ON LIGHT MAPPINGS WITHOUT
PERFECT FIBERS ON COMPACTA

By

Roman POL(*)

Abstract. We give some conditions ensuring that a light mapping $f : X \to Y$ between compacta must be injective on a non-trivial continuum and we discuss related questions concerning the invariance of strong infinite-dimensionality.

1. Introduction.

We shall consider in this paper only separable metrizable spaces, and a compactum is such a compact space. A perfect space is a non-empty compactum without isolated points.

A mapping $f : X \to Y$ between compacta is light if f is continuous and all fibers $f^{-1}(y)$ are zero-dimensional.

Our terminology concerning continua, or dimension follows Kuratowski [7] and Nagata [8].

We shall prove (in sec. 2) the following. results, originally motivated by some questions about strongly infinite-dimensional spaces, discussed in sec. 4.

1.1 THEOREM. *Let $f : X \to Y$ be a light mapping between compacta such that for each zero-dimensional set $A \subset X$ the set of points $x \in X$ with $f^{-1}f(x) \setminus A$ non-perfect contains a compactum of dimension ≥ 2. Then f is injective on a non-trivial continuum in X.*

1.2. COROLLARY. *Let $f : X \to Y$ be a continuous mapping with countable fibers defined on a compactum X of dimension at least 2. Then there is a non-trivial continuum in X on which f is injective.*

Torunczyk [11], [12] obtained deep results related to this topic, assuming Y

is finite-dimensional. In sec. 3 we combine one of the Torunczyk's theorems with Corollary 1.2. We use also in this section a striking curve constructed by H. Cook [3] to illustrate necessary restrictions on the assumptions. However, in case of infinite-dimensional range Y, some natural questions are left open, cf. Remark 3.6.

The main tool in the proof of Theorem 1.1 is a fundamental result of R. H. Bing [2] concerning hereditarily indecomposable continua, i.e. such continua C that, whenever two subcontinua of C meet, one of them is contained in the other, cf. [7].

Since the indecomposability does not enter the statements of the results in this paper, it would be interesting to clarify, to what extent its appearance in this context is accidental.

2. Proof of Theorem

1.1. The proof consists of two steps. In the first one we obtain the assertion assuming in addition that the range of f is a hereditarily indecomposable continuum (we can assume less about f in this situation). Then, in the next step, we pass to a general situation using a theorem of R. H. Bing [2].

We consider X with a fixed metric, and dist (x, S) is the distance of a point x from a set S in X, cf. [7;21.IV].
Recall that, since the fibers of f are zero-dimensional, f maps non-trivial continua to non-trivial continua.

(I) Let $L \subset Y$ be a hereditarily indecomposable continuum, let $K = f^{-1}(L)$ and let $K_{(0)}$ be the union of all one-point components of K. If the set of points x in K for which the closure of the set $f^{-1}f(x) \setminus K_{(0)}$ is not perfect contains a non-trivial continuum H, then f is injective on some non-trivial continuum in K.

PROOF. (I). Aiming at a contradiction, let us assume that f is not injective on any non-trivial continuum in K. We shall find $z \in f(H)$ such that
(1) $f^{-1}f(x) \setminus K_{(0)}$ has no relatively isolated points.
Since $H \cap K_{(0)} = \varnothing$, the set in (1) is nonempty and therefore its closure would be perfect, contradicting the choice of H.

The complement of $K_{(0)}$ is σ-compact [Ku;V.47.VI] and one can choose a sequence $S_1, S_2, \ldots$ of compact sets disjoint from $K_{(0)}$ such that each point in $K \setminus K_{(0)}$ belongs to S_i for infinitely many i and the diameters of S_i converge to 0. Let

$$U_i = \{x \in K : \text{dist}(x, S_i) < 1/i\}.$$

We shall define inductively non-trivial continua

$$f(H) \supset C_1 \supset C_2 \supset \dots$$

such that, for each i,

(2) either $f(S_i) \cap C_i = \varnothing$, or for each $y \in C_i$ the set $f^{-1}(y) \cap (U_i \setminus K_{(0)})$ contains at least two points. To begin with we set $C_0 = f(H)$, with $S_0 = U_0 = \varnothing$, and assume that C_i has been already defined.

If C_i is not contained in $f(S_{i+1})$, let C_{i+1} be any non-trivial continuum in $C_i \setminus f(S_{i+1})$.

Suppose that $C_i \subset f(S_{i+1})$. Then, by our assumption, f can not be injective on the compactum $f^{-1}(C_i) \cap S_{i+1}$, which f maps onto the continuum C_i, and therefore there exist $c \in C_i$ and distinct points a, $b \in S_{i+1}$ with $c = f(a) = f(b)$. Since a, $b \notin K_{(0)}$, one can find disjoint non-trivial continua T_a, T_b contained in U_{i+1} with $a \in T_a, b \in T_b$. The non-trivial subcontinua $f(T_a), f(T_b)$ and C_i of L contain the point c, and since L is hereditarily indecomposable, the intersection

$$C_{i+1} = f(T_a) \cap f(T_b) \cap C_i$$

is a non-trivial continuum.

For each $y \in C_{i+1}$ the fiber $f^{-1}(y)$ intersects both T_a, T_b and since $T_a \cup T_b \subset U_{i+1} \setminus K_{(0)}$ the inductive step is completed.

We shall check that any point $z \in C_1 \cap C_2 \cap \dots \subset f(H)$ satisfies (1). Let $x \in f^{-1}(z) \setminus K_{(0)}$ and let U be any open set in K containing x. One can find i with $x \in S_i$ and $U_i \subset U$. Since $z \in f(S_i) \cap C_i$, the second part of condition (2) guarantees that U_i, and hence also U, intersects $f^{-1}(z) \setminus K_{(0)}$ in at least two points. This demonstrates (1), and ends the proof of (I).

(II). We shall combine now (I) with a theorem of Bing [2] which provides a sequence $B_1, B_2, \dots$ of compacta in Y such that

(3) any pair of disjoint closed sets in Y is separated by some B_i,

(4) each non-trivial continuum in B_i is hereditarily indecomposable.

Let

$$E_1 = f^{-1}(B_1), \quad E_{i+1} = f^{-1}(B_{i+1} \setminus (B_1 \cup \dots \cup B_i)),$$

and let

(5) $G = \bigcup_i G_i$, where $G_i = \{x : \{x\}$ is the component of x in $E_i\}$.

Since the sets $E_1 \cup \dots \cup E_i = f^{-1}(B_1 \cup \dots \cup B_i)$ are compact, the sets G_i, and also their union G, are zero-dimensional. Let us set (where cl stands for the closure)

$$A = G \setminus \bigcup\{cl(f^{-1}f(x) \setminus G) : x \in X\}.$$

Then A is zero-dimensional, and for each $x \in X$,

(6) $$f^{-1}f(x) \setminus A = cl(f^{-1}f(x) \setminus G).$$

From the assumptions about f, applied to A, and property (6), we conclude that there is a compactum Z in X of dimension ≥ 2 such that for each $x \in Z$ the closure $cl(f^{-1}f(x) \setminus G)$ is not perfect.

Condition (3) implies that $Y \setminus \underset{i}{\cup} B_i$ is zero-dimensional, and so is its preimage under the zero-dimensional map f – the set $X \setminus \underset{i}{\cup} E_i$. Therefore $Z \cap \underset{i}{\cup} E_i$ has positive dimension, and since the sets $E_1 \cup ... \cup E_i$ are compact, there exists i such that $Z \cap E_i$ contains a non-trivial continuum H. Notice that $H \cap G_i = \varnothing$ (cf. (5)), and hence

(7) $$H \subset (Z \setminus G) \cap E_i.$$

Let L be the maximal continuum in B_i containing the non-trivial continuum $f(H)$ and let $K = f^{-1}(L)$. Then L is hereditarily indecomposable, by (4). Let $K_{(0)}$ be the set defined in (I). Any continuum in $f^{-1}(B_i)$ intersecting K is contained in K, and since E_i is open in $f^{-1}(B_i)$, it follows that (cf. (5))

(8) $$K_{(0)} \cap E_i = G \cap E_i.$$

For every $x \in H$, by (7), (8) and the choice of Z, the closure of the set $f^{-1}f(x) \setminus K_{(0)}$ is not perfect. Therefore, one can use the statement (I) to conclude that f is injective on some non-trivial continuum in K.

3. A Corollary to Torunczyk's Theorem on Light Mappings with finite-Dimensional Range.

The following result was proved by H. Torunczyk [11], Theorem 1 (cf. also [12]).

3.1. THEOREM (Torunczyk). *Let* $f : X \to Y$ *be a light mapping of a compactum X into a finite-dimensional compactum Y. Then there is a zero-dimensional set A in X such that f restricted to $X \setminus A$ is finite-to-one.*

If X is a compactum of dimension at least 3, then the complement of any zero-dimensional set in X contains a compactum of dimension at least 2. Thus, the theorem of Torunczyk and Corollary 1.2 yield the following conclusion.

3.2. COROLLARY. *Let $f : X \to Y$ be a light mapping of a compactum X of dimension at least 3 into a finite-dimensional compactum Y. Then f is injective on a non-trivial continuum in X.*

3.3. REMARK. Torunczyk [11], Corollary 2 proved also that, under the assumptions of Theorem 3.1, f is injective on a compactum of dimension at least equal to

$$\dim X - (1 + \text{the integer part of } \frac{1}{2} \dim Y)$$

This gives the assertion of Corollary 3.2, whenever this integer is positive.

Using a curve defined by Cook [3] one can obtain the following two examples which shed some light on the assumptions in Theorem 1.1 and Corollary 3.2.

3.4. EXAMPLE. *There exists a light mapping $f : S \to I^2$ from a 2-dimensional Cantor manifold S onto the unit square which is not injective on any non-trivial continuum in S.*

3.5. EXAMPLE. *There exists a light mapping without perfect fibers $g : T \to I^4$ defined on a 2-dimensional Cantor manifold T which is not injective on any non-trivial continuum in T.*

More specifically, Cook [3] defined a one-dimensional continuum M with the following properties:
(1) M is hereditarily indecomposable,
(2) each non-trivial continuum in M contains a continuum which can be mapped onto a continuum which is not a continuous image of any plane continuum.

The properties of M (M_1 in Cook's notation) are stated in Theorem 9 and the Note at the end of section 3 in [3].

Let us notice that the square $M \times M$ also has the property described in (2). Indeed, let $p : M \times M \to M$ be the projection and let K be a non-trivial continuum in $M \times M$. If $p(K)$ is a singleton, K can be considered as a subcontinuum of M. If $p(K)$ is non-trivial, at first one can find a continuum L as in (2), and next one can use (1) and Theorem 4 in [3] to get a continuum K' in K with $p(K') = L$.

In effect we conclude that
(3) no non-trivial subcontinuum of $M \times M$ embeds in the plane.

Example 3.4 can now be described instantly. Since M is a curve, three is a light mapping $u : M \to I$ onto the unit interval, and $f = u \times u : M \times M \to I^2$ has the required property, by (3).

Example 3.5 requires a few additional adjustments.

Let C be the Cantor set in I. Let us attach C to $u^{-1}(C)$ by the map u, i.e. let us consider the quotient space N for the upper-semicontinuous decomposition of M into the compacta $u^{-1}(t)$ with $t \in C$ and singletons $\{x\}$ with $x \notin u^{-1}(C)$. Let $q : M \to N$ be the quotient mapping. We get the induced mapping $v : N \to I$, with $u = v \circ q$. Notice that the map v is injective on the Cantor set $q(u^{-1}(C))$.

We set $T = N \times N$. The mapping $g : T \to I^4$, is the composition $g = (d \times d) \circ (v \times v)$ where d is a continuous finite-to-one mapping $d : I \to I^2$ such that $d(C) = I^2$ (to get such d, one can start with any finite-to-one continuous mapping to C onto I^2, extending successively this map over the open contigeous intervals of C in I, $J_1, J_2, \ldots$, so that each $d(J_i)$ is a polygonal line which avoids the points of intersections of the polygonal lines already defined and intersects their union in at most finitely many points).

3.6. REMARK. We conjecture that without some additional assumptions on Y, the assertion of Corollary 3.2 is not true.

4. Countable-to-One Mappings on Infinite-Dimensional Compacta.

A separable metrizable space X is a C-space (a weakly infinite-dimensional space) if for each sequence of open covers (two-element open covers, respectively) $\mathcal{U}_1, \mathcal{U}_2, \cdots$ of X there exist disjoint open families $\mathcal{V}_1, \mathcal{V}_2, \cdots$ such that each $\mathcal{V}_i$ refines $\mathcal{U}_i$ and the union $\cup_i \mathcal{V}_i$ covers X, cf. [1].

The compacta which are not weakly infinite-dimensional, we shall call them strongly infinite-dimensional, are exactly the compacta which can be mapped essentially onto the Hilbert cube I^∞. Evidently, C-spaces are weakly infinite-dimensional (it is not known if the converse is true). The Hilbert cube is strongly infinite-dimensional, cf. [8].

D. W. Henderson [6] proved that every strongly infinite-dimensional compactum contains a non-trivial continuum, each non-trivial subcontinuum of which is strongly infinite-dimensional. We shall see in a moment that analogous result is also true for C-spaces. In effect, using Corollary 1.2 we shall get the following result.

4.1. THEOREM. *Let* $f : K \to L$ *be a countable-to-one mapping between*

compacta. If K is strongly infinite-dimensional, or it is not a C-space, then so is L, and moreover, f is injective on a continuum which is strongly infinite-dimensional, or is not a C-space, respectively.

To prove this we need for C-spaces a counterpart to the theorem of Henderson. To this end we shall explain that a simple proof of this theorem indicated in [9], sec. 11, can be carried out in a more general setting, including the property C. It is convenient to distinguish some properties essential for that reasoning, and having this in mind, we shall call a nonempty class $\mathscr{K}$ of separable metrizable spaces admissible if $\mathscr{K}$ has the following properties:

(i)   *if $X \in \mathscr{K}$ and Y is homeomorphic to a closed subset of X, then $Y \in \mathscr{K}$;*

(ii)   *a space which is a countable union of members of $\mathscr{K}$ is in $\mathscr{K}$;*

(iii)   *if $f : X \to Y$ is a perfect mapping, Y is zero-dimensional, and all fibers $f^{-1}(y)$ are in $\mathscr{K}$, then $X \in \mathscr{K}$;*

(iv)   *if $A \subset X$, $A \in \mathscr{K}$ and all closed in X sets disjoint from A are in $\mathscr{K}$ then $X \in \mathscr{K}$.*

Both C-spaces and weakly infinite-dimensional spaces, form admissible classes, cf. [4], [5].

4.2. PROPOSITION. *Let $\mathscr{K}$ be an admissible class and let X be a compactum not belonging to $\mathscr{K}$. Then there is a non-trivial continuum in X all whose non-trivial subcontinua do not belong to $\mathscr{K}$.*

PROOF. In the proof of Proposition 11.2 in [9] the class of weakly infinite-dimensional spaces can be replaced by any admissible class of spaces, without any change of the reasonings (for this part the property (iv) is not necessary). In effect, for any admissible class $\mathscr{K}$, we get a subset A of the Hilbert cube I^{∞} belonging to $\mathscr{K}$ such that A intersects each non-trivial continuum in I^{∞} belonging to $\mathscr{K}$.

Let X be any compactum not in $\mathscr{K}$. We can assume that $X \subset I^{\infty}$. Since $A \cap X \in \mathscr{K}$, property (iv) gives us a compactum $Z \subset X \setminus A$ which does not belong to $\mathscr{K}$, then any non-trivial continuum in Z has the required property (let us notice that all zero-dimensional sets are in every admissible class, by (iii), and hence Z contains such a continuum).

This completes the proof of Proposition 4.2, and hence also of Theorem 4.1.

Theorem 4.1 can be proved more directly, in the spirit of Sklyarenko's

paper [10], but I do not know any reasoning which would avoid the theorem of Henderson, or its analogues.

5. A Remark on a Theorem of Sklyarenko.

The following observation, related to a theorem of Sklyarenko [10], gives some additional information associated with Theorems 1.1 and 4.1. We formulate the result for strongly infinite-dimensional spaces, but the proof applies to any admissible class, as defined in sec. 4.

5.1. PROPOSITION. *Let* $f: X \to Y$ *be a continuous mapping between compacta without any perfect fiber. If* Y *is strongly infinite-dimensional, then* f *is injective on a strongly infinite-dimensional continuum in* X.

PROOF. We denote by diam$A$ the diameter of a set A with respect to a fixed metric in X.

Aiming at a contradiction, assume that that f is not injective on any strongly infinite-dimensional compactum in X. Let $S_1, S_2, \ldots$ be the closures of elements of any countable base in X.

Inductively, we shall define strongly infinite-dimensional compacta in Y,

$$C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots,$$

such that, for any n,

(*) either $f(S_n) \cap C_n = \varnothing$, or for each $t \in C_n$, $f^{-1}(t) \cap S_n$ contains at least two points. To begin with, set $C_0 = Y, S_0 = \varnothing$, and assume that we have defined C_n. Let us consider

$$K = f(S_{n+1}) \cap C_n.$$

If K is weakly infinite-dimensional, choose a strongly infinite-dimensional compactum $C_{n+1} \subset C_n \setminus K$. Suppose that K is strongly infinite-dimensional and set

$$K_p = \{y \in K : \text{diam}(S_{n+1} \cap f^{-1}(y)) \geq 1/p\},$$

and $K_0 = K \setminus \cup\{K_p : p = 1, 2, \cdots\}$. Then, for $p \geq 1, K_p$ is a compactum, and f is injective on $S_{n+1} \cap f^{-1}(K_0)$. By the assumption, there is no strongly infinite-dimensional compactum contained in $K_0 \subset f(S_{n+1})$, and therefore, there is $p \geq 1$ with K_p strongly infinite-dimensional (cf. sec. 4, conditions (ii) and (iv) in the definition of admissible classes). We set $C_{n+1} = K_p$.

Now, to reach a contradiction, we check that for any $t \in C_1 \cap C_2 \cap \ldots$ the

fiber $f^{-1}(t)$ is perfect. Indeed, if $S_n \cap f^{-1}(t) \neq \varnothing$, then $f(S_n) \cap C_n \neq \varnothing$ and by (*), S_n contains at least two points from $f^{-1}(t)$, and therefore, $f^{-1}(t)$ has no relatively isolated points.

Since any strongly infinite-dimensional compactum contains such a continuum, the proof is completed.

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Roman Pol
Institute of Mathematics
Warsaw University
Banacha 2
02-097 Warsaw
Poland

Added in proof: M.Levin, Fund, Math, 197(1995), 93-98, obtained results strengthening Proposition 4.2.