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ON THE MODULI OF PERIODIC INSTANTONS

Dedicated to Professor Seiya Sasao on his 60th birthdary

By

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1. Introduction and statement of the result

Let M be a smooth oriented 4-manifold, which admits an open subset K with one end N and an open submanifold W_0 with two ends N_-, N_+. W_1, W_2, $\cdots$ denote copies of W_0. The 4-manifold M will be called end-periodic if it admits a decomposition $M=K\cup_N W_0\cup_N W_1\cup\cdots$, where $N\subset K$ is identified with the end N_- of W_0 and the end N_+ of W_0 is identified with the end N_- of W_1 and so on. Let Y be the compact oriented 4-manifold which is obtained from W_0 by identifying the two ends. The manifold Y has a Z-cover $\tilde{Y}=\cdots\cup_N W_{-1}\cup_N W_0\cup_N W_1\cup_N\cdots$ with projection $\pi: \tilde{Y}\to Y$.

A geometric object on M, a vector bundle, a connection, a differential operator, a Riemannian metric etc. will be called end-periodic if its restriction on $End\ M=W_0\cup_N W_1\cup_N\cdots$ is the pull back by π of an object on Y. By making use of a smooth function $t: W_0\to[0, 1]$ such that $t|_{N_-}=0$ and $t|_{N_+}=1$, we obtain a smooth step function $\tau: M\to[0, \infty)$ such that $\tau|_K=0$ and $\tau(x)=n+t(x)$ if $x\in W_n$, $n=1, 2, \cdots$.

Let $P\to M$ be an end-periodic principal $SU(2)$-bundle, and A_0 be an end-periodic connection on P which is isomorphic over $End\ M$ to the product connection on $End\ M\times SU(2)$. Then by the lemma 7.1 in [11],

$$l=(1/8\pi^2)\int_M tr(F_{A_0}\wedge F_{A_0})$$

is an integer, where $tr(\cdot)$ is the trace on the adjoint representation of the group $SU(2)$. Let $E\to M$ be an end-periodic vector bundle. Put $L^2_{\mathrm{loc}}(E)=\{$section u of E; $u\in L^2(E|_A)$ for every measurable $A\Subset M\}$ and denote by $\|\cdot\|_{A_0}$ the norm by the covariant derivative $\nabla_{A_0}: C_0^\infty(E)\to C_0^\infty(E\otimes T^*M)$ of compactly supported smooth sections. Further $\nabla_{A_0}^{(j)}$ denote the j times iterated derivative $\nabla_{A_0}\cdots\nabla_{A_0}$. For $\delta>0$, we put

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$$\mathcal{A}_l(\delta)=\{A_0+a\ ;\ a\in L^2_{5,\mathrm{loc}}(adP\otimes T^*M)$$
$$\text{with norm} \int_M e^{\tau\delta}\{\textstyle\sum_{j=0}^5|\nabla_{A_0}^{(j)}a|^2\}<\infty\}\ ,$$

and define the small gauge group $\mathcal{G}_l=\{h\in L^2_{6,\mathrm{loc}}(Aut\ P)\ ;\ \|\nabla_{A_0}h\|_{A_0}<\infty\}$, where adP is the associated bundle $P\times_{Ad}su(2)$, and $Aut\ P$ denotes the automorphisms of the principal bundle P. Here we have used the adjoint representation Ad: $SU(2)/Z_2\to End(\mathrm{su}(2))$ and the embedding $C^\infty(P\times_{Ad}SU(2)/Z_2)\to C^\infty(P\times_{Ad}End(\mathrm{su}(2)))$ of effective gauge transformations, and their Sobolev completions (4 in [8]).

Let $\mathcal{A}_l^*(\delta)\subset\mathcal{A}_l(\delta)$ denote the subset of irreducible connections, and g_0 be an end-periodic metric on the tangent bundle TM and $\mathcal{E}$ be the set of asymptotically periodic metrics ((6.1) in [11]). Consider a $\mathcal{G}_l$-equivariant map

$$\rho:\ \mathcal{A}_l(\delta)\times\mathcal{E}\ni(A,\ \varphi)\longrightarrow P_-(g_0)(\varphi^{-1})^*F_A\in L^2_{4,\mathrm{loc}}(adP\otimes P_-\wedge^2T^*M)\ ,$$

where P_- denotes the projection to the anti-self dual parts. Let $\hat{\mathfrak{m}}_l=\rho^{-1}(0)/\mathcal{G}_l$ and $\bar{\pi}:\ \hat{\mathfrak{m}}_l\to\mathcal{E}$ be the projection.

Now the manifold $S^1\times\boldsymbol{R}^3$ is end periodic (Proposition 1 in Section 2). Then our main result is

THEOREM. *For a Baire set of $\varphi\in\mathcal{E}$, the moduli space $\bar{\pi}^{-1}(\varphi)\cap\mathcal{A}_l^*/\mathcal{G}_l$ is a smooth manifold of dimension $8l-3$.*

We can choose a connection A_0 for each l (Proposition 9 in Section 3), and can replace the Sobolev space $L^2_{5,\mathrm{loc}}$ by $L^2_{2,\mathrm{loc}}$ (Remark 1 in Section 4). Almost all arguments in [11] are fitted with the case, $M=S^1\times\boldsymbol{R}^3$ except for the admissibility which is (1) $\pi_1(W)$ does not represent non trivially in $SU(2)$, (2) $H_1(N;\boldsymbol{R})=H_2(N;\boldsymbol{R})=0$, (3) the intersection pairing on $H_2(Y;\boldsymbol{R})$ be positive definite. C. H. Taubes proved in [11].

THEOREM (1.4 in [11]). *Let M be a smooth, end-periodic and admissible 4-manifold. Suppose that $\pi_1(M)$ has only the trivial representation into $SU(2)$. If $H_2(K;Z)$ has positive definite, unimodular intersection pairing, then this pairing is diagonalizable over Z. If the intersection pairing on $H_2(K;\boldsymbol{Z})$ is only known to be positive definite, then the intersection pairing on $H_2(M;Z)$ is unimodular and diagonalizable over Z in the following sense: There is a sequence of free abelian groups $\Lambda_{-1}\subset\Lambda_0\subset\Lambda_1\subset\cdots\subseteqq H_2(M;Z)$ with $\varinjlim\Lambda_n=H_2(M;Z)$ such that (1) $\Lambda_{-1}\otimes\boldsymbol{R}=H_2(K;\boldsymbol{R})$ and (2) the intersection pairing on Λ_n is unimodular and diagonalizable.*

Further he obtained the striking

THOREM (1.1 in [11]). *There exists an uncountable family of diffeomorphism* *classes of oriented* 4*-manifolds which are homeomorphic to* $\boldsymbol{R}^4$.

In [11] the admissibility is used in many parts. Then our main task is to deduce substitutes directly from the topological structure of the manifold instead of the admissibility condition. The main theorem is proved in Section 4.

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2. Preparations

PROPOSITION 1. *The space* $M=S^1\times\boldsymbol{R}^3$ *is end-periodic.*

PROOF. Let $D^3_{3/2}$ be the disc of radius 3/2 in $\boldsymbol{R}^3$. Then using the following diffeomorphism

$$S^1\times(\boldsymbol{R}^3-D^3_{3/2})\approx S^1\times S^2\times(3/2,\ \infty),\ (x,\ y)\longrightarrow(x,\ y/\|y\|,\ \|y\|)$$

we see that the space $S^1\times\boldsymbol{R}^3$ admits a decomposition as a smooth manifold,

$$S^1\times D^3_{3/2}\cup(S^1\times S^2\times(1,\ 3))\cup(S^1\times S^2\times(2,\ 5))\cup\cdots$$
$$\cup(S^1\times S^2\times(2n,\ 2n+3))\cup(S^1\times S^2\times(2n+2,\ 2n+5))\cup\cdots.$$

Now for our argument we put

$$K=S^1\times D^3_{3/2}\cup S^1\times S^2\times(1,\ 3),$$
$$N=S^1\times S^2\times(2,\ 3)=N_-,\ W_0=S^1\times S^2\times(2,\ 5),\ N_+=S^1\times S^2\times(4,\ 5),$$

and so on, and put further $End\ M=W_0\cup W_1\cup\cdots=S^1\times S^2\times(2,\ \infty)$.

PROPOSITION 2. *Any principal* $SU(2)$*-bundle over* M *is trivial.*

PROOF. The identity map $1_M: M\rightarrow M$ is homotopic to the projection $S^1\times \boldsymbol{R}^3\rightarrow S^1\times(0)$, and the space $S^1\times\boldsymbol{R}^3$ yields the Lindelöf property. Then a principal $SU(2)$-bundle $P\rightarrow M$ is isomorphic to the bundle $(P|_{S^1})\times\boldsymbol{R}^3$, where $P|_{S^1}$ is the restriction of the bundle $P\rightarrow M$ onto the subspace $S^1\times(0)\subset S^1\times\boldsymbol{R}^3$. Since the group $SU(2)$ is connected, the bundle $P|_{S^1}$ is trivial.

PROPOSITION 3. *Let* $[S^1\times S^2,\ SU(2)]$ *be the set of homotopy classes. Then we have a bijection*

$$[S^1\times S^2,\ SU(2)]\longrightarrow\pi_3(SU(2))\cong\boldsymbol{Z}\ \textit{integers}.$$

PROOF. By 8.9 Satz in [2], the following sequence is exact:

$$*\longleftarrow[S^1\vee S^2,\ SU(2)]\longleftarrow[S^1\times S^2,\ SU(2)]\longleftarrow[S^1\wedge S^2,\ SU(2)]\longleftarrow *.$$

Thus this proposition is obtained.

Let Y be a compact oriented 4-manifold which is obtained from W_0 by identify N_+ with N_- via the obvious map. Then $Y=S^1\times S^2\times S^1$. Let $[\gamma]$ be the cohomology class which corresponds to the last circle S^1.

PROPOSITION 4. *The Euler-characteristic and the signature of* Y *is zero, and the map* $[\gamma]\cup : H^1_{dR}(Y)\rightarrow H^2_{dR}(Y)$ *has the* 1*-dimensional kernel* $\{\lambda[\gamma] : \lambda\in R\}$, *where* H_{dR} *denotes the deRham cohomology.*

PROOF. $\chi(Y)=1-2+2-2+1=0$.

The signature is the signature of $\{H^2(Y)\ni x\rightarrow(x\cup x)[Y]\in Z\}$, then it is zero. Let γ' be the cohomology class which corresponds to the first circle S^1. Then $[\gamma]\cup[\gamma]=0$ and $[\gamma']\cup[\gamma]$ corresponds to $S^1\times(pt)\times S^1$.

3. Index calculations

The manifold Y admits a Z-cover $\tilde{Y}=\cdots\cup W_{-1}\cup W_0\cup W_1\cup\cdots$, with the projection $\pi : \tilde{Y}\rightarrow Y$. For an end-periodic bundle $E\rightarrow M$, and $1\leqq p\leqq\infty$, $0\leqq k<\infty$, $\delta\in\boldsymbol{R}$, $L^p_{k,\delta}(E)$ denotes the completion of $C^\infty_0(E)$ in the norm

$$\|s\|_{L^p_{k,\delta}}=\left[\int_M d\ vol(e^{\tau\delta}\textstyle\sum_{j=0}^k|\nabla^{(j)}s|^p\right]^{1/p},$$

where τ: *End* $M\rightarrow[0, \infty)$ is the smooth step function.

PROPOSITION 5. *For all but a discrete set of* $\delta\in\boldsymbol{R}$ *and* $k\geqq 4$, *the complex*

$$0\longrightarrow L^p_{k+2,\delta}(M)\xrightarrow{d}L^p_{k+1,\delta}(T^*M)\xrightarrow{P_-d}L^p_{k,\delta}(P_-\wedge^2T^*M)\longrightarrow 0$$

is Fredholm.

PROOF. The proposition 4 in [11] can be applied to our case without any change.

PROPOSITION 6. *For the anti-self-dual deRham complex in Proposition* 5,

$$H^0(L^2_{\cdot,\delta}(d,\ P_-d))=ker(d : L^2_{k+2,\delta}(M)\longrightarrow L^2_{k+1,\delta}(T^*M))=0.$$

PROOF. For $\delta\geqq 0$, the constants are not in $L^2_{\cdot,\delta}$.

PROPOSITION 7. $H^1(L^2_{\cdot,\delta}(d,\ P_-d))=0$ *for* $k>2$.

PROOF. By the Sobolev imbedding $L^2_{k+1,\delta}\subset L^2_{k+1}\subset C^l$ for $k+1-2>l$, and for $w\in ker\ P_-d$, *the equality*

$$0=\int_M |P_- dw|^2=(1/2)\int_M |dw|^2$$

shows that $dw=0$ on M. Here the second equality is obtained as follows:

$$|P_- dw|^2=((1/2)(1-*)dw)\wedge((1/2)(*-1)dw)$$

and

$$dw\wedge dw=d(dw\wedge w), \qquad (*dw)\wedge(*dw)=(*dw,\ dw)\omega_\rho=dw\wedge dw,$$

ω_ρ is a volume form. Now we consider the integral $\int_{\bar{K}_n} d(dw\wedge w)$, where $\bar{K}_n$ denotes the closure of $K_n=K\cup_N W_0\cup_N \cdots \cup_N W_n$. Then the integral is equal to $\int_{\delta\bar{K}_n} dw\wedge w$ which converges to zero as $n\to\infty$. Thus w represents a class $[w]$ $\in H^1_{dR}(M;\boldsymbol{R})$, and a homomorphism

$$r: H^1(L^2_{\cdot,\delta}(d,\ P_- d)) \longrightarrow H^1_{dR}(M;\boldsymbol{R})$$

is obtained. If $w\sim 0$ *in* $C^1(M;\boldsymbol{R})$, then there exists a $z\in C^2(M;\boldsymbol{R})$ such that $w=dz$. Now the lemma 5.2 in [11] can be applied,

$$\int_M e^{\tau\delta}|z-\bar{z}|^2<Z\int_M e^{\tau\delta}|w|^2<\infty \qquad \text{for some } \bar{z}\in\boldsymbol{R}.$$

Then $d(z-\bar{z})=w$ and $dw=0$, therefore $w\sim 0$ in $L^2_{k+1,\delta}(T^*M)$. Hence r is monic. Any class $[w]\in H^1(L^2_{\cdot,\delta}(d,\ P_- d))$ can be represented as $w=fd\theta+\cdots$, θ is the local coordinate in the circle S^1. Then

$$\int_M e^{\tau\delta}|f|^2<\infty \quad \text{and} \quad f\longrightarrow 0 \quad \text{as } \tau\longrightarrow\infty. \tag{1}$$

On the other hand $H^1_{dR}(M;\boldsymbol{R})\cong\boldsymbol{R}$ is generated by the class $[d\theta]$. If $w-d\theta=(\partial g/\partial\theta)d\theta+\cdots$, then $(f-1)d\theta=g_\theta d\theta$, $g_\theta=(\partial g/\partial\theta)$. Therefore $g_\theta\to -1$ as $\tau\to\infty$ by (1). Now g is a periodic function of θ and so g_θ can be expanded as a Fourier series

$$g_\theta=\sum_{n=1}^\infty(a_n\cos n\theta+b_n\sin n\theta). \tag{2}$$

Here the constant term is zero. Therefore the Fourier series can not converge to the constant function -1. This is a contradiction. Thus r is not surjective and the proposition is proved.

Now we proceed to compute the second cohomology group $H^2(L^2_{\cdot,\delta}(d,\ P_- d))$. The homomorphism $r_n: H^2(L^2_{\cdot,\delta}(d,\ P_- d))\to H^2_{0,dR}(K_n;\boldsymbol{R})$, (5.13) in [11] can be defined by the property $H_2(K_n;R)=0$ instead of the condition $H^2(N;\boldsymbol{R})=0$ which is one of the admissiblity condition in [11]. For our case we have to choose k with $k-2>1$, i.e. $k>3$. The lemmas 5.2, 5.4, 5.5, 5.8 and 5.9 for the proof of the lemma 5.3 in [11] can be obtained in our case without the admissibility condition except for the proof of the lemma 5.9. So we state it again and prove.

LEMMA. *Consider the complex*

$$0 \longrightarrow C^\infty(Y) \xrightarrow{d_z} C^\infty(T^*Y) \xrightarrow{P_-d_z} C^\infty(P_-\wedge^2 T^*Y) \longrightarrow 0,$$

where $z\in \boldsymbol{C}$ *obeys* $|z|=1$ *and* $d_z f = df + z^t dz^{-t}\wedge f$, *is a* 1-*form on* Y. *Then coker* $(P_-d_z)=0$.

PROOF. The manifold $Y=S^1\times S^2\times S^1$ has a positive scalar curvature, and is conformally flat. Then it is a self-dual 4-dimensional manifold (Example 1 in the section 1 of [1]). Now d_z is a covariant derivative with respect to a connection $(-\log z)dt$. Then its curvature is zero and so the connection is self-dual. Thus the second cohomology of the complex vanishes.

PROPOSITION 8. $H^2(L^2_{\cdot,\delta}(d,\ P_-d))=0$ *for* $k\geqq 4$.

PROOF. By the Lemma 5.4 in [11], the homomorphism r_n is injective for sufficiently large n. By making use of Poincaré duality, $H^2_{0,dR}(K_n)\cong H_n(K_n)=0$. Then by the Lemma 5.6 in [11], we obtained the proposition.

4. Gauge theory and Moduli space

By Proposition 2 in the section 1, any $SU(2)$-principal bundle over M is isomorphic to the product bundle $M\times SU(2)$. We have in particular $P|_{\mathrm{End}\,M}\cong \pi^*(Y\times SU(2))$, i.e. it is end-periodic, where $\pi:\mathrm{End}\,M\to Y$ is the projection.

PROPOSITION 9. *For each integer* l, *there exists a principal* $SU(2)$-*bundle* P *and an end-periodic connection* A_0 *on* P *which is isomorphic over* End M *to the product connection on* End $M\times SU(2)$ *satisfying*

$$p\equiv(1/8\pi^2)\int_M \mathrm{tr}\,(F_{A_0}\wedge F_{A_0})=l\,.$$

PROOF. As in the proof of the Lemma 7.1 in [11], consider a compact 4-manifold Q. In our case the manifold Q is $S^1\times D^3_+\cup S^1\times D^3_-$. Here we refer to the construction of the definition 4.2 in [10]. Let $\pi: F_Q\to Q$ be the projection of the orthonormal frame bundle, and $(f_i)_{1\leq i\leq l}$ be a set of orthonormal frames at l distinct points in $S^1\times D^3_+$. We choose sufficiently small Gaussian coordinate neighborhoods U_i of $\pi(f_i)$ such that $U_i\subset S^1\times D^3_+$ for all i, and put

$U_0=Q-\bigcup_{i-1}^{l}\pi(f_i)$. Then $\bigcup_{i=0}^{l}U_i$ is a covering of Q, and by the construction due to Taubes we obtain a bundle $P(y)$ which satisfies the relation $-c_2(P(y))=l$, where y denotes an l-tuple $((f_i, \lambda_i)_{i=1}^{l})$ for some positive numbers $\{\lambda_i\}$. By the relation

$$(1/4\pi^2)\int_Q \mathrm{tr}\,(F_{A_0}\wedge F_{A_0})=\int_Q \{c_1^2(P(y))-2c_2(P(y))\},$$

p is equal to l, where A_0 denotes the connection which is given by the definition 4.2 above. Since $A_0=\theta+\sum_{i-1}^{l}\psi_{f_i}^*(\beta_{\sqrt{\lambda_i}}w_{\lambda_i}^2)$ (θ is the flat product connection) gives the product connection in $S^1\times D_-^3$, we can replace $S^1\times D_-^3$ by End M. Thus the proposition is proved.

PROOF OF THEOREM. The smooth manifold structure is obtained by the argument in sections 6, 7 and 8 in [11]. The manifold Q in the proof of Proposition 9 is self-dual and the connections on Q is self-dual and so, on the manifold Q the index of the elliptic operator $P_-(g_0)\ (\varphi^{-1})^*d_A+e^{-\tau\delta}d_A{}^*e^{\tau\delta}$ is $8l$, where we use the equations $\chi(Q)=\tau(Q)=0$.

On the other hand, by Propositions 6, 7, 8 in section 3, the index of the anti-self-dual deRham complex

$$0\longrightarrow L_{k+2,\delta}^2(M)\xrightarrow{\ d\ }L_{k+1,\delta}^2(T^*M)\xrightarrow{\ P_-d\ }L_{k,\delta}^2(P_-\wedge^2T^*M)\longrightarrow 0$$

is zero. Let $\mathfrak{m}_l'=\rho^{-1}(0)/\mathcal{G}_l'$, where $\mathcal{G}_l'=\{g\in\mathcal{G}_l\,;\,r(g)=1\}$ [11], and $\bar{\pi}'\colon \mathfrak{m}_l'\to\mathcal{E}$ be the projection. By the excision property of the index, the index of the elliptic operator above is just the dimension of the manifold $(\bar{\pi}')^{-1}(\varphi)\cap\mathcal{A}_l^*/\mathcal{G}_l'$ (c. f. the proof of the Lemma 8.4 in [11]). Since the projection $(\bar{\pi}')^{-1}(\varphi)\cap\mathcal{A}_l^*/\mathcal{G}_l'\to\bar{\pi}^{-1}(\varphi)\cap\mathcal{A}_l^*/\mathcal{G}_l$ is a principal $SO(3)$-bundle (c. f. the proof of the Lemma 7.3 in [11]), the dimension of our moduli is $8l-3$.

REMARK 1. By the Proposition (4.2.16) in [3], our theorem holds for $k=2$. In fact we can use the regularity of the linear elliptic operator $e^{-\tau\delta}d_B{}^*e^{\tau\delta}+P_-d_B$, where B is a smooth connection and so the operator has smooth coefficients.

REMARK 2. Existence of self-dual connections.

Using the connection A_0 in Proposition 9, we will get $a\in L_{3,\mathrm{loc}}^2(adP\otimes T^*M)$ such that $A=A_0+a$ is self-dual. It suffices then to solve the equation

$$(P_-D_{A_0})^*D_{A_0}u+P_-(*D_{A_0}u\wedge *D_{A_0}u)=-P_-F_{A_0}\quad ((3.4)\text{ in [9], (2.2) in [6]}).$$

By the consideration in the section 4 of [9], the solution is constructed inductively. The first step is to solve $P_-D_{A_0}(*D_{A_0}u)=-P_-F_{A_0}$. Now $A_0=0$ on the half $S^1\times D_-^3$ in Q, then we can assume that $u_1=0$ on the half $S^1\times D_-^3$. For the equations

$$P_-D_{A_0}(*D_{A_0}u_k)=2\sum_{j=1}^{k-2} P_-(*D_{A_0}u_j\wedge *D_{A_0}u_j)-P_-(*D_{A_0}u_{k-1}\wedge *D_{A_0}u_{k-1}),$$

we can assume that $u_k=0$ on the half $S^1\times D_-^3$.

Next we replace the half $S^1\times D_-^3$ by End M. Then

$$\int_M \{\sum_{i=0}^4 |\nabla_{A_0}{}^{(i)}a|^2\}=\int_Q \{\sum_{i=0}^4 |\nabla_{A_0}{}^{(i)}a|^2\}<\infty .$$

Therefore by the argument in sections 5~9 of [9], it can be shown that A is self-dual.

REMARK 3. Reducible self-dual connections.

Let A be a reducible connection on an $SU(2)$-bundle over the space $S^1\times \boldsymbol{R}^3$. Then A reduces to an $S(U(1)\times U(1))$-connection, and the curvature F_A has the form $\begin{bmatrix} d\alpha & 0 \\ 0 & -d\alpha \end{bmatrix}$, where $d\alpha$ is a 2-form. Now a self-dual connection is a Yang-Mills connection. Therefore $d\alpha$ is harmonic. Since $H^2(S^1\times \boldsymbol{R}^3, \boldsymbol{R})=0$, $F_A=0$ and so the connection is flat. The moduli space of reducible flat connections is just the space $\mathrm{Hom}(\pi_1(S^1), S(U(1)\times U(1))/\mathrm{Ad}\, SU(2)$, which can be identified with a one dimensional manifold $U(1)/Z_2$, the upper hemicircle (c.f. the Lemmas 6.2 and 6.5 in [6]).

References

[1] Atiyah, M.F., Hitchin, N.J. and Singer, I.M., Self-duality in four-dimensional Riemannian geometry, Proc. R. Soc. London A. **362** (1978), 425-461.

[2] Dold, A., Halbexacte homotopie functoren, Lecture Notes in Math. 12, 1966.

[3] Donaldson, S.K. and Kronheimer, P.B., The geometry of four-manifolds, Oxford, 1990.

[4] Freed, D.S. and Uhlenbeck, K.K., Instantons and four manifolds, Springer, 1984.

[5] Harrington, B.J. and Shepard, H.K., Periodic euclidean solutions and the finite-temperature Yang-Mills gas, Phys. Review D.17.8 (1978), 2122-2125.

[6] Itoh, M., Self-dual Yang-Mills equations and Taubes' theorem, Tsukuba J. Math. **8**(1) (1984), 1-29.

[7] Matsunaga, H., Note on periodic instantons, Mem. Fac. Sci. Shimane Univ. **25** (1991), 1-5.

[8] Parker, T.H., Gauge theories on four dimensional Riemannian manifolds, Commun. Math. Phys. **85** (1982), 563-602.

[9] Taubes, C.H., Self-dual connections on non-self-dual 4-manifolds, J. Differential Geometry, 17 (1982), 139-170.

[10] Taubes, C. H., Self-dual connections on 4-manifolds with indefinite intersection matrix, J. Differential Geometry **19** (1984), 517-560.
[11] Taubes, C. H., Gauge theory on asymptotically periodic 4-manifolds, J. Differential Geometry **25** (1987), 363-430.

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Added in proof. In the proof of the lemma in Section 3, we assume that the manifold $Y=S^1\times S^2\times S^1$ is conformally flat, but this is not clear. Then we give here a correct proof. The twist map $S^1\times S^2\times S^1\rightarrow S^2\times S^1\times S^1$ is orientation preserving. We give on the projective line S^2 the Fubini-Study metric, which coincides with the metric obtained by the stereographic projection onto Euclidean plane R^2. On the other hand we introduce the flat metric on the torus $S^1\times S^1$. Then we obtain a product Kähler metric on the manifold Y and its scalar curvature is positive. Now we apply the Theorem 3.1 in [M. Itoh, On the moduli space of anti-self-dual Yang-Mills connections on Kähler surfaces, Publ. RIMS Kyoto Univ. **19**(1983), 15-32]. Then for each irreducible ASD connection on the Kähler surface Y $h^2=0$. Now we reverse the orientation on Y, then for the complex in the lemma the second cohomology vanishes.