TSUKUBA J. MATH.
Vol. 17 No. 2 (1993), 363–375

ON DILATION THEOREMS OF OPERATOR ALGEBRAS

Il Bong JUNG* and Bok Ja KIM**

1. Introduction.

Let $\mathcal{H}$ be a separable, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For a linear manifold $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$, a *form* on $\mathcal{A}$ is a linear functional on $\mathcal{A}$. For $x, y \in \mathcal{H}$, $x \otimes y$ denotes the form on $\mathcal{L}(\mathcal{H})$ defined by $x \otimes y(S) = (Sx, y)$, for any $S \in \mathcal{L}(\mathcal{H})$ (cf. [2]). An *elementary form* on a linear manifold $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$ is the restriction $x \otimes y | \mathcal{A}$ to $\mathcal{A}$ for $x, y \in \mathcal{H}$ (cf. [13]). It is well-known that there are several Hausdroff locally convex topologies on $\mathcal{L}(\mathcal{H})$ (cf. [9]). In particular, a *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the weak*-topology on $\mathcal{L}(\mathcal{H})$. The theory of dual algebra is closely related to the study of simultaneous equations of weak*-continuous elementary forms (cf. [1], [3], [7], and [10]). Recently several functional analysists have been studied systems of simultaneous equations of weak*-continuous elementary forms on a singly generated dual algebra (cf. [5]). This study has been applied to invariant subspaces, dilation theory, and reflexivity for contraction operators (cf. [5]). In particular, Bercovici-Foias-Pearcy (cf. [4]) obtained several dilation theorems of contraction operators. As a sequel study, Jung-Jo (cf. [12]) studied universal dilation theorems of a contraction operator with some properties. Moreover, M. Marsalli (cf. [13]) studied the dilation theory of general dual algebras with applications to the reflexivity.

This paper is a sequel study of those in [13]. In section 2, we introduce properties $(\tau_{m,n})$ which are concerned with the system of simultaneous equations of vector forms and obtain some related fundamental structure theorems. In section 3, we obtain some new dilation theorems of operator algebras with properties $(\tau_{m,n})$, which are applied to singly generated dual algebras. In section 4, we characterize properties $(\tau_{1,n})$ to dilations of operator algebras. Finally, in section 5, using these results, we obtain a dilation theorem of a contraction operator in the class $A_{1,n}$ which will be defined below and appeared frequently in the theory of dual algebras.

* The first author was partially supported by a Post-Doctoral Fellowship by KOSEF, 1991.
** The second author was partially supported by TGRC-KOSEF, 1991.
Received March 25, 1992.

2. Simultaneous equations of vector forms.

Throughout this paper $\mathcal{A}$ denotes a unital subalgebra of $\mathcal{L}(\mathcal{H})$ (note that the closedness of $\mathcal{A}$ is not considered). $\boldsymbol{N}$ denotes the set of natural numbers and $\boldsymbol{C}$ the complex plane. We write $\mathrm{Lat}\,\mathcal{A}$ for the set of all invariant subspaces for any operators in $\mathcal{A}$. Without confusion, a subspace of a Hilbert space means a norm-closed subspace. For $x\in\mathcal{H}$, $[\mathcal{A}x]$ denotes the subspace of $\mathcal{H}$ generated by a set $\mathcal{A}x=\{Ax\,|\,A\in\mathcal{A}\}$. We write

$$\mathcal{A}^{(n)}=\{A^{(n)}=\underbrace{A\oplus\cdots\oplus A}_{(n)}\,|\,A\in\mathcal{A}\} \tag{2.1}$$

which is called the n-th *ampliation* of $\mathcal{A}$. A subspace $\mathcal{L}$ of $\mathcal{H}$ is said to be *semi-invariant* for $\mathcal{A}$ if $P_{\mathcal{L}}AP_{\mathcal{L}}BP_{\mathcal{L}}=P_{\mathcal{L}}ABP_{\mathcal{L}}$, for any $A, B\in\mathcal{A}$, where $P_{\mathcal{L}}$ is the orthogonal projection onto $\mathcal{L}$. Note that a subspace $\mathcal{L}$ of $\mathcal{H}$ is semi-invariant for $\mathcal{A}$ if and only if there exist subspaces $\mathcal{M}$ and $\mathcal{N}$ in $\mathrm{Lat}\,\mathcal{A}$ such that $\mathcal{N}\subset\mathcal{M}$ and $\mathcal{L}=\mathcal{M}\ominus\mathcal{N}$ (i.e. $\mathcal{M}\cap\mathcal{N}^{\perp}$). Let $\mathcal{A}$ and $\mathcal{B}$ be subalgebras of $\mathcal{L}(\mathcal{H})$. Then $\mathcal{A}$ is said to be a *dilation* of $\mathcal{B}$ (and $\mathcal{B}$ is a *compression* of $\mathcal{A}$) if there exists a semi-invariant subspace $\mathcal{L}$ of $\mathcal{H}$ for $\mathcal{A}$ such that $P_{\mathcal{L}}\mathcal{A}P_{\mathcal{L}}$ $(=\{P_{\mathcal{L}}AP_{\mathcal{L}}: A\in\mathcal{A}\})=\mathcal{B}$, where $P_{\mathcal{L}}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{L}$. If $T\in\mathcal{L}(\mathcal{H})$ and $\mathcal{L}\subset\mathcal{H}$ is a semi-invariant subspace for $\{T\}$, we write $T_{\mathcal{L}}$ for the compression of $\{T\}$ to $\mathcal{L}$.

If there is no confusion, throughout this paper the topology τ is one of the following topologies; weak operator topology, operator-normed topology, strong operator topology, weak*-topology (or equivalently, ultra-weak operator topology), or ultra-strong operator topology on $\mathcal{L}(\mathcal{H})$. In particular, we write ω for the weak operator topology and ω^* for the weak*-topology on $\mathcal{L}(\mathcal{H})$.

The following definition should be compared with [5, Definition 2.01].

DEFINITION 2.1. Suppose m and n are any cardinal numbers with $1\leqq m, n\leqq\aleph_0$. A subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ has property $(\tau_{m,n})$ if for any system $\{\phi_{ij}\}_{0\leqq i<m,\,0\leqq j<n}$ of τ-continuous forms on $\mathcal{A}$, there exist $\{x_i\}_{0\leqq i<m}$ and $\{y_j\}_{0\leqq j<n}$ in $\mathcal{H}$ such that for $0\leqq i<m$, $0\leqq j<n$, $\phi_{ij}=x_i\otimes y_j$ on $\mathcal{A}$.

We recall from [5, Definition 2.01] that a dual algebra $\mathcal{A}$ has property $(\boldsymbol{A}_{m,n})$ if it has property $(\omega^*_{m,n})$. Recall from [5, Definition 9.13] that a dual algebra $\mathcal{A}$ has property $(\boldsymbol{B}_{m,n})$ if it has property $(\omega_{m,n})$.

The following statements come from some fundamental theorems in the theory of dual algebras.

PROPOSITION 2.2. *Suppose $\mathcal{A}$ is a subalgebra of $\mathcal{L}(\mathcal{H})$.*

(a) *Assume that the adjoint operation $\Phi(A)=A^*$ from $\mathcal{A}$ onto $\mathcal{A}^*(=\{A^* | A\in\mathcal{A}\})$ is continuous under the given topology τ in $\mathcal{L}(\mathcal{H})$. Suppose m and n are any cardinal numbers such that $1\leqq m$, $n\leqq\aleph_0$. Then $\mathcal{A}$ has property $(\tau_{m,n})$ if and only if $\mathcal{A}^*$ has property $(\tau_{n,m})$.*

(b) *If $\mathcal{M}$ is a τ-closed subalgebra with property $(\tau_{m,n})$ for some cardinal numbers m and n with $1\leqq m$, $n\leqq\aleph_0$ and $\mathcal{N}$ is a τ-closed subalgebra of $\mathcal{M}$, then $\mathcal{N}$ has property $(\tau_{m,n})$.*

(c) *If $\mathcal{A}$ has property $(\tau_{1,1})$, then an ampliation $\mathcal{A}^{(n)}$ has property $(\tau_{1,n})$ for any cardinal number n with $1\leqq n\leqq\aleph_0$.*

(d) *If $\mathcal{A}$ has property $(\tau_{1,n})$ for some cardinal number n with $1\leqq n\leqq\aleph_0$, then $\mathcal{A}^{(n)}$ has property $(\tau_{n,n})$.*

(e) *If $\mathcal{A}$ has property $(\tau_{1,1})$, then $\mathcal{A}^{(n^2)}$ has property $(\tau_{n,n})$ for any cardinal number n with $1\leqq n\leqq\aleph_0$.*

PROOF. (a) Let $\{\phi_{ji}\}_{0\leqq i<m,\,0\leqq j<n}$ be a system of τ-continuous forms on $\mathcal{A}^*$. Set $\varphi_{ij}=\bar{\phi}_{ji}\circ\Phi$, $0\leqq i<m$, $0\leqq j<n$, where $\bar{\phi}_{ji}(A)=\overline{\phi_{ji}(A)}$ for $A\in\mathcal{A}^*$. Then φ_{ij} is a τ-continuous form on $\mathcal{A}$, so that there exist sequences $\{x_i\}_{0\leqq i<m}$ and $\{y_j\}_{0\leqq j<n}$ in $\mathcal{H}$ such that $\varphi_{ij}=x_i\otimes y_j$. Moreover, it is easy to show that $\phi_{ji}=y_j\otimes x_i$, on $\mathcal{A}^*$, $0\leqq j<n$, $0\leqq i<m$.

(b) Let $\{\phi_{ij}\}_{0\leqq i\leqq m,\,0\leqq j<n}$ be a system of τ-continuous forms on $\mathcal{N}$. Since $\mathcal{A}$ is a locally convex space under the given topology τ, by [6, Proposition 14.13], there exists a system $\{\psi_{ij}\}_{0\leqq i<m,\,0\leqq j<n}$ of τ-continuous forms on $\mathcal{M}$ such that $\psi_{ij}|\mathcal{N}=\phi_{ij}$, $0\leqq i<m$, $0\leqq j<n$. Hence there exist x_i, $y_j\in\mathcal{H}$, $0\leqq i<m$, $0\leqq j<n$ such that $\psi_{ij}=x_i\otimes y_j$ on $\mathcal{M}$. So it follows trivially that $\phi_{ij}=x_i\otimes y_j$ on $\mathcal{N}$.

(c) Let $\{\phi_i\}_{0\leqq i<n}$ be a system of τ-continuous forms on $\mathcal{A}^{(n)}$. Define $\psi_i(A)=\phi_i(A^{(n)})$, for $A\in\mathcal{A}$, $0\leqq i<n$. In fact, it is not difficult to show that $A\to A^{(n)}$ is a τ-continuous linear map from $\mathcal{A}$ onto $\mathcal{A}^{(n)}$. Hence ψ_i is a τ-continuous form on $\mathcal{A}$. So there exist sequences $\{x_i\}_{0\leqq i<n}$ and $\{y_i\}_{0\leqq i<n}$ in $\mathcal{H}$ such that $\psi_i=x_i\otimes y_i$ on $\mathcal{A}$. Now we set

$$\tilde{x}=(\underbrace{x_0,\ x_1,\ \cdots}_{(n)}) \tag{2.2a}$$

and

$$\tilde{y}_i=(\overbrace{\underbrace{0,\ \cdots,\ 0}_{(i)},\ y_i,\ 0,\ \cdots}^{(n)}),\qquad 0\leqq i<n. \tag{2.2b}$$

Then it is easy to show that $\phi_i=\tilde{x}\otimes\tilde{y}_i$ on $\mathcal{A}^{(n)}$, $0\leqq i<n$.

(d) Let $\{\phi_{ij}\}_{0\leqq i,j<n}$ be a system of τ-continuous forms on $\mathcal{A}^{(n)}$. Define $\psi_{ij}(A)=\phi_{ij}(A^{(n)})$ for $A\in\mathcal{A}$, $0\leqq i,\ j<n$. Then ψ_{ij} is a τ-continuous form on $\mathcal{A}$.

By hypothesis, for fixed i with $0\leqq i<n$, there exist $x_i\in\mathcal{H}$ and $\{y_{ij}\}_{0\leqq j<n}$ in $\mathcal{H}$ such that $\phi_{ij}=x_i\otimes y_{ij}$. Now we set

$$\tilde{x}_i=(\overbrace{\underbrace{0,\cdots,0}_{(i)},x_i,0,\cdots}^{(n)}),\qquad 0\leqq i<n, \tag{2.3a}$$

and

$$\tilde{y}_j=(\overbrace{y_{0j},y_{1j},\cdots}^{(n)}),\qquad 0\leqq j<n. \tag{2.3b}$$

Then it is easy to show that $\phi_{ij}=\tilde{x}_i\otimes\tilde{y}_j$ on $\mathcal{A}^{(n)}$, $0\leqq i,j<n$.

(e) Since $(\mathcal{A}^{(n)})^{(n)}$ is identified with $\mathcal{A}^{(n^2)}$, by (c) and (d) this statement is proved.

3. Properties $(\tau_{m,n})$ and dilation theorems.

Suppose $m\in\boldsymbol{N}$. For a set $\{x_i\}_{i=1}^m$ in $\mathcal{H}$, we denote $[\mathcal{A}x_i]_{i=1}^m$ by the smallest subspace containing $\bigcup_{i=1}^m\{Ax_i\mid A\in\mathcal{A}\}$, i.e.,

$$[\mathcal{A}x_i]_{i=1}^m=\overline{\{\textstyle\sum_{i=1}^m A_ix_i\mid A_i\in\mathcal{A}\}}. \tag{3.1}$$

LEMMA 3.1. *Let x_i, $y_j\in\mathcal{H}$, $1\leqq i\leqq m$, $1\leqq j\leqq n$, m, $n\in\boldsymbol{N}$ and let P denote the orthogonal projection onto $[\mathcal{A}x_i]_{i=1}^m\ominus([\mathcal{A}x_i]_{i=1}^m\ominus[\mathcal{A}^*y_j]_{j=1}^n)$. Then we have*

(a) *$P\mathcal{H}$ is semi-invariant for $\mathcal{A}$,*

(b) *$(PAPu,v)=(APu,v)=(Au,v)$, where $A\in\mathcal{A}$, $u\in[\mathcal{A}x_i]_{i=1}^m$, $v\in[\mathcal{A}^*y_j]_{j=1}^n$,*

(c) *$PAPx_i=PAx_i$ for $A\in\mathcal{A}$, $i=1,\cdots,m$,*

(d) *if $(PAPx_i,v)=0$ for $A\in\mathcal{A}$, $v\in P\mathcal{H}$, $i=1,\cdots,m$, then $v=0$, and*

(e) *$\{\sum_{i=1}^m PA_iPx_i\mid A_i\in\mathcal{A}\}$ is dense in $P\mathcal{H}$.*

PROOF. (a) It is obvious that

$$P\mathcal{H}=[\mathcal{A}x_i]_{i=1}^m\ominus([\mathcal{A}x_i]_{i=1}^m\ominus[\mathcal{A}^*y_j]_{j=1}^n) \tag{3.2}$$

is semi-invariant for $\mathcal{A}$.

(b) In the proof of [13, Lemma 9], let us change $[\mathcal{A}x]$ to $[\mathcal{A}x_i]_{i=1}^m$, $[\mathcal{A}^*y]$ to $[\mathcal{A}^*y_j]_{j=1}^n$, x to x_i, $1\leqq i\leqq m$, and y to y_j, $1\leqq j\leqq n$. And if we follow the same way with [13, Lemma 9], we can prove (b).

(c) and (d). The proofs can be found in that of [13, Lemma 9(c)].

(e) Let $v\in P\mathcal{H}$. Then $v=v_1\oplus v_2$ for $v_1\in[P\mathcal{A}Px_i]_{i=1}^m$ and $v_2\in([P\mathcal{A}Px_i]_{i=1}^m)^\perp$. By (d) we have $v\in[P\mathcal{A}Px_i]_{i=1}^m$. Hence $P\mathcal{H}\subset[P\mathcal{A}Px_i]_{i=1}^m$. So (3.1) proves (e).

We denote $\mathcal{K}$ a separable, complex Hilbert space and $\mathcal{B}$ a unital subalgebra of $\mathcal{L}(\mathcal{K})$. For a (bounded) linear operator X, $\mathcal{D}(X)$ will denote the domain of X and $\mathcal{R}(X)$ the range of X.

THEOREM 3.2. *Assume that a unital subalgebra* $\mathcal{A}\subset\mathcal{L}(\mathcal{H})$ *has property* $(\tau_{m,n})$ *where* $m, n\in\boldsymbol{N}$. *Let* $\mathcal{B}$ *be a unital subalgebra of* $\mathcal{L}(\mathcal{K})$. *Suppose that* $\Phi: (\mathcal{A}, \tau)\to(\mathcal{B}, \omega)$ *is a continuous, surjective, homomorphism. Then for any sets* $\{u_i\}_{1\leq i\leq m}$ *and* $\{v_j\}_{1\leq j\leq n}$ *in* $\mathcal{K}$, *there exist* $\mathcal{M}, \mathcal{N}\in\mathrm{Lat}\,\mathcal{A}$ *with* $\mathcal{N}\subset\mathcal{M}$ *and a closed, injective linear transformation* $X: \mathcal{D}(X)\to\mathcal{M}\ominus\mathcal{N}$ *such that*

(a) *the linear manifold* $\mathcal{D}(X)$ *is dense in* $[\mathcal{B}u_i]_{i=1}^m\ominus([\mathcal{B}u_i]_{i=1}^m\ominus[\mathcal{B}^*v_j]_{j=1}^n)$,

(b) $\mathcal{R}(X)$ *is dense in* $\mathcal{M}\ominus\mathcal{N}$,

(c) $PAPXz=XQ\Phi(A)Qz$, *for* $A\in\mathcal{A}$, $z\in\mathcal{D}(X)$, *where* P *is the orthogonal projection onto* $\mathcal{M}\ominus\mathcal{N}$ *and* Q *is the orthogonal projection onto* $[\mathcal{B}u_i]_{i=1}^m\ominus([\mathcal{B}u_i]_{i=1}^m\ominus[\mathcal{B}^*v_j]_{j=1}^n)$, *and*

(d) $\{v_j\}_{j=1}^n\subset\mathcal{R}(X^*)$.

PROOF. The first idea of the proof comes from [4, Theorem 3.4]. If we define

$$\phi_{ij}(A)=(\Phi(A)u_i, v_j), \qquad \text{for} \quad A\in\mathcal{A},\ 1\leq i\leq m,\ 1\leq j\leq n, \tag{3.3}$$

then ϕ_{ij} is a τ-continuous form on $\mathcal{A}$. (Indeed, $A_\alpha\xrightarrow{\tau}A$ implies that $\Phi(A_\alpha)\xrightarrow{\omega}\Phi(A)$. Hence $(\Phi(A_\alpha)u_i, v_j)\to(\Phi(A)u_i, v_j)$, $1\leq i\leq m$, $1\leq j\leq n$). Since $\mathcal{A}$ has property $(\tau_{m,n})$, there exist $\{x_i\}_{1\leq i\leq m}$ and $\{y_j\}_{1\leq j\leq n}$ in $\mathcal{H}$ such that

$$\phi_{ij}(A)=(Ax_i, y_j) \qquad \text{for} \quad A\in\mathcal{A}. \tag{3.4}$$

Let $\mathcal{M}=[\mathcal{A}x_i]_{i=1}^m$ and $\mathcal{N}=[\mathcal{A}x_i]_{i=1}^m\ominus[\mathcal{A}^*y_j]_{j=1}^n$. Then by Lemma 3.1 (a) we obtain two orthogonal projections P and Q such that $P\mathcal{H}=\mathcal{M}\ominus\mathcal{N}$ is semi-invariant for $\mathcal{A}$ and

$$Q\mathcal{K}=[\mathcal{B}u_i]_{i=1}^m\ominus([\mathcal{B}u_i]_{i=1}^m\ominus[\mathcal{B}^*v_j]_{j=1}^n) \tag{3.5}$$

is semi-invariant for $\mathcal{B}$. Let us consider the correspondence

$$X_0: \sum_{i=1}^m Q\Phi(A_i)Qu_i\longrightarrow\sum_{i=1}^m PA_iPx_i \tag{3.6}$$

for any $A_i\in\mathcal{A}$, $1\leq i\leq m$. We shall show that X_0 is well-defined, one to one, linear transformation. Let $A_i\in\mathcal{A}$, $1\leq i\leq m$, and let

$$\sum_{i=1}^m Q\Phi(A_i)Qu_i=0. \tag{3.7}$$

For any $B_j\in\mathcal{B}$, $1\leq j\leq n$, by Lemma 3.1 (b) we have

$$(\sum_{i=1}^m\Phi(A_i)Qu_i, \sum_{j=1}^n B_j^*v_j)=0, \tag{3.8}$$

$$\text{i.e.,}\quad \sum_{i=1}^m\sum_{j=1}^n(B_j^*\Phi(A_i)Qu_i, v_j)=0.$$

Since Φ is surjective homomorphism, this is equivalent to that for any $S_j\in\mathcal{A}$,

$1 \leqq j \leqq n$,

(3.9) $\sum_{i=1}^{m} \sum_{j=1}^{n} (\Phi(S_j)\Phi(A_i)Qu_i, v_j)=0$,

i.e., $\sum_{i=1}^{m} \sum_{j=1}^{n} (\Phi(S_jA_i)Qu_i, v_j)=0$,

i.e., $\sum_{i=1}^{m} \sum_{j=1}^{n} (\Phi(S_jA_i)u_i, v_j)=0$ by Lemma 3.1(b),

i.e., $\sum_{i=1}^{m} \sum_{j=1}^{n} (S_jA_ix_i, y_j)=0$ by (3.3) and (3.4),

i.e., $\sum_{i=1}^{m} \sum_{j=1}^{n} (S_jA_iPx_i, y_j)=0$, by Lemma 3.1(b),

i.e., $(\sum_{i=1}^{m} A_iPx_i, \sum_{j=1}^{n} S_j^* y_j)=0$.

This means that $\sum_{i=1}^{m} PA_iPx_i=0$. Thus X_0 is well-defined, one-to-one, linear transformation.

We now show X_0 is closable and its closure X is injective. To do so, it suffices to show that if

(3.10a) $\sum_{i=1}^{m_k} Q\Phi(A_i^{(k)})Qu_i \longrightarrow u' (k \to \infty)$

and

(3.10b) $\sum_{i=1}^{m_k} PA_i^{(k)}Px_i \longrightarrow x' (k \to \infty)$,

then we have $u'=0$ if and only if $x'=0$. To show this, a similar method with the above proof is used. So we schetch only the proof. Indeed, $u'=0$ if and only if for any $S_j \in \mathcal{A}$, $1 \leqq j \leqq n$,

(3.11) $(u', \sum_{j=1}^{n} \Phi(S_j)^* v_j)=0$,

i.e., $\lim_k (\sum_{i=1}^{m_k} Q\Phi(A_i^{(k)})Qu_i, \sum_{j=1}^{n} \Phi(S_j)^* v_j)=0$,

i.e., $\lim_k (\sum_{i=1}^{m_k} PA_i^{(k)}Px_i, \sum_{j=1}^{n} S_j^* y_j)=0$,

i.e., $(x', \sum_{j=1}^{n} S_j^* y_j)=0$.

This is equivalent to $x'=0$. Therefore X_0 is closable and its closure X is one-to-one. Furthermore, it follows from Lemma 3.1(e) that $\mathcal{D}(X_0)$ is dense in $Q\mathcal{K}$. Hence $\mathcal{D}(X)$ is dense in $Q\mathcal{K}$. Since

(3.12) $\mathcal{R}(X_0)=\{\sum_{i=1}^{m} PA_iPx_i \mid A_i \in \mathcal{A}, 1 \leqq i \leqq m\}$

is dense in $\mathcal{M} \ominus \mathcal{N}$, by Lemma 3.1(e) $\mathcal{R}(X)$ is dense in $\mathcal{M} \ominus \mathcal{N}$. For $A, S_i \in \mathcal{A}$, $1 \leqq i \leqq m$, we have

$$\begin{aligned} PAPX_0(\sum_{i=1}^{m} Q\Phi(S_i)Qu_i) &= \sum_{i=1}^{m} PAPS_iPx_i \\ &= \sum_{i=1}^{m} PAS_iPx_i \\ &= X_0(\sum_{i=1}^{m} Q\Phi(AS_i)Qu_i) \end{aligned} \quad (3.13)$$

$$=X_0Q\Phi(A)Q(\sum_{i=1}^m Q\Phi(S_i)Qu_i).$$

This proves that

$$PAPX_0z=X_0Q\Phi(A)Qz \quad \text{for} \quad A\in\mathcal{A} \text{ and } z\in\mathcal{D}(X_0). \tag{3.14}$$

To show (c), we now suppose $z\in\mathcal{D}(X)$. Then there exists a sequence $\{z_n\}_{n=1}^\infty$ in $\mathcal{D}(X_0)$ such that $z_n\to z$ and $Xz_n\to Xz$. Let $A\in\mathcal{A}$. Then $Q\Phi(A)Qz_n\to Q\Phi(A)Qz$ and $PAPXz_n\to PAPXz$. Since $\{z_n\}_{n=1}^\infty\subset\mathcal{D}(X_0)$, by (3.14) we have $PAPXz_n=PAPX_0z_n=X_0Q\Phi(A)Qz_n$. Because $\{Q\Phi(A)Qz_n\}_{n=1}^\infty\subset\mathcal{D}(X_0)$, we have that $Q\Phi(A)Qz_n\to Q\Phi(A)Qz$ and $XQ\Phi(A)Qz_n\to PAPXz$. Since X is closed, $\Phi(A)Qz\in\mathcal{D}(X)$ and $XQ\Phi(A)Qz=PAPXz$.

Finally, we show (d). To do so, it suffices to show that for any $A_i\in\mathcal{A}$, $1\leqq i\leqq m$, $1\leqq j\leqq n$,

$$(X(\sum_{i=1}^m Q\Phi(A_i)Qu_i),\ y_j)=(\sum_{i=1}^m Q\Phi(A_i)Qu_i,\ v_j). \tag{3.15}$$

But by Lemma 3.1(b), we have

$$\begin{aligned}(X(\sum_{i=1}^m Q\Phi(A_i)Qu_i),\ y_j)&=(\sum_{i=1}^m PA_iPx_i,\ y_j)\\ &=(\sum_{i=1}^m A_ix_i,\ y_j)\\ &=(\sum_{i=1}^m \Phi(A_i)u_i,\ v_j) \text{ by (3.3) and (3.4)}\\ &=(\sum_{i=1}^m Q\Phi(A_i)Qu_i,\ v_j).\end{aligned} \tag{3.16}$$

Hence the proof is complete.

Recall that a set $\{e_i\}_{1\leqq i\leqq n}$ of vectors in a Hilbert space $\mathcal{K}$ is an *n-cyclic set* for an operator A in $\mathcal{L}(\mathcal{K})$ if $\mathcal{K}$ is the smallest invariant subspace for A containing $\{e_i\}_{1\leqq i\leqq n}$. For $T\in\mathcal{L}(\mathcal{K})$, $(\mathcal{A}_T, \tau)$ denotes a unital τ-closed subalgebra of $\mathcal{L}(\mathcal{K})$ generated by T under the given topology τ.

The following is a generalization of [11, Theorem 2.1] which was the main tool of the work [11]. Indeed, if τ is a weak*-topology on $\mathcal{L}(\mathcal{H})$, the following gives immediately [11, Theorem 2.1]. We shall recall some definitions about the theory of dual algebras in section 5.

COROLLARY 3.3. *Let $T\in\mathcal{L}(\mathcal{H})$. Suppose that $(\mathcal{A}_T, \tau)$ has property $(\tau_{m,n})$, where $m, n\in\boldsymbol{N}$. Suppose $A\in\mathcal{L}(\mathcal{K})$. Assume that $\Phi: (\mathcal{A}_T, \tau)\to(\mathcal{A}_A, \omega)$ is a continuous, surjective, homomorphism. If the operator A possesses an m-cyclic set $\{e_1, \cdots, e_m\}$ of vectors in $\mathcal{K}$ and its adjoint operator A^* has an n-cyclic set $\{f_1, \cdots, f_n\}$ of vectors in $\mathcal{K}$, then there exist invariant subspaces $\mathcal{M}$, $\mathcal{N}$ with $\mathcal{M}\supset\mathcal{N}$ and a closed, one-to-one, linear transformation $X: \mathcal{D}(X)\to\mathcal{M}\ominus\mathcal{N}$ such that*

(a) *the linear manifold $\mathcal{D}(X)$ is dense in $\mathcal{K}$ and contains $\{e_1, \cdots, e_m\}$,*

(b) *the range* $\mathcal{R}(X)$ *of* X *is dense* $\mathcal{M}\ominus\mathcal{N}$,

(c) $T_{\mathcal{M}\ominus\mathcal{N}}Xz=X\Phi(T)_z$, *for all* $z\in\mathcal{D}(X)$, *and*

(d) $\{f_j\}_{j=1}^n\subset\mathcal{R}(X^*)$.

PROOF. It follows from Theorem 3.2 that there exist invariant subspaces $\mathcal{M}$, $\mathcal{N}$ with $\mathcal{M}\supset\mathcal{N}$ and a closed, one-to-one, linear transformation X: $\mathcal{D}(X)\to\mathcal{M}\ominus\mathcal{N}$ such that (a), (b), (c) and (d) in Theorem 3.3. In particular, note from the proof of Theorem 3.2 that

$$\overline{\mathcal{D}(X)}=[\mathcal{A}_A e_i]_{i=1}^m\ominus([\mathcal{A}_A e_i]_{i=1}^m\ominus[\mathcal{A}_A^* f_j]_{j=1}^n). \tag{3.17}$$

But since A possesses an m-cyclic set $\{e_1, \cdots, e_m\}$ and A^* possesses an n-cyclic set $\{f_1, \cdots, f_n\}$, we have

$$\mathcal{K}=[\mathcal{A}_A e_i]_{i=1}^m=[\mathcal{A}_A^* f_j]_{j=1}^n. \tag{3.18}$$

Hence according to (3.17) and (3.18) we have $\overline{\mathcal{D}(X)}=\mathcal{K}$ and $\overline{\mathcal{R}(X)}=\mathcal{M}\ominus\mathcal{N}$. Furthermore, since $PTPXz=XQ\Phi(T)Qz$ for $z\in\mathcal{D}(X)$, where P and Q are projections in Theorem 3.2, it is easy to show that $T_{\mathcal{M}\ominus\mathcal{N}}Xz=X\Phi(T)z$ for all $z\in\mathcal{D}(X)$. Hence the proof is complete.

4. Properties $(\tau_{1,n})$ and dilation theorems.

The following theorem is a generalization of [13, Theorem 11].

THEOREM 4.1. *Suppose that* $\mathcal{A}$ *is a unital subalgebra of* $\mathcal{L}(\mathcal{H})$. *Then* $\mathcal{A}$ *has property* $(\omega_{1,m})$ *for* $m\in\boldsymbol{N}$ *if and only if for every* $n\in\boldsymbol{N}$ *and every pair* $\tilde{x}$, $\tilde{y}_p\in\mathcal{H}^{(mn)}$, $1\leqq p\leqq m$, *there exist* $\mathcal{M}$, $\mathcal{N}\in\mathrm{Lat}\mathcal{A}$ *with* $\mathcal{N}\subset\mathcal{M}$ *and a linear transformation* X: $\mathcal{D}(X)\to\mathcal{M}\ominus\mathcal{N}$ *such that*

(a) $PAPXQ\tilde{x}=XQA^{(mn)}Q\tilde{x}$, *for any* $A\in\mathcal{A}$, *where* Q *is the orthogonal projection onto* $[\mathcal{A}^{(mn)}\tilde{x}]\ominus([\mathcal{A}^{(mn)}\tilde{x}]\ominus[\mathcal{A}^{(mn)*}\tilde{y}_p]_{p=1}^m)$, *and* P *is the orthogonal projection onto* $\mathcal{M}\ominus\mathcal{N}$,

(b) $\mathcal{D}(X)$ *is dense in* $[\mathcal{A}^{(mn)}\tilde{x}]\ominus([\mathcal{A}^{(mn)}\tilde{x}]\ominus[\mathcal{A}^{(mn)*}\tilde{y}_p]_{p=1}^m)$,

(c) $\mathcal{R}(X)$ *is dense in* $\mathcal{M}\ominus\mathcal{N}$, *and*

(d) $\{\tilde{y}_p\}_{p=1}^n\subset\mathcal{R}(X^*)$.

PROOF. Suppose that $\mathcal{A}\subset\mathcal{L}(\mathcal{H})$ has property $(\omega_{1,m})$. If we define Φ: $(\mathcal{A}, \omega)\to(\mathcal{A}^{(mn)}, \omega)$ by $\Phi(T)=T^{(mn)}$, then it is easy to show that Φ is a homeomorphic algebra isomorphism. Then Φ satisfies the hypothesis in Theoroem 3.2. Hence we can use Theorem 3.2. Since Φ preserves the identity operator, applying (3.13) we have that $PAPXQ\tilde{x}=XQ\Phi(A)Q\tilde{x}$ for $A\in\mathcal{A}$. And (b), (c) and (d) are obvious by Theorem 3.2 (b), (c) and (d).

Conversely, given a system of weakly continuous forms $\{\phi_p\}_{1\leq p\leq m}$ on $\mathcal{A}$, by [5, Proposition 1.7] there exist $x_p^{(1)}, x_p^{(2)}, \cdots, x_p^{(n)}$ and $y_p^{(1)}, y_p^{(2)}, \cdots, y_p^{(n)}$ in $\mathcal{H}$ such that

(4.1) $$\phi_p(A)=\sum_{i=1}^{n}(Ax_p^{(i)}, y_p^{(i)}), \quad \text{for all } A\in\mathcal{A}.$$

Let us set

(4.2a) $$\tilde{x}=(x_1^{(1)}, \cdots, x_1^{(n)}, \cdots, x_p^{(1)}, x_p^{(2)}, \cdots, x_p^{(n)}, \cdots, x_m^{(1)}, \cdots, x_m^{(n)})$$

and

(4.2b) $$\tilde{y}_p=(\underbrace{0, \cdots, 0}_{n(p-1)}, y_p^{(1)}, y_p^{(2)}, \cdots, y_p^{(n)}, 0, \cdots, 0)$$

in $\mathcal{H}^{(mn)}$. Then we have

(4.3) $$\phi_p(A)=(A^{(mn)}\tilde{x}, \tilde{y}_p).$$

By the hypothesis, note that there exists a linear transformation X satisfying (a), (b), (c) and (d). Let $x'=XQ\tilde{x}$ and choose y'_p such that $\tilde{y}_p=X^*y'_p$, $1\leq p\leq m$. Then it follows from Lemma 3.1 and (4.3) that for $1\leq p\leq m$ we have

(4.4)
$$\begin{aligned}\phi_p(A)&=(A^{(mn)}\tilde{x}, \tilde{y}_p) \quad \text{by (4.3)}\\ &=(QA^{(mn)}Q\tilde{x}, \tilde{y}_p) \quad \text{by Lemma 3.1}\\ &=(XQA^{(mn)}Q\tilde{x}, \tilde{y}_p)\\ &=(PAPXQ\tilde{x}, \tilde{y}_p)\\ &=(PAPx', y'_p)\\ &=(APx', Py'_p).\end{aligned}$$

Hence $\mathcal{A}$ has property $(\omega_{1,m})$ and the proof is complete.

For a unital subalgebra $\mathcal{A}\subset\mathcal{L}(\mathcal{H})$, we write $\tilde{\mathcal{H}}\equiv\sum\oplus_{i=1}^{\infty}\mathcal{H}_i$ and $\tilde{A}\equiv\sum\oplus_{i=i}^{\infty}A_i$, where $\mathcal{H}_i=\mathcal{H}$ and $A_i=A\in\mathcal{A}$, $1\leq i<\infty$. And we denote $\tilde{\mathcal{A}}=\{\tilde{A}: A\in\mathcal{A}\}$.

THEOREM 4.2. *Assume that $\mathcal{A}$ is a unital subalgebra of $\mathcal{L}(\mathcal{H})$. Suppose $m\in\boldsymbol{N}$. Then $\mathcal{A}$ has property $(\omega_{1,m}^*)$ if and only if for every $\tilde{x}, \tilde{y}_p\in\tilde{\mathcal{H}}^{(m)}$, $1\leq p\leq m$, there exist $\mathcal{M}, \mathcal{N}\in\mathrm{Lat}\mathcal{A}$ with $\mathcal{N}\subset\mathcal{M}$ and a linear transformation $X: \mathcal{D}(X)\to\mathcal{M}\ominus\mathcal{N}$ such that*

(a) *$PAPXQ\tilde{x}=XQ\tilde{A}^{(m)}Q\tilde{x}$, for $A\in\mathcal{A}$, where Q is the orthogonal projection onto $[\tilde{\mathcal{A}}^{(m)}\tilde{x}]\ominus([\tilde{\mathcal{A}}^{(m)}\tilde{x}]\ominus[\tilde{\mathcal{A}}^{(m)*}\tilde{y}_p]_{p=1}^{m})$ and P is the orthogonal projection onto $\mathcal{M}\ominus\mathcal{N}$,*

(b) *$\mathcal{D}(X)$ is dense in $[\tilde{\mathcal{A}}^{(m)}\tilde{x}]\ominus([\tilde{\mathcal{A}}^{(m)}\tilde{y}_p]\ominus[\tilde{\mathcal{A}}^{(m)*}\tilde{y}_p]_{p=1}^{m})$,*

(c) $\mathcal{R}(X)$ *is dense in* $\mathcal{M}\ominus\mathcal{N}$, *and*

(d) $\{\tilde{y}_p\}_{n=1}^{m}\subset\mathcal{R}(X^*)$.

PROOF. Suppose that $\mathcal{A}$ has property $(\omega_{1,m}^*)$. Note that $\Phi:(\mathcal{A},\omega^*)\to(\tilde{\mathcal{A}},\omega)$ defined by $\Phi(A)=\tilde{A}$ is a homeomorphic algebra isomorphism. Since Φ satisfies the hypothesis in Theroem 3.2, it holds.

Conversely, given a system of weak*-continuous forms $\{\phi_p\}_{1\leq p\leq m}$, there exist $x_p^{(1)}, x_p^{(2)}, \cdots$ and $y_p^{(1)}, y_p^{(2)}, \cdots$ in $\mathcal{H}$ such that

$$\phi_p(A)=\sum_{i=1}^{\infty}(Ax_p^{(i)}, y_p^{(i)}), \quad \text{for} \quad A\in\mathcal{A} \tag{4.5a}$$

with

$$\sum_{i=1}^{\infty}\|x_p^{(i)}\|^2<\infty \quad \text{and} \quad \sum_{i=1}^{\infty}\|y_p^{(i)}\|^2<\infty. \tag{4.5b}$$

We now set

$$\tilde{x}=(x_1^{(1)}, \cdots, x_m^{(1)}, x_1^{(2)}, \cdots, x_m^{(2)}, \cdots) \tag{4.6a}$$

and

$$\tilde{y}_p=(\underbrace{0, \cdots, 0}_{(p-1)}, y_p^{(1)}, \underbrace{0, \cdots, 0}_{(m-1)}, y_p^{(2)}, \underbrace{0, \cdots, 0}_{(m-1)}, y_p^{(3)}, 0, \cdots) \tag{4.6b}$$

for $1\leq p\leq m$. Then it is not difficult to show that $\phi_p(A)=(\tilde{A}\tilde{x}, \tilde{y}_p)$ for $1\leq p\leq m$. Hence it follows from the hypothesis that for $\tilde{x}$ and $\tilde{y}_p$, $1\leq p\leq m$, there exists a linear transformation X satisfying (a), (b), (c) and (d). Let $x'=XQ\tilde{x}$ and choose y_p' such that $\tilde{y}_p=X^*y_p'$, $1\leq p\leq m$. Then by the method of (4.4) we can prove that $\phi_p(A)=(APx', Py_p')$ for $1\leq p\leq m$, $A\in\mathcal{A}$. Hence $\mathcal{A}$ has property $(\omega^*_{1,m})$ and the proof is complete.

5. Applications to dual operator algebras.

Let $\boldsymbol{D}$ be the open unit disc in the complex plane $\boldsymbol{C}$ and let $\boldsymbol{T}$ be the boundary of $\boldsymbol{D}$. The space $L^p=L^p(\boldsymbol{T})$, $1\leq p\leq\infty$, is the usual Lebesgue function space relative to normalized Lebesgue measure on $\boldsymbol{T}$. In particular, we denote by $H^p=H^p(\boldsymbol{T})$, $1\leq p\leq\infty$, the Hardy space. Throughout this section $\mathcal{H}$ is an infinite dimensional separable, complex Hilbert space. A contration $T\in\mathcal{L}(\mathcal{H})$ (i.e., $\|T\|\leq 1$) is *absolutely continuous* if in the canonical decomposition $T=T_1\oplus T_2$, where T_1 is a unitary operator and T_2 is a completely nonunitary contraction, T_1 is either absolutely continuous or acts on the space (0) (cf. [14]).

Let T be an absolutely continuous contration in $\mathcal{L}(\mathcal{H})$ and let $(\mathcal{A}_T, \omega^*)$ be the dual algebra generated by T. Then it follows from Foias-Sz.-Nagy func-

tional calculus [5, Theorem 4.1] that there is an algebra homomorphism Φ: $H^\infty \to (\mathcal{A}_T, \omega^*)$ defined by $\Phi(f)=f(T)$ for every f in H^∞. The mapping Φ is norm-decreasing weak*-continuous and the range of Φ is weak*-dense in $\mathcal{A}_T$. Recall that $\boldsymbol{A}=\boldsymbol{A}(\mathcal{H})$ is the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which Φ is an isometry. If m and n are any cardinal numbers such that $1\leqq m$, $n\leqq\aleph_0$, we write $\boldsymbol{A}_{m,n}=\boldsymbol{A}_{m,n}(\mathcal{H})$ for the set of all T in $\boldsymbol{A}(\mathcal{H})$ such that the singly generated dual algebra $(\mathcal{A}_T, \omega^*)$ has property $(\omega^*_{m,n})$(cf. [5]).

For $T\in\boldsymbol{A}$ and any nonnegative integer n, we define a linear functional

$$C_T^{(n)} : (\mathcal{A}_T, \omega^*) \longrightarrow \boldsymbol{C} \tag{5.1a}$$

by

$$C_T^{(n)}h(T)=\hat{h}(n) \quad \text{for every } h\in H^\infty, \tag{5.1b}$$

where $\hat{h}(n)$ is the n-th Fourier coefficient of h (cf. [8]).

LEMMA 5.1. *If $T\in\boldsymbol{A}$, then $C_T^{(n)}$ is a ω^*-continuous form on $(\mathcal{A}_T, \omega^*)$ with $\|C_T^{(n)}\|=1$ for any nonnegative integer n.*

PROOF. Note that $\mathcal{A}_T=\{h(T)\,|\,h\in H^\infty\}$. Let $h_\alpha(T)$ be a net in $\mathcal{A}_T$ converging to $h(T)$ under the ω^*-topology on $\mathcal{L}(\mathcal{H})$. Since $T\in\boldsymbol{A}$, it is obvious that $\|h_\alpha-h\|_\infty\to 0$. Hence we have that $|\hat{h}_\alpha(n)-\hat{h}(n)|\to 0$ for any nonnegative integer n.

Moreover, by the definition of the Fourier coefficient of $h\in H^\infty$, it follows easily that $\|C_T^{(n)}\|=1$.

THEOREM 5.2. *Suppose that $T\in\boldsymbol{A}_{n,1}(\mathcal{H})$, $1\leqq n\leqq\aleph_0$, and T has a cyclic vector in $\mathcal{H}$. Then there exist semi-invariant subspaces $\mathcal{K}$ and $\mathcal{K}_*$ for T and T^* (respectively) with* $\dim\mathcal{K}\geqq n$ *and a closed, one-to-one, linear transformation X: $\mathcal{D}(X)\to\mathcal{K}$ such that*

(a) *$\mathcal{D}(X)$ is dense in $\mathcal{K}$,*

(b) *$\mathcal{R}(X)$ is dense in $\mathcal{K}_*$, and*

(c) *$T^*_{\mathcal{K}_*}Xz=XT_{\mathcal{K}}z$ for any $z\in\mathcal{D}(X)$.*

PROOF. If we define Φ: $\mathcal{A}_{T^*}\to\mathcal{A}_T$ by $h(T^*)\mapsto h(T)$ for any $h\in H^\infty$, then $\Phi=\Phi_T\circ\Phi_{T^*}^{-1}$ is a weak*-continuous isometric isomorphism. Note that $T^*\in\boldsymbol{A}_{1,n}$. And consider ω^*-continuous linear form $C_{T^*}^{(i)}$, $0\leqq i<n$ (see Lemma 5.1). Then there exist z, t_j, $0\leqq j<n$, in $\mathcal{H}$ such that

$$C_{T^*}^{(j)}=z\otimes t_j \quad \text{on } \mathcal{A}_{T^*}. \tag{5.2}$$

Let $\mathcal{K}=[\mathcal{A}_{T*}z]$ and let x be a cyclic vector for T. Then $\mathcal{K}=[\mathcal{A}_T x]\ominus([\mathcal{A}_T x]\ominus[\mathcal{A}_{T*}z])$ is a semi-invariant subspace for T. Moreover, we have

$$(5.3)\qquad (T^{*k}z,\ t_j)=(T^{*k}z,\ P_{\mathcal{K}}t_j)\qquad \text{for}\quad k=0,\ 1,\ \cdots,\ 0\leqq j<n$$

which implies that

$$(5.4)\qquad z\otimes t_j=z\otimes P_{\mathcal{K}}t_j\qquad \text{on } \mathcal{A}_{T*}.$$

For a simple notation, let us set $\tilde{t}_j=P_{\mathcal{K}}t_j$. Then by (5.2) and (5.4) we have

$$(5.5)\qquad C_{T*}^{(j)}=z\otimes\tilde{t}_j\qquad \text{on } \mathcal{A}_{T*}.$$

Hence by (5.5) and (5.1b) we have

$$(5.6)\qquad (T^{*k+1}z,\ \tilde{t}_0)=C_{T*}^{(0)}(T^{*k+1})=0\qquad \text{for } k=0,\ 1,\ \cdots.$$

So $(f(T^*)z,\ (T\mid\mathcal{K})\tilde{t}_0)=0$ for all $f\in H^\infty$. Since $\{f(T^*)z\mid f\in H^\infty\}$ is dense in $\mathcal{K}$, $(T\mid\mathcal{K})\tilde{t}_0=0$. Also

$$(5.7)\qquad (T^{*m}z,\ (T\mid\mathcal{K})\tilde{t}_j)=(T^{*m+1}z,\ \tilde{t}_j)=C_{T*}^{(j)}(T^{*m+1})$$
$$=\begin{cases}1, & \text{if } j=m+1\\ 0, & \text{otherwise.}\end{cases}$$

So $(T\mid\mathcal{K})\tilde{t}_0\neq 0$. Continuing this process, we obtain a strictly increasing set-sequence $\{\mathrm{Ker}(T\mid\mathcal{K})^i\}_{i=1}^n$ of subsets in $\mathcal{K}$. Hence it implies that $\dim\mathcal{K}\geqq n$.

To use Theorem 4.1, consider $x,\ z\in\mathcal{K}$. Then there exist $\mathcal{M},\ \mathcal{N}\in\mathrm{Lat}(T)$ with $\mathcal{N}\subset\mathcal{M}$ and a closed, one-to-one, linear transformation $X:\ \mathcal{D}(X)\to\mathcal{M}\ominus\mathcal{N}$ such that $\mathcal{D}(X)$ is dense in $[\mathcal{A}_{T*}z]=\mathcal{K}$ and $\mathcal{R}(X)$ is dense in $\mathcal{M}\ominus\mathcal{N}$. Put $\mathcal{M}\ominus\mathcal{N}=\mathcal{K}_*$. Then by Theorem 4.1 it is obvious that $T_{\mathcal{K}_*}^*Xz=XT_{\mathcal{K}}z$ for any $z\in\mathcal{D}(X)$. Hence the proof is complete.

REMARK. Applying Theorem 4.2, we can obtain a characterization of memberships for the classes $\boldsymbol{A}_{1,n}$.

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Current Adress:
Dept. of Mathematics and Statistics
University of Edinburgh
Edinburgh, EH9 3JZ, U. K.

Permament Adress:
Dept. of Mathematics
College of Natural Sciences
Kyungpook National University
Taegu 702-701, Korea

Dept. of Mathematics
College of Natural Sciences
Kyungpook National university
Taegu 702-701, Korea