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# SCHWARTZ KERNEL THEOREM FOR THE FOURIER HYPERFUNCTIONS

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## §0. Introduction

The purpose of this paper is to give a direct proof of the Schwartz kernel theorem for the Fourier hyperfunctions. The Schwartz kernel theorem for the Fourier hyperfunctions means that with every Fourier hyperfunction $K$ in $\mathcal{F}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ we can associate a linear map

$$\mathcal{K}:\mathcal{F}(\boldsymbol{R}^{n_2})\longrightarrow\mathcal{F}'(\boldsymbol{R}^{n_1})$$

and vice versa, which is determined by

$$\langle\mathcal{K}\varphi,\ \psi\rangle=K(\psi\otimes\varphi),\qquad \psi\in\mathcal{F}(\boldsymbol{R}^{n_1}),\ \varphi\in\mathcal{F}(\boldsymbol{R}^{n_2}).$$

For the proof we apply the representation of the Fourier hyperfunctions as the initial values of the smooth solutions of the heat equation as in [3] which implies that if a $C^\infty$-solution $U(x, t)$ satisfies some growth condition then we can assign a unique compactly supported Fourier hyperfunction $u(x)$ to $U(x, t)$ (see Theorem 1.4). Also we make use of the following real characterizations of the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions in [1, 3, 5]

$$\mathcal{F}=\left\{\varphi\in C^\infty\ \middle|\ \sup_{\alpha,x}\frac{|\partial^\alpha\varphi(x)|\exp k|x|}{h^{|\alpha|}\alpha!}<\infty\quad\text{for some } h,\ k>0\right\}$$

$$=\left\{\varphi\in C^\infty\ \middle|\ |\sup|\varphi(x)|\exp k|x|<\infty,\quad \sup|\hat{\varphi}(\xi)|\ \exp h|\xi|<\infty\right.$$
$$\left.\text{for some } h,\ k>0\right\}$$

Also, we closely follow the direct proof of the Schwartz kernel theorem for the distributions as in Hörmander [2].

## §1. Preliminaries

We denote by $x=(x_1, x_2)\in\boldsymbol{R}^n$ for $x_1\in\boldsymbol{R}^{n_1}$ and $x_2\in\boldsymbol{R}^{n_2}$, and use the multi-index notation; $|\alpha|=\alpha_1+\cdots+\alpha_n$, $\partial^\alpha=\partial^{\alpha_1}\cdots\partial^{\alpha_n}$ for $\alpha=(\alpha_1,\cdots,\alpha_n)\in\boldsymbol{N}_0^n$ where

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$\boldsymbol{N}_0$ is the set of nonnegative integers.

For $f \in L^1(\boldsymbol{R}^n)$ the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\xi)=\int e^{-ix\cdot\xi}f(x)dx\,, \qquad \xi\in\boldsymbol{R}^n\,.$$

We first give two equivalent definitions of the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions in [1, 3, 5] as follows:

DEFINITION 1.1 ([3]). An infinitely differentiable function $\varphi$ is in $\mathcal{F}(\boldsymbol{R}^n)$ if there are positive constants $h$ and $k$ such that $\varphi\in\mathcal{F}_{h,k}$, where

$$\mathcal{F}_{h,k}=\left\{\varphi\in C^\infty \,\middle|\, |\varphi|_{h,k}=\sup_{\alpha,x}\frac{|\partial^\alpha\varphi(x)|\exp k|x|}{h^{|\alpha|}\alpha!}<\infty\right\}$$

DEFINITION 1.2 ([1]). The space $\mathcal{F}$ of test functions for the Fourier hyperfunctions consists of all $C^\infty$ functions such that for some $h$, $k>0$

$$\sup_x|\varphi(x)|\exp k|x|<\infty\,,$$

$$\sup_\xi|\hat{\varphi}(\xi)|\exp h|\xi|<\infty\,.$$

We denote by $E_t(x)$ the $n$-dimensional heat kernel;

$$E_t(x)=\begin{cases}(4\pi t)^{-n/2}\exp(-|x|^2/4t)\,, & t>0\,,\\ 0\,, & t\leqq 0\,.\end{cases}$$

We now need the following Proposition 1.3 and Theorem 1.5 to prove the Main theorem in § 2.

PROPOSITION 1.3 ([4]). *There are positive constants $C$ and $a$ such that*

$$|\partial^\alpha E_t(x)|\leqq C^{|\alpha|+1}t^{-(n+|\alpha|)/2}(\alpha!)^{1/2}\exp(-a|x|^2/4t)\,,$$

*where $a$ can be taken as close as desired to* 1 *and* $0<a<1$.

From Proposition 1.3 we can easily obtain the following

COROLLARY 1.4. *There exist positive constants $C$, $C'>0$ such that for every $\varepsilon>0$ and sufficiently small $t>0$*

$$|E_t(x)|_{C\varepsilon^{-1/2},\varepsilon}\leqq C'\varepsilon^{-n/2}\exp[\varepsilon(1/t+|x|)]\,.$$

PROOF. By Proposition 1.3 we can easily see that there exist positive constants $C$, $C'>0$ such that for every $\varepsilon>0$

$$\sup_{\alpha,x}\frac{|\partial_y^\alpha E_t(x-y)|\exp\varepsilon|y|}{(C\varepsilon^{-1/2})^{|\alpha|}\alpha!}\leqq C'\varepsilon^{-n/2}\exp(\varepsilon/2t)\exp(2\varepsilon^2 t)\exp\varepsilon|x|\,.$$

In fact, we have

$$
\begin{aligned}
&|\partial_y^\alpha E_t(x-y)| \\
&\leqq C^{|\alpha|+1} t^{-(n+|\alpha|)/2} \alpha!^{1/2} \exp(-a|x-y|^2/4t) \\
&\leqq C^{|\alpha|+1} (\varepsilon^{-1/2})^{|\alpha|} \varepsilon^{-n/2} \exp \varepsilon/2t [(n+|\alpha|)!]^{1/2} \alpha!^{1/2} \exp(-a|x-y|^2/4t) \\
&\leqq (\sqrt{2} C \varepsilon^{-1/2})|\alpha| C' \alpha! \exp(-a|x-y|^2/4t).
\end{aligned}
$$

Thus, we obtain that for every $\varepsilon>0$ and small $t>0$

$$
|E_t(x-\cdot)|_{C\varepsilon^{-1/2}, \varepsilon} \leqq C' \varepsilon^{-n/2} \exp[\varepsilon(1/t+|x|)],
$$

which completes the proof.

THEOREM 1.5 ([3]). *Let $u \in \mathcal{F}'$ and $T>0$. Then $U(x, t)=u_y(E(x-y, t))$ is a $C^\infty$ function in $\boldsymbol{R}^n \times (0, T)$ and satisfies the following:*

(i) *$(\partial/\partial t-\Delta)U(x, t)=0$ in $\boldsymbol{R}^n \times (0, T)$.*

(ii) *For every $\varepsilon>0$ there exists a constant $C>0$ such that*

$$
|U(x, t)| \leqq C \exp[\varepsilon(1/t+|x|)] \qquad \text{in } \boldsymbol{R}^n \times (0, T).
$$

(iii) *$\lim_{t \to 0+} U(x, t)=u$ in $\mathcal{F}'$ i.e.,*

$$
u(\varphi)=\lim_{t \to 0+} \int U(x, t)\varphi(x)dx, \qquad \varphi \in \mathcal{F}.
$$

*Conversely, every $C^\infty$ function $U(x, t)$ in $\boldsymbol{R}^n \times (0, T)$ satisfying* (i) *and* (ii) *can be expressed in the form $U(x, t)=u_y(E(x-y, t))$ with a unique element $u \in \mathcal{F}'$.*

THEOREM 1.6 ([3]). *If $\varphi \in \mathcal{F}(\boldsymbol{R}^n)$ then it follows that $\varphi * E_t$ converges to $\varphi$ in $\mathcal{F}(\boldsymbol{R}^n)$ when $t \to 0^+$.*

We shall prove the associativity for convolution in $\mathcal{F}(\boldsymbol{R}^n)$.

THEOREM 1.7. *If $u \in \mathcal{F}'(\boldsymbol{R}^n)$ and $\varphi, \psi \in \mathcal{F}(\boldsymbol{R}^n)$ then*

$$
(u*\varphi)*\psi=u*(\varphi*\psi).
$$

The proof is an easy consequence of the following

THEOREM 1.8. *If $\varphi \in \mathcal{F}_{h_1, k_1}(\boldsymbol{R}^n)$, $\psi \in_{h_2, k_2}(\boldsymbol{R}^n)$, then the Riemann sum*

$$
\sum_{j \in \boldsymbol{Z}^n} \varphi(x-js)s^n\psi(js) \tag{1.1}
$$

*converges to $\varphi*\psi(x)$ in $\mathcal{F}_{h, k}$ when $s \to 0$ for $h>\max\{h_1, h_2, 2\sqrt{2}\}$, $k<\min\{k_1, k_2\}$.*

Before proving Theorem 1.8 we show the following refinement of Definition

1.2 which is the main theorem in [1].

LEMMA 1.9. *Let* $h>2\sqrt{2}$ *and* $k>0$. *Then the following conditions are equivalent*:

(i) $\varphi\in\mathcal{F}_{h,k}$

(ii) $\sup_x |\varphi(x)|\exp k|x|<\infty$

$\sup_x |\partial^\alpha\varphi(x)|\leqq C(h/2\sqrt{2})^{|\alpha|}\alpha!$.

(iii) *There exists an integer* $a>2\sqrt{2}$ *such that*

$$\sup_x |\varphi(x)|\exp k|x|<\infty, \tag{1.2}$$

$$\sup_\xi |\hat{\varphi}(\xi)|\exp(2\sqrt{2}|\xi|/ah)<\infty. \tag{1.3}$$

PROOF. It follows from Theorem 2.1 in [1] that (ii) is a sufficient condition for $\varphi\in\mathcal{F}_{h,k}$. So it suffices to prove the implications (i)⇒(iii) and (iii)⇒(ii)

(i)⇒(iii): It suffices to show (1.3). We obtain from (i) that

$$\begin{aligned}|\xi^\alpha\hat{\varphi}(\xi)| &= \left|\int e^{-ix\cdot\xi}\partial^\alpha\varphi(x)dx\right| \\ &\leqq C_0 h^{|\alpha|}\alpha!\int\exp(-k|x|)dx \\ &\leqq C_1(ah/2\sqrt{2})^{|\alpha|}\alpha!(2\sqrt{2}/a)^{|\alpha|} \quad \text{for all } \alpha\end{aligned}$$

where $a>2\sqrt{2}$. Hence

$$\sum_\alpha 1/\alpha!(2\sqrt{2}|\xi|/ah)^{|\alpha|}|\hat{\varphi}(\xi)|\leqq C_1\sum_\alpha(2\sqrt{2}/a)^{|\alpha|}<\infty$$

Therefore we obtain (1.3).

(iii)⇒(ii): By Hölder's inequality we have

$$\begin{aligned}|\partial^\alpha\varphi(x)|^{4a} &= \frac{1}{(2\pi)^{4an}}\left|\int e^{ix\cdot\xi}\xi^\alpha\hat{\varphi}(\xi)d\xi\right|^{4a} \\ &\leqq \frac{1}{(2\pi)^{4an}}\int(|\xi|^{|\alpha|}|\hat{\varphi}(\xi^{1/2})^{4a}d\xi\left(\int|\hat{\varphi}(\xi)|^{2a/4a-1}d\xi\right)^{4a-1} \\ &\leqq C\sup_\xi\frac{|\xi|^{4a|\alpha|}}{\exp(2\sqrt{2}|\xi|/h)} \\ &\leqq C'(h/2\sqrt{2})^{4a|\alpha|}(\alpha!)^{4a}.\end{aligned}$$

Thus we obtain (ii).

LEMMA 1.10. *Let* $k>0$ *and* $j=(j_1,\cdots,j_n)$, $j_i\in N_0$, *and let* $0<s<A$ *for some fixed* $A$. *Then*

$$\sum_{j\in N_0^n} s^n \exp(-ks|j|)<C$$

*where* $C$ *is independent of* $s$ *and* $|j|$ *is a Euclidean norm.*

PROOF. Note that $\sqrt{n}|j|\geqq\sum_i j_i$ for $j\in N_0^n$ and that the function $x/(1-\exp(-kx))$ is strictly increasing for $x>0$. If $0<s<A$ then

$$\begin{aligned}&\sum_{j\in N_0^n} s^n \exp(-ks|j|)\\ &\leqq \sum_{j_1\in N_0} s\exp(-ksj_1/\sqrt{n})\times\cdots\times\sum_{j_n\in N_0} s\exp(-ksj_n/\sqrt{n})\\ &=\Big(\frac{2s}{1-\exp(-ks/\sqrt{n})}\Big)^n\\ &<\Big(\frac{2A}{1-\exp(-kA/\sqrt{n})}\Big)^n.\end{aligned}$$

PROOF OF THEOREM 1.8. Choose $h, k>0$ such that $h>\max\{h_1, h_2, 2\sqrt{2}\}$ and $k<\min\{k_1, k_2\}$. Let $f_s(x)=\sum_j \varphi(x-js)s^n\psi(js)$, $s>0$. By Lemma 1.9 we shall show that for any $\varepsilon>0$ there exists a constant $\delta>0$ such that if $s<\delta$ then

$$\sup_x |f_s(x)-\varphi*\psi(x)|\exp k|x|<\varepsilon, \tag{1.4}$$

$$\sup_\xi |\hat{f}_s(\xi)-\widehat{\varphi*\psi}(\xi)|\exp(2\sqrt{2}|\xi|/ah)<\varepsilon \tag{1.5}$$

where $a>2\sqrt{2}$. From now on we take $a=4\sqrt{2}$. Choose $k'$ such that $k<k'<\min\{k_1, k_2\}$.

If $s<A$ then $f_s\in\mathcal{F}_{h,k'}$ by Lemma 1.10. In fact,

$$\begin{aligned}|f_s|_{h,k'}&\leqq\sum_j\frac{|\partial^\alpha\varphi(x-js)|s^n|\psi(js)|}{h^{|\alpha|}\alpha!}\exp k'|x-js|\exp k'|js|\\ &\leqq C\sum_j s^n\exp(-(k_2-k')|js|)\\ &\leqq M_1\end{aligned}$$

where $M_1$ is independent of $s<A$. Similarly we obtain $\varphi*\psi\in\mathcal{F}_{h,k'}$. For any $\varepsilon>0$ choose $R=R_\varepsilon>0$ such that

$$\exp(-(k'-k)R)<\varepsilon, \quad \exp\Big(-\frac{1}{2}\Big(\frac{1}{h_1}-\frac{1}{h}\Big)R\Big)<\varepsilon.$$

Thus for all $s<A$ we obtain

$$\begin{aligned}\sup_{|x|\geqq R}|f_s(x)-\varphi*\psi(x)|\exp k|x|&\leqq\sup_{|x|\geqq R}(|f_s(x)|+|\varphi*\psi(x)|)\exp k|x|\\ &\leqq C\sup_{|x|\geqq R}\exp(-k'|x|)\exp k|x|\end{aligned} \tag{1.6}$$

$$\leq C \exp(-(k'-k)R)$$
$$\leq C\varepsilon .$$

Note that for any $s>0$ the function $f_s(x)$ is continuous on the compact set $\{x \mid |x| \leqq R\}$ and the sequence $\{f_s \mid 0<s<A\}$ is bounded and equicontinuous. In fact, for $|x| \leqq R$ we have

$$(1.7) \qquad |f_s| \leqq C' \sum_j \exp(-k_1|x-js|)s^n \exp(-k_2|js|)$$
$$\leqq C' e^{k_1 R} \sum_j \exp(-(k_1+k_2)|js|)s^n$$
$$\leqq M_2$$

where $M_2$ is independent of $s<A$. The last inequality is also obtained by Lemma 1.10. Also, for any $\varepsilon>0$ there exists $\delta_1>0$ such that if $|x_1-x_2|<\delta_1$ then

$$(1.8) \qquad |f_s(x_1)-f_s(x_2)| = \sum_j |\varphi(x_1-js)-\varphi(x_2-js)|s^n|\psi(js)|$$
$$= \sum_j |\nabla\varphi(\xi)|\,|x_1-x_2|s^n|\psi(js)|$$
$$\leqq M|x_1-x_2|$$

where the second inequality is obtained from (1.7). Thus, by Arzela-Ascoli's theorem we obtain that for $|x| \leqq R$ the sequence $\{f_s\}$ converges uniformly to $\varphi*\psi(x)$, i.e., for any $\varepsilon>0$ there exists $\delta_2>0$ such that if $s<\delta_2$ then

$$(1.9) \qquad \sup_{|x| \leq R} |f_s(x)-\varphi*\psi(x)| \exp k|x| < \varepsilon$$

If $\delta=\min\{A, \delta_2\}$ then (1.4) is obtained from (1.6) and (1.9). On the other hand, if $g_s(\xi)=\sum_j s^n \exp(-i(js)\cdot\xi)\psi(js)$ we obtain for some $B>0$ the sequence $\{g_s \mid 0<s<B\}$ is bounded and equicontinuous as (1.7) and (1.8). Thus for $|\xi| \leqq R$ the sequence $\{g_s\}$ converges uniformly to $\hat{\psi}(\xi)$, i.e., for any $\varepsilon>0$ there exists $\delta_3>0$ such that if $s<\delta_3$ then

$$(1.10) \qquad \sup_{|\xi| \leq R} |g_s(\xi)-\hat{\psi}(\xi)| < \varepsilon.$$

From the above fact we obtain (1.5). In fact, if $s<\delta=\min\{\delta_3, B\}$ then

$$(1.11) \qquad \sup_{\xi} |\hat{f}_s(\xi)-\widehat{\varphi*\psi}(\xi)| \exp(|\xi|/2h)$$
$$= \sup_{\xi} \Big|\sum_j \hat{\varphi}(\xi)\exp(-i(js\cdot\xi))s^n\psi(js)-\hat{\varphi}(\xi)\hat{\psi}(\xi)\Big| \exp(|\xi|/2h)$$
$$\leqq \sup_{\xi} |\hat{\varphi}(\xi)| \exp(|\xi|/2h)|g_s(\xi)-\hat{\psi}(\xi)|$$

$$\leq C \sup_{|\xi|\leq R} \exp\Big(-\frac{1}{2}\Big(\frac{1}{h_1}-\frac{1}{h}\Big)|\xi|\Big)|g_s(\xi)-\hat{\phi}(\xi)|$$

$$+C \sup_{|\xi|\geq R} \exp\Big(-\frac{1}{2}\Big(\frac{1}{h_1}-\frac{1}{h}\Big)|\xi|\Big)(|g_s(\xi)|+|\hat{\phi}(\xi)|)$$

$$\leq C' \sup_{|\xi|\leq R} |g_s(\xi)-\hat{\phi}(\xi)|+C\exp\Big(-\frac{1}{2}\Big(\frac{1}{h_1}-\frac{1}{h}\Big)R\Big)$$

$$\leq M_4\varepsilon\,,$$

which completes the proof.

THEOREM 1.11. *If* $u\in\mathcal{F}'(\boldsymbol{R}^n)$ *then* $u*E_t$ *converges to* $u$ *in* $\mathcal{F}'(\boldsymbol{R}^n)$ *as* $t\to 0^+$.

PROOF. We note that $u(\phi)=u*\check{\phi}(0)$ if $\phi\in\mathcal{F}(\boldsymbol{R}^n)$ *and* $\check{\phi}(x)=\phi(-x)$. This gives

$$(u*E_t)(\phi)=(u*E_t)*\check{\phi}(0)=u*(E_t*\check{\phi})(0)=u(E_t*\phi)\,.$$

By Theorem 1.6 $E_t*\phi$ converges to $\phi$ in $\mathcal{F}(\boldsymbol{R}^n)$ as $t\to 0^+$. So it follows that $(u*E_t)(\phi)$ converges to $u(\phi)$ as claimed.

## § 2. Main Theorem

We are now in a position to state and prove the Schwartz kernel theorem for the space $\mathcal{F}'$.

THEOREM 2.1. *If* $K\in\mathcal{F}'(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ *then a linear map* $\mathcal{K}$ *determined by*

$$\langle\mathcal{K}\varphi,\phi\rangle=K(\phi\otimes\varphi),\qquad \phi\in\mathcal{F}(\boldsymbol{R}^{n_1}),\ \varphi\in\mathcal{F}(\boldsymbol{R}^{n_2}) \tag{2.1}$$

*is continuous in the sense that* $\mathcal{K}\varphi_j$ *converges to* 0 *in* $\mathcal{F}'(\boldsymbol{R}^{n_1})$ *if* $\varphi_j$ *converges to* 0 *in* $\mathcal{F}(\boldsymbol{R}^{n_2})$. *Conversely, for every such linear map* $\mathcal{K}$ *there is one and only one Fourier hyperfunction* $K$ *such that* (2.1) *is valid.*

PROOF. If $K\in\mathcal{F}'(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ then (2.1) defines a Fourier hyperfunction $\mathcal{F}\varphi$, since the map $\phi\to K(\phi\otimes\varphi)$ is continuous. Also $\mathcal{K}$ is continuous, since the map $\varphi\to K(\phi\otimes\varphi)$ is continuous.

Let us now prove the converse. We first prove the uniqueness, i.e., if

$$u(\phi\otimes\varphi)=0\qquad\text{for } \phi\in\mathcal{F}(\boldsymbol{R}^{n_1}),\ \varphi\in\mathcal{F}(\boldsymbol{R}^{n_2}),$$

then $u=0$ in $\mathcal{F}'(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$.

It follows from Theorem 1.11 that $u*E_t$ converges to $u$ in $\mathcal{F}'(\boldsymbol{R}^n)$ as $t\to 0^+$. However, $u*E_t=0$, since $E_t(x_1-y_1,x_2-y_2)$ is the product of a function of $y_1$ and one of $y_2$. Hence $u=0$ in $\mathcal{F}'$.

We now prove the existence. Since $\mathcal{K}$ is continuous, the bilinear form on $\mathcal{F}_{h_1,k_1}(\boldsymbol{R}^{n_1})\times\mathcal{F}_{h_2,k_2}(\boldsymbol{R}^{n_2})$

$$(\psi, \varphi)\longmapsto\langle\mathcal{K}\varphi, \psi\rangle$$

is separately continuous, therefore continuous, since $\mathcal{F}_{h,k}$ is a Fréchet space for all $h$, $k>0$. Hence we obtain that there is a constant $C(h_1, k_1, h_2, k_2)$ such that

$$|\langle\mathcal{K}\varphi, \psi\rangle|\leqq C|\psi|_{h1,k1}|\varphi|_{h2,k2}. \tag{2.2}$$

Set for $(x_1, x_2)\in\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}$ and small $t>0$

$$K_t(x_1, x_2)=\langle\mathcal{K}E_{t,2}(x_2-\cdot), E_{t,1}(x_1-\cdot)\rangle \tag{2.3}$$

where $E_{t,j}(x_j)$ is the $n_j$-dimensional heat kernel.

We now show that $K_t$ has a limit in $\mathcal{F}'(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ as $t\to0$, and then show that (2.1) is also satisfied by the limit. It follows from (2.2) and Corollary 1.4 that for every $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ such that

$$|K_t(x_1, x_2)|\leqq C_\varepsilon\exp\varepsilon(1/t+|x|).$$

Since

$$\partial E_t/\partial t=\Delta_x E_t, \qquad t>0$$

we have

$$\partial K_t/\partial t=\Delta_x K_t.$$

It follows from Theorem 1.5 that there exists a limit $K_0\in\mathcal{F}'$ such that $K_t$ converges to $K_0$ in $\mathcal{F}'(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$.

Let $\varphi_j\in\mathcal{F}(\boldsymbol{R}^{n_j})$, $j=1, 2$ and form

$$\langle K_t, \varphi_1\otimes\varphi_2\rangle=\iint K_t(x_1, x_2)\varphi_1(x_1)\varphi_2(x_2)dx_1dx_2.$$

We have

$$\iint K_t(x_1, x_2)\varphi_1(x_1)\varphi_2(x_2)dx_1dx_2$$

$$=\iint\langle\mathcal{K}E_{t,2}(\cdot-x_2)\varphi_2(x_2), E_{t,1}(\cdot-x_1)\varphi_1(x_1)\rangle dx_1dx_2.$$

Approximating the above integral by the Riemann sum we obtain from Lemma 1.8 that

$$\langle K_t, \varphi_1\otimes\varphi_2\rangle=\langle\mathcal{K}(\varphi_2*E_{t,2}), \varphi_1*E_{t,1}\rangle.$$

Since $\varphi_j*E_{t,j}$ converges to $\varphi_j$ in $\mathcal{F}(\boldsymbol{R}^{n_j})$ as $t\to0$, it follows from (2.2) that the right hand side converges to $\langle\mathcal{K}\varphi_2, \varphi_1\rangle$ as $t\to0$. Thus

$$\langle K_0, \varphi_1\otimes\varphi_2\rangle=\langle\mathcal{K}\varphi_2, \varphi_1\rangle$$

which completes the proof.

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