H)$, and let $b'' \in B''_H$. Assume that σ maps w'_i, n'' , and b'' to w_i, n and b respectively. Thus

$$\sigma : n''b''w'_i \rightarrow nbw_i \in wB_Gs \subseteq B_GsB_G \cup B_GwsB_G.$$

Hence $n''b''w'_i \in \sigma^{-1}(B_GsB_G \cup B_GwsB_G)$

$$= \sigma^{-1}(\mathbf{B}_G s \mathbf{B}_G) \cup \sigma^{-1}(\mathbf{B}_G w s \mathbf{B}_G).$$

To finish, we need only show that for any $w \in \mathbf{W}$, $\sigma^{-1}(\mathbf{B}_G w \mathbf{B}_G) = \mathbf{B}_H'' w \mathbf{B}_H''$. Now, $\mathbf{B}_H'' w \mathbf{B}_H'' \subseteq \sigma^{-1}(\mathbf{B}_G w \mathbf{B}_G)$ is clear. Let $x'' \in \sigma^{-1}(\mathbf{B}_G w \mathbf{B}_G)$. Then $\sigma(x'') = \text{bnc}$ where n is a representative of w in $N_G / (\mathbf{B}_G \cap N_G)$. Choose b'' and c'' in \mathbf{B}_H'' and n'' a representative of w in $N_H'' / (\mathbf{B}_H'' \cap N_H'')$ such that σ maps $x_0'' \equiv b'' n'' c''$ to bnc . Then $x_0''^{-1} = k \in \mathbf{K}_H \subseteq \mathbf{B}_H''$, and we see that $x'' = k x_0'' \in \mathbf{K}_H \mathbf{B}_H'' w \mathbf{B}_H'' = \mathbf{B}_H'' w \mathbf{B}_H''$. Therefore $\sigma^{-1}(\mathbf{B}_G w \mathbf{B}_G) \subseteq \mathbf{B}_H'' w \mathbf{B}_H''$.

v. For all $s \in \mathbf{S}$, $s \mathbf{B}_H'' s \not\subseteq \mathbf{B}_H''$.

We know $s \mathbf{B}_G s \not\subseteq \mathbf{B}_G$ so we have $w_{i_0} b_0 w_{i_0} \notin \mathbf{B}_G$ for some representative w_{i_0} of s and some $b_0 \in \mathbf{B}_G$. Choose $b'' \in \mathbf{B}_H''$ and $w_i'' \in N_H''$ such that σ takes b'' to b_0 and w_i'' to w_{i_0} . Then

$$\sigma : w_i'' b'' w_i'' \rightarrow w_{i_0} b_0 w_{i_0} \notin \mathbf{B}_G,$$

and so $w_i'' b'' w_i'' \notin \mathbf{B}_H''$.

Acknowledgements. We would like to thank E. Abe, R. V. Moody, V. G. Kac, and the referee for helpful suggestions.

References

- [A-M] E. Abe and J. Morita, Some Tits systems with affine Weyl groups in Chevalley groups over Dedekind domains, *J. Algebra* 115 (1988), 450–465.
- [C] C. Chevalley, Sur certains groupes simple, *Tôhoku Math. J.* 7 (1955), 14–66.
- [K] V. G. Kac, *Infinite dimensional Lie algebras*, Third ed., Cambridge, 1990.
- [Mar] R. Marcuson, Tits' systems in generalized nonadjoint Chevalley groups, *J. Algebra* 34 (1975), 84–96.
- [Mat] O. Mathieu, Construction d'un groupe de Kac-Moody et applications, *Composito Math.* 69(1989), 37–60.
- [M-T] R. V. Moody and K. L. Teo, Tits' systems with crystallographic Weyl groups, *J. Algebra* 21 (1972), 178–190.
- [Mor₁] J. Morita, Tits' systems in Chevalley groups over Laurent polynomial rings, *Tsukuba J. Math.* 3 (1979), 41–51.
- [Mor₂] J. Morita, Coverings of generalized Chevalley groups associated with affine Lie algebras, *Tsukuba J. Math.* 6 (1982), 1–8.
- [Mor₃] J. Morita, Commutator relations in Kac-Moody groups, *Proc. Jpn. Acad. Ser. A* 63 (1987), 21–22.
- [M-R] J. Morita and U. Rehman, A Matsumoto-type theorem for Kac-Moody groups, *Tôhoku Math. J.* 42 (1990), 537–560.
- [P-K] D. H. Peterson and V. G. Kac, Infinite flag varieties and conjugacy theorems, *Proc. Natl. Acad. Sci. USA* 80 (1983), 1778–1782.
- [S] R. Steinberg, *Lectures on Chevalley Groups*, Yale University Lecture Notes, New Haven, CT, 1967.

- [T₁] J. Tits, Théorème de Bruhat et sous-groupes paraboliques, C. R. Acad. Sci. Paris, Série A, 254 (1962), 2910–2912.
- [T₂] J. Tits, Uniqueness and presentation of Kac-Moody groups over fields, J. Algebra 105 (1987), 542–573.

MATHCOMP
3207 Alma Street
Palo Alto, California 94306
USA