TSUKUBA J. MATH.
Vol. 20 No. 1 (1996), 21-32

NEW EXPLICIT EXAMPLES OF COMPLETE SEMI-SYMMETRIC HYPERSURFACES OF HYPERBOLIC TYPE

By

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Abstract. We associate to every holomorphic function of one complex variable a semi-symmetric hypersurface in the four-dimensional Euclidean space. Under simple explicit conditions for this function the resulting Riemannian space is complete and of hyperbolic type (in the terminology of Szabó).

1. Introduction and main theorem

Riemannian manifolds (M,g) whose Riemannian curvature tensor R satisfies the algebraic condition $R_{XY}\cdot R=0$ for all vector fields X and Y on M are called *semi-symmetric spaces*. This condition means that, at each point $p\in M$, the curvature tensor R_p is the same as that of a symmetric space. This space may vary with the point p. Trivial examples are symmetric spaces and all two-dimensional Riemannian manifolds.

A hypersurface $M\subset \boldsymbol{R}^{n+1}$ is said to be semi-symmetric if it is a semi-symmetric space with respect to the induced Riemannian metric. In 1968, K. Nomizu ([K]) proved that a connected and complete semi-symmetric hypersurface of the Euclidean space $\boldsymbol{R}^{n+1}$ whose type number is greater than two in at least one point, is of the form $S^k\times \boldsymbol{R}^{n-k}$ where S^k is a hypersphere in a Euclidean subspace $\boldsymbol{R}^{k+1}$ of $\boldsymbol{R}^{n+1}$ and $\boldsymbol{R}^{n-k}$ is a Euclidean subspace orthogonal to $\boldsymbol{R}^{k+1}$. This result inspired his conjecture that every irreducible, complete semi-symmetric space of dimension greater than or equal to three is locally symmetric. But this conjecture was refuted in 1972 by H. Takagi who constructed a connected and complete hypersurface in $\boldsymbol{R}^4$ which satisfies the above curvature condition, but which is not locally symmetric ([T]). See also [Se], [TrV] for

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1991 Mathematics Subject Classification: 53A07, 53C20, 53C25, 53C35.
Keywords and phrases: semi-symmetric hypersurfaces.
Received January 21, 1994. Revised May 31, 1994

other counterexamples.

A systematic treatment of complete semi-symmetric hypersurfaces in Euclidean spaces was presented by Szabó ([Sz2]). He considered hypersurfaces whose type number is smaller than or equal to two everywhere. Using the Gauss equation, it is easily seen that all these spaces are semi-symmetric. He distinguished three classes: hypersurfaces of trivial type, of parabolic type and of hyperbolic type. He gave a full classification for these three classes and also an explicit construction. Moreover, he showed, using his general construction procedure, that the example by Takagi is of hyperbolic type.

To our knowledge, the example of Takagi has been, up to now, the only explicit example of a complete semi-symmetric hypersurface of hyperbolic type. Nevertheless, the procedure by Szabó can be used to construct a large family of such hypersurfaces of dimension three (which is the only case worth studying, cf. Theorem 3.2). Indeed, we will prove:

MAIN THEOREM. *Let $F(z)$ be a holomorphic function of one complex variable z defined on the whole complex plane and such that $F''(z)\neq 0$ everywhere. Let $P(x,y), Q(x,y)$ be the real functions of the two real variables x and y defined by $F(x+iy)=P(x,y)+iQ(x,y)$. Define the function $f(X,Y,Z)$ by*

$$f(X,Y,Z)=P\Big(\frac{X+YZ}{1+Z^2},\frac{Y-XZ}{1+Z^2}\Big)-ZQ\Big(\frac{X+YZ}{1+Z^2},\frac{Y-XZ}{1+Z^2}\Big). \tag{1.1}$$

Then $W=f(X,Y,Z)$ determines a complete semi-symmetric hypersurface of hyperbolic type in $\boldsymbol{R}^4$.

We remark that each function $F(z)$ from the Main Theorem can be constructed as follows: choose an arbitrary holomorphic function $\phi(z)$ defined on the whole complex plane and take $F(z)$ as a two-step primitive function of exp $(\phi(z))$ ([R, Theorem 13.11]). Furthermore, for an arbitrary holomorphic function $F(z)$ of one complex variable, the hypersurface $W=f(X,Y,Z)$, where f is given by (1.1), still defines a semi-symmetric hypersurface.

For more information and more examples of semi-symmetric spaces we refer to the fundamental papers by Z. I. Szabó ([Sz1], [Sz3]) and to the recent results in [B], [BKV] and [K]. (We mention that the terminology used in the recent papers differs from that introduced by Szabó and that, in fact, new types of semi-symmetric spaces appear if the completeness assumption is dropped.)

The paper is organized as follows. In Section 2 we give the basic definitions concerning complete immersed semi-symmetric hypersurfaces and in Section 3 we briefly review Szabó's construction for complete semi-symmetric hypersurfaces of hyperbolic type. In Section 4 we apply this construction to prove the above theorem.

The author wishes to thank O. Kowalski and L. Vanhecke for their help and for useful discussions.

2. Complete immersed semi-symmetric hypersurfaces

In this section, we review briefly the basic definitions and results about complete immersed semi-symmetric hypersurfaces. For a more detailed treatment and for further references we refer to the article by Szabó ([Sz2]).

Let (M^n, g) be a complete semi-symmetric manifold which is isometrically immersed in Euclidean space $\boldsymbol{R}^{n+1}$. We denote by ∇ (resp. D) the Levi Civita connection of M^n (resp. of $\boldsymbol{R}^{n+1}$). For the curvature tensor R of M^n we use the sign convention $R_{XY}Z = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. Then we have

$$R_{XY}Z = g(A(X), Z)A(Y) - g(A(Y), Z)A(X)$$

where A is the shape operator of the hypersurface M with respect to a local unit normal vector field u to M in $\boldsymbol{R}^{n+1}$, i.e., $A(X) = -D_X u$ for $X \in TM$. The rank of A at a point $p \in M$ is called the *type number* at p and is denoted by $k(p)$. Because of Nomizu's result mentioned in the introduction we suppose that $k(p) \leq 2$ everywhere. If this number is 0 or 1 at a point p, then clearly $R_p = 0$. So, in what follows we consider an (open) subset of M on which the type number is equal to two. On this open set, the tangent space to M at a point $p \in M$ can be decomposed as

$$T_p M = \tilde{V}_p^0 + \tilde{V}_p^1$$

where $\tilde{V}_p^0$ is the nullity vector space of the curvature tensor, i.e., $\tilde{V}_p^0 = \{X \in T_p M | R_{XY}Z = 0 \text{ for all } Y, Z \in T_p M\}$, and $\tilde{V}_p^1$ is its orthogonal complement. $\tilde{V}_p^0$ has dimension $n-2$ and $\tilde{V}_p^1$ has dimension 2. The distributions $\tilde{V}^0$ and $\tilde{V}^1$ defined in this way are differentiable on the open set. Moreover, $\tilde{V}^0$ is integrable and its integral manifolds are totally geodesic and locally Euclidean submanifolds. They are open subsets of (n-2)-dimensional Euclidean subspaces of $\boldsymbol{R}^{n+1}$. The distribution $\tilde{V}^1$ is not integrable in general, but it is parallel along the integral manifolds of $\tilde{V}^0$.

Consider now a local system of smooth unit vector fields $\{m_1, \cdots, m_{n-2}\}$ tangent to $\tilde{V}^0$ which are pairwise orthogonal and such that $\nabla_{m_\alpha} m_\beta = 0$. For $X \in \tilde{V}^1$ we have

$$\nabla_X m_\alpha = \tilde{B}_\alpha(X) + \text{a linear combination of } m_1, \ldots, m_{n+2}$$

where $\tilde{B}_\alpha(X) \in \tilde{V}^1$. We extend the endomorphisms $\tilde{B}_\alpha$ to the whole tangent space by defining $\tilde{B}_\alpha(Y) = 0$ for all $Y \in \tilde{V}^0$. Concerning the fields of endomorphisms $\tilde{B}_\alpha$, Szabó shows that there are two possibilities: either they are all zero along an integral manifold of $\tilde{V}^0$ or there exists a system of unit vector fields $\{\overline{m}_1, \cdots, \overline{m}_{n-2}\}$ tangent to $\tilde{V}^0$ satisfying $\nabla_{\overline{m}_\alpha} \overline{m}_\beta = 0$ such that $\tilde{B}_1 \neq 0$ and $\tilde{B}_2 = \ldots = \tilde{B}_{n-2} = 0$ along this integral manifold. In the latter case, either $\tilde{B}_1$ has only zero eigenvalues or it has two imaginary eigenvalues. (Here the completeness of (M^n, g) plays an essential role.)

We can now give some definitions. Let M^n be a connected complete immersed hypersurface in $\boldsymbol{R}^{n+1}$ such that $k(p) \leq 2$ for every point $p \in M$. Let v_1 be the open set on which $k(p) = 2$. Then in the interior v_0 of $M \setminus v_1$ the Riemann curvature tensor vanishes. Let $v_2 \subseteq v_1$ be the open set where not all $\tilde{B}_\alpha$ are zero. Then all tensors $\tilde{B}_\alpha$ vanish in the interior v_t of $v_1 \setminus v_2$. The open set v_t is called the *pure trivial part* of M. Finally, let $v_h \subseteq v_2$ be the open set where the unique non-zero endomorphism field $\tilde{B}_1$ has two imaginary eigenvalues. v_h is called the *pure hyperbolic part* of M and v_p, the interior of $v_2 \setminus v_h$, the *pure parabolic part* of M. On v_p the endomorphism field $\tilde{B}_1$ has only zero eigenvalues but is non-trivial. It is clear that the open set $v_0 \cup v_t \cup v_p \cup v_h$ is everywhere dense in M. Moreover, the open sets v_t, v_p and v_h always contain the integral manifolds of $\tilde{V}^0$ passing through their points, and these integral manifolds are complete.

We can now define three different types of complete immersed hypersurfaces M with type number smaller than or equal to two (and hence semi-symmetric).

DEFINITION 2.1. A complete immersed hypersurface M^n with type number everywhere smaller than or equal to two is said to be of

1) *trivial type* if $v_2 = \emptyset$, i.e., M^n contains only v_0 and possibly a pure trivial part;
2) *parabolic type* if $v_t = v_h = \emptyset$ and $v_p \neq \emptyset$, i.e., M^n contains only v_0 and a non-empty pure parabolic part;
3) *hyperbolic type* if $M^n = v_h$, i.e., M^n contains only the pure hyperbolic part.

Szabó ([Sz2]) studies these three classes of manifolds in detail and presents general procedures to construct explicit examples of such spaces. Moreover,he gives the full classification of complete semi-symmetric hypersurfaces inthe Euclidean space $\boldsymbol{R}^{n+1}$.

REMARK. It is possible to generalize the notions of trivial, parabolic and hyperbolic type to complete semi-symmetric spaces which are foliated by Euclidean spaces of codimension two and which are not necessarily imbedded as hypersurfaces in some Euclidean space (see [Sz3]). The metrics for the spaces of parabolic type (in this generalized sense) are given *explicitly* in [BKV]. Furthermore, it is proved there that these spaces are characterized by the property that their scalar curvature is constant along each integral manifold of the nullity distribution $\tilde{V}^0$.

3. Szabó's construction for complete semi-symmetric hypersurfaces of hyperbolic type

In this section we focus on complete semi-symmetric hypersurfaces of hyperbolic type, and, in particular, on Szabó's procedure for the construction of such spaces. For the proofs and for more details we refer once again to [Sz2]. First we mention

PROPOSITION 3.1. *The sectional curvature K_σ of a plane section σ in a hypersurface of hyperbolic type is always non-positive. So every complete and simply connected immersed hypersurface M^n of hyperbolic type is diffeomorphic to $\boldsymbol{R}^n$.*

Next, as concerns the construction of hypersurfaces of hyperbolic type, we note that we have to consider only three-dimensional hypersurfaces because of the following theorem.

THEOREM 3.2. *Every connected, simply connected and complete immersed hypersurface M^n of hyperbolic type is of the form $M^n = M^3 \times \boldsymbol{R}^{n-3}$, where M^3 is an immersed hypersurface of hyperbolic type in a Euclidean subspace $\boldsymbol{R}^4$ and $\boldsymbol{R}^{n-3}$ is a Euclidean subspace orthogonal to $\boldsymbol{R}^4$.*

Related to the three-dimensional hypersurfaces of hyperbolic type is the notion of a *plane-uncoverable line-fibration* of a simply connected open set U of $\boldsymbol{R}^3$.

Let U be an open set of $\boldsymbol{R}^3$ which is simply connected. We suppose that a one-fold covering of U with straight lines exists such that the unit vector field u tangent to these lines is differentiable. Such a covering will be called a *line-fibration* of U. For every point $p \in U$, let V_p^1 be the orthogonal complement of u_p

and V_p^0 the one-dimensional subspace of T_pM spanned by u_p. Define the field B of endomorphisms of the distribution V^1 by $B(X) = D_X u$, where D is the usual covariant derivative of $\boldsymbol{R}^3$. Then either $B^2 = 0$ holds along a line of the fibration or B has two imaginary eigenvalues along that line. If $B^2 = 0$ holds for every line of the line-fibration, then there exists a one-parameter family of planes such that the lines of the fibration all lie in some plane of this family and the lines lying in the same plane are parallel. Conversely, if through every line l of a line-fibration there exists a plane H such that H covers parallel lines from the fibration around l, then the equation $B^2 = 0$ holds for the line-fibration. These observations lead to

DEFINITION 3.3. If $B^2 = 0$ holds for every line of a line-fibration of a simply connected open set U of $\boldsymbol{R}^3$, then the line-fibration is called *plane-coverable*. If the endomorphism B has two imaginary eigenvalues along every line of the fibration, then the line-fibration is called *plane-uncoverable*.

Now consider a complete three-dimensional hypersurface M^3 of hyperbolic type in $\boldsymbol{R}^4$. The integral curves of $\tilde{V}^0$ in M^3 are straight lines in $\boldsymbol{R}^4$. Fix such a line $\tilde{l}$ and take a three-dimensional Euclidean subspace S of $\boldsymbol{R}^4$ which is not orthogonal to $\tilde{l}$. The orthogonal projection $\pi : M^3 \to S$ then maps an open neighbourhood $\tilde{U}$ of $\tilde{l}$ diffeomorphically onto an open set U of S such that the projections of the integral curves of $\tilde{V}^0$ form a line-fibration of U. This fibration is called the *projected line-fibration*. Szabó proves

PROPOSITION 3.4. *If M^3 is of hyperbolic type, then the projected line-fibration is plane-uncoverable.*

Conversely, it holds

THEOREM 3.5. *Let $U \subset \boldsymbol{R}^3$ be an open set with a plane-uncoverable line-fibration. Then around every line of the fibration there exists a differentiable function $f(x,y,z)$ such that the points $(x,y,z,f(x,y,z))$ represent a hypersurface of hyperbolic type and the lines of the fibration correspond to the integral curves of the distribution $\tilde{V}^0$ on this hypersurface.*

Starting from a plane-uncoverable line-fibration with unit tangent vector field u on some simply connected open subset U of $\boldsymbol{R}^3$, one must construct the appropriate function $f(x,y,z)$ on U. Szabó deduces the following necessary and sufficient conditions for the function f:

LEMMA 3.6. *The hypersurface (x,y,z,f(x,y,z)) is of hyperbolic type with respect to the line-fibration of U if and only if*

$$D_u df = 0\,, \tag{3.1}$$

$$\operatorname{rank} D^2 f = 2\,. \tag{3.2}$$

Take a plane M^2 in $\boldsymbol{R}^3$ not containing any straight line of the fibration. Let (x,y) be a Cartesian coordinate system of M^2 and let t be such that $\dfrac{\partial}{\partial t} = u$ and M^2 is given by $t = 0$. Then (x,y,t) is a coordinate system on a neighbourhood of M^2 in U. Condition (3.1) implies in particular $u(uf) = 0$, so f must be of the form

$$f(x,y,t) = \rho(x,y)t + \lambda(x,y) \tag{3.3}$$

in this coordinate neighbourhood, where ρ and λ are functions of the variables x and y only.

Expressing the condition (3.1) in terms of the functions ρ and λ, one finds that the partial differential equations

$$a_1^1 \frac{\partial^2 \lambda}{\partial x^2} + 2a_2^1 \frac{\partial^2 \lambda}{\partial x \partial y} + a_2^2 \frac{\partial^2 \lambda}{\partial y^2} = 0\,, \tag{3.4}$$

$$\begin{aligned} \frac{\partial \rho}{\partial x} + (B_1^1 \Phi_1 + B_1^2 \Phi_2)\rho &= B_1^1 \frac{\partial \lambda}{\partial x} + B_1^2 \frac{\partial \lambda}{\partial y}\,, \\ \frac{\partial \rho}{\partial y} + (B_2^1 \Phi_1 + B_2^2 \Phi_2)\rho &= B_2^1 \frac{\partial \lambda}{\partial x} + B_2^2 \frac{\partial \lambda}{\partial y} \end{aligned} \tag{3.5}$$

must be satisfied. Here the B_j^i determine the field of endomorphisms B at the points of M^2 by

$$\begin{aligned} B(\frac{\partial}{\partial x}) &= D_{\frac{\partial}{\partial x}} u_{(x,y,0)} = B_1^1 \frac{\partial}{\partial x} + B_1^2 \frac{\partial}{\partial y} - (B_1^1 \Phi_1 + B_1^2 \Phi_2)u, \\ B(\frac{\partial}{\partial y}) &= D_{\frac{\partial}{\partial y}} u_{(x,y,0)} = B_2^1 \frac{\partial}{\partial x} + B_2^2 \frac{\partial}{\partial y} - (B_2^1 \Phi_1 + B_2^2 \Phi_2)u \end{aligned} \tag{3.6}$$

and Φ_1 and Φ_2 are given by

$$\Phi_1 = \langle \frac{\partial}{\partial x}, u \rangle_{(x,y,0)}\;, \quad \Phi_2 = \langle \frac{\partial}{\partial y}, u \rangle_{(x,y,0)}\,. \tag{3.7}$$

Further, the matrix field (a_j^i) on M^2 is given by

$$(a_j^i) = \begin{pmatrix} -B_2^1 & (B_1^1 - B_2^2)/2 \\ (B_1^1 - B_2^2)/2 & -B_1^2 \end{pmatrix}. \tag{3.8}$$

As the line-fibration is plane-uncoverable, the discriminant Δ of the characteristic equation $\mu^2 - trB\,\mu + \det B = 0$ is negative. Hence $\det(a^i_j) = -B^1_2 B^2_1 - (1/4)(B^1_1 - B^2_2)^2 = -\Delta > 0$, which implies that the matrix field (a^i_j) is definite everywhere.

Finally, the condition (3.2) is equivalent to

$$\text{(3.9)} \qquad \operatorname{rank}\begin{pmatrix} \partial^2\lambda/\partial x^2 & \partial^2\lambda/\partial x\partial y \\ \partial^2\lambda/\partial x\partial y & \partial^2\lambda/\partial y^2 \end{pmatrix} = 2.$$

(Without this condition the resulting hypersurface may admit points p with type number $k(p)$ smaller than two. Then it is of mixed type and $V_0 \cup V_h$ is an everywhere dense open subset of M.)

The conditions (3.4), (3.5) and (3.9) are necessary and sufficient to ensure that the function f given by formula (3.3) in terms of the coordinates (x,y,t) determines a hypersurface of hyperbolic type in $\boldsymbol{R}^4$. The last step then requires expressing f in the standard Euclidean coordinates on $\boldsymbol{R}^3$. Indeed, as u is not orthogonal to M^2 everywhere, the coordinates (x,y,t) are not Cartesian coordinates. We take the standard (Cartesian) coordinates (X,Y,Z) on $\boldsymbol{R}^3$ as follows: the plane $t = 0$ is given by $Z = 0$ and on this plane the coordinates (X,Y) and (x,y) are the same.

Note that the above procedure works in general only locally. Nevertheless, as we will see in the next section, it is possible to obtain *complete* explicit examples using the above correspondence between plane-uncoverable line-fibrations of $\boldsymbol{R}^3$ and hypersurfaces of hyperbolic type in $\boldsymbol{R}^4$.

4. The new examples: proof of the main theorem

In this section we construct the new examples given in the Main Theorem, following Szabó's procedure as explained in the previous section.

The first thing to do is to find a plane-uncoverable line-fibration of $\boldsymbol{R}^3$. For that purpose, we foliate $\boldsymbol{R}^3$ by the one-parameter family of one-sheeted hyperboloids given by

$$K_k : x^2 + y^2 = k^2(1+z^2), \qquad k \in \boldsymbol{R}^+,$$

to which one has to add the z-axis, corresponding to $k = 0$. All surfaces K_k are *ruled* surfaces. We now determine the straight lines through a point $(x_0, y_0, 0)$ where $x_0{}^2 + y_0{}^2 \neq 0$ contained in the hyperboloid K_{k_0} with $k_0 = (x_0{}^2 + y_0{}^2)^{1/2}$. These straight lines are given by the intersection of K_{k_0} with the tangent plane to K_{k_0} at the point $(x_0, y_0, 0)$:

$$x^2 + y^2 = k_0^2(1+z^2),$$
$$x_0 x + y_0 y = k_0^2.$$

Their directions (α, β, γ) satisfy

$$\alpha^2 + \beta^2 = k_0^2\gamma^2,$$
$$x_0\alpha + y_0\beta = 0,$$

and hence they are given by $(-y_0, x_0, 1)$ and $(-y_0, x_0, -1)$. In the case $x_0 = y_0 = 0$, the corresponding pair of lines coincides with the z-axis. We choose the family of straight lines corresponding to the first of the above directions. It is clear that they form a line-fibration of $\boldsymbol{R}^3$. Moreover, this line-fibration is plane-uncoverable. This will follow from Definition 3.3 and the explicit calculation of the tensor field B further on.

Now, we shall define a smooth unit vector field u tangent to the given line-fibration. So, take $(x,y,z) \in \boldsymbol{R}^3$. Then there is a unique point $(x_0, y_0, 0)$ lying on the same line of the fibration. Hence, for some $T \in \boldsymbol{R}$,

$$x = x_0 - y_0 T,$$
$$y = y_0 + x_0 T,$$
$$z = T,$$

or

$$x_0 = (x + yz)/(1+z^2),$$
$$y_0 = (y - xz)/(1+z^2).$$

Then we put

$$\begin{aligned} u_{(x,y,z)} = u_{(x_0,y_0,0)} &= (x_0^2 + y_0^2 + 1)^{-1/2}\left(-y_0\frac{\partial}{\partial x} + x_0\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \\ &= (1+z^2)^{-1/2}(x^2+y^2+z^2+1)^{-1/2} \\ &\quad \left((xz - y)\frac{\partial}{\partial x} + (yz + x)\frac{\partial}{\partial y} + (1+z^2)\frac{\partial}{\partial z}\right). \end{aligned}$$

This unit vector field appears also in [Sz2] and it is shown to belong to Takagi's example.

The next step in Szabó's construction is to take a plane M^2 in $\boldsymbol{R}^3$ with a Cartesian coordinate system such that u is never tangent to M^2 and to calculate the associated tensor field B and the associated functions Φ_1 and Φ_2. In our present situation we can take the plane $z = 0$ as M^2 with (x,y) as Cartesian coordinates. The functions Φ_1, Φ_2 from (3.7) are given on M^2 by

$$\Phi_1 = -y(x^2+y^2+1)^{-1/2}, \qquad \Phi_2 = x(x^2+y^2+1)^{-1/2}.$$

For the tensor field B on M^2 given by (3.6) we note that we have

$$D_{\frac{\partial}{\partial x}} u_{(x,y,0)} = -x(x^2+y^2+1)^{-1} u_{(x,y,0)} + (x^2+y^2+1)^{-1/2} \left.\frac{\partial}{\partial y}\right|_{(x,y,0)},$$

$$D_{\frac{\partial}{\partial y}} u_{(x,y,0)} = -y(x^2+y^2+1)^{-1} u_{(x,y,0)} - (x^2+y^2+1)^{-1/2} \left.\frac{\partial}{\partial y}\right|_{(x,y,0)},$$

and so B is given in matrix form by

$$B = (x^2+y^2+1)^{-1/2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(The formula given in [Sz2] is incorrect.) We see that B has only imaginary eigenvalues which proves that the line-fibration is plane-uncoverable. The matrix field (a^i_j) from (3.8) is then given by

$$(a^i_j) = (x^2+y^2+1)^{-1/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By Szabó's construction, it suffices now to find solutions of the following system of partial differential equations:

$$\frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial y^2} = 0, \tag{4.1}$$

$$\begin{aligned} \frac{\partial \rho}{\partial x} + \frac{x}{x^2+y^2+1}\rho &= \frac{1}{\sqrt{x^2+y^2+1}} \frac{\partial \lambda}{\partial y}, \\ \frac{\partial \rho}{\partial y} + \frac{y}{x^2+y^2+1}\rho &= -\frac{1}{\sqrt{x^2+y^2+1}} \frac{\partial \lambda}{\partial x}. \end{aligned} \tag{4.2}$$

Then the graph of the function $f(x,y,t) = \rho(x,y)t + \lambda(x,y)$, i.e., $W = f(x(X,Y,Z), y(X,Y,Z), t(X,Y,Z))$, is a semi-symmetric hypersurface of the Euclidean space $\boldsymbol{R}^4$ with standard Euclidean coordinates (X,Y,Z,W). The relation between the coordinate system (x,y,t) on $\boldsymbol{R}^3$ and the standard Euclidean coordinates (X,Y,Z) on $\boldsymbol{R}^3$ is given by

$$\begin{aligned} t &= Z(1+Z^2)^{-1/2}(X^2+Y^2+Z^2+1)^{1/2}, \\ x &= (X+YZ)/(1+Z^2), \\ y &= (Y-XZ)/(1+Z^2). \end{aligned} \tag{4.3}$$

If the harmonic function $\lambda(x,y)$ is defined on the whole of $\boldsymbol{R}^2$, then so is the solution $\rho(x,y)=(x^2+y^2+1)^{-1/2}\int_{x_0}^{x}(\partial\lambda/\partial y)dx$ of (4.2) and the associated hypersurface is complete because it is closed in $\boldsymbol{R}^4$. From (3.9) it follows moreover that the pure hyperbolic part of the hypersurface consists of those points for which $\det\begin{pmatrix}\partial^2\lambda/\partial x^2 & \partial^2\lambda/\partial x\partial y\\ \partial^2\lambda/\partial x\partial y & \partial^2\lambda/\partial y^2\end{pmatrix}\neq 0$.

Now, let $F(z)$ be any holomorphic function of one complex variable z defined on the whole complex plane. Put $z=x+iy$ and define two real-valued functions $P(x,y), Q(x,y)$ by $F(x+iy) = P(x,y) + iQ(x,y)$. P and Q satisfy the Cauchy-Riemann conditions

$$\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y},\qquad \frac{\partial P}{\partial y}=-\frac{\partial Q}{\partial x}.$$

In particular, both P and Q satisfy (4.1) in the whole plane $\boldsymbol{R}^2[x,y]$.
If we put $\lambda(x,y)=P(x,y)$ and $\rho(x,y)=-(x^2+y^2+1)^{-1/2}Q(x,y)$, then the system (4.1), (4.2) is obviously satisfied on the whole plane $\boldsymbol{R}^2$, and we can put $f(x,y,t)=P(x,y)-(x^2+y^2+1)^{-1/2}Q(x,y)t$. Using the transition formulas (4.3), we find after some calculation

$$f(X,Y,Z)=P\Big(\frac{X+YZ}{1+Z^2},\frac{Y-XZ}{1+Z^2}\Big)-Z\,Q\Big(\frac{X+YZ}{1+Z^2},\frac{Y-XZ}{1+Z^2}\Big),$$

which is the desired formula (1.1). Here $f(X,Y,Z)$ is defined on the whole of $\boldsymbol{R}^3$, hence the associated hypersurface $W=f(X,Y,Z)$ is complete.

Finally we note that $\det\begin{pmatrix}\partial^2\lambda/\partial x^2 & \partial^2\lambda/\partial x\partial y\\ \partial^2\lambda/\partial x\partial y & \partial^2\lambda/\partial y^2\end{pmatrix}\neq 0$ whenever $F''(x+iy)\neq 0$.
This concludes the proof of the Main Theorem.

REMARK 1. We obtain Takagi's example if we start with the holomorphic function $F(z)=\frac{i}{2}z^2$.

REMARK 2. If we take $\lambda(x,y)=Q(x,y)$, we obtain for the function f the following expression:

$$f(X,Y,Z)=Q\Big(\frac{X+YZ}{1+Z^2},\frac{Y-XZ}{1+Z^2}\Big)+Z\,P\Big(\frac{X+YZ}{1+Z^2},\frac{Y-XZ}{1+Z^2}\Big).$$

This solution corresponds to (1.1) by a change of the holomorphic function $F(z)$ to $-iF(z)$.