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UNIVERSAL SPACES FOR SONE FAMILIES OF RIM-SCATTERED SPACES

By

S.D. ILIADIS

1. Introduction.

1.1. Definitions and notations. All spaces considered in this paper are separable and metrizable and the ordinals are countable.

Let F be a subset of a space X. By $Bd(F)$, $Cl(F)$, $Int(F)$ and $|F|$ we denote the boundary, the closure, the interior and the cardinality of F, respectively. An open (respectively, closed) subset U of X' is called *regular* iff $U=Int(Cl(U))$ (respectively, $U=Cl(Int(U))$). If X is a metric space, then the diameter of F is denoted by $diam(F)$. A map f of a space X into a space Y is called *closed* iff the subset $f(F)$ of Y is closed for every closed subset F of X.

A *compactum* is a compact metrizable space; a *continuum* is a connected compactum. A space is said to be *scattered* iff every non-empty subset has an isolated point.

A space Y is said to be an *extension* of X iff X is a dense subset of Y. A space Y is said to be a *compactification* of X iff Y is a compact extension of X. Let Y and Z be extensions of X. A map π of Y into Z is called a *natural projection* iff $\pi(x)=x$ for every $x\in X$. Obviously, if there exist a natural projection of Y into Z, then it is uniquely determined.

A space T is said to be *universal* for a family A of spaces iff both the following conditions are satisfied: (α) $T\in A$, (β) for every $X\in A$, there exists an embedding of X in T. If ony condition (β) is satisfied, then T is called a *containing space* for a family A.

A *partition* of a space X is a set D of closed subsets of X such that (α) if $F_1, F_2\in D$ and $F_1\neq F_2$, then $F_1\cap F_2=\emptyset$, and (β) the union of all elements of D is X. The *natural projection* of X onto D is the map π defined as follows, if $x\in X$, then $\pi(x)=F$, where F is the uniquely determined element of D containing x. The *quotient space* of the partition D is the set D with a topology which is the maximal on D for which the map π is continuous. (We observe that we use the same notation for a partition of aspace and for the correspond-

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ing quotient space). The partition D is called *upper semi—continuous* iff for every $F\in D$ and for every open subset U of X containing F there exists an open subset V of X which is union of elements of D such that $F\subseteqq V\subseteqq U$.

Obviously, in order to define a oartition D of a space X it is sufficient to define the non-degenerate elements of D. Let D' be a subset of D (generally, let D' be a set of subsets of a space X). We denote by $(D')^*$ the union of all elements of D'.

An ordinal α is called *isolated* iff it has the form $\beta+1$, where β is an ordinal. A non-isolated ordinal is called a limit ordinal (hence, the ordinal zero is a limit ordinal).

Every ordinal α is uniquely represented as the union of a limit ordinal β and of a non-negative integer m. In what follows, the ordinal β is denoted by $\beta(\alpha)$ and the integer m is denoted by $m(\alpha)$. Also, by $\gamma(\alpha)$ we denote the ordinal $\beta+2m+min\{\beta, 1\}$ and by $m^{+}(\alpha)$ we denote the integer $m+min\{\beta, 1\}$. The set $\{0, 1, \cdots\}$ is denoted by N.

Let M be a subset of a space X. For every ordinal α we define, by induction, a subset $M^{(\alpha)}$ of M as follows: $M^{(0)}=M$, $M^{(1)}$ is the set of all limit points of M in M. $M^{(\alpha)}=(M^{(\alpha-1)})^{(1)}$ if $\alpha>1$ is an isolated ordinal and $M^{(\alpha)}=\bigcap_{\beta<\alpha} M^{(\beta)}$ if $\alpha>1$ is a limit ordinal. The set $M^{(\alpha)}$ is called *α—derivative of* M (See $[K_2]$, v.I, § 24. IV).

We say that M has *type* $\leqq\alpha$, and we write $type(M)\leqq\alpha$ iff $M^{(\alpha)}=\emptyset$. If α is the least such ordinal, we say that M has *type* α, and we write $type(M)=\alpha$. Obviously, $type(M)=0$ iff $M=\emptyset$.

We say that a scattered subset M has *type* α (respectively, $\leqq\alpha$) *at the point* $a\in M$ and we write $type(a, M)=\alpha$ (respectively, $type(a, M)\leqq\alpha$) iff $a\notin M^{(\alpha)}$ and $a\in M^{(\beta)}$ for every $\beta<\alpha$ (respectively, $a\notin M^{(\alpha)}$). (See $[I_3]$).

We denote by $com\text{-}type(a, M)$ (*compact type of* M *at the point* a) the minimal ordinal γ for which there exists a compactification K of M such that $type(a, K)=\gamma$. (See [I-Z]). By $max(M)$ we denote the set of all points a of M for which $com-type(x, M)\leqq com-type(a, M)$ for every $x\in M$.

We say that M has *locally compact type* γ (respectively, *compact type* γ) which is denoted by $loc\text{-}com\text{-}type(M)$ (respectively, by $com\text{-}type(M)$) iff γ is the minimal ordinal for which there exists a locally compact extension of M (respectively, a compactification of M) having type γ. (See [I-Z]).

We observe that:

(1) A subset M of a space X is scattered iff there exists an ordinal α such that $type(M)\leqq\alpha$.

(2) Every scattered space is countable.

(3) A compactum is scattered iff it is countable.

(4) The type of a non-empty countable compactum is an isolated ordinal.

(5) There exist compacta having type α for every isolated ordinal α. (See [M-S]).

(6) The number of compacta having type α, where α is an ordinal, is countable. (See [M-S]).

We denote by L_n, $n=1, 2, \cdots$, the set of all ordered n-tuples $i_1 \cdots i_n$, where $i_t=0$ or $1, t=1, \cdots, n$. Also, we set $L_0=\{\emptyset\}$ and $L=\bigcup_{n=0}^{\infty} L_n$. For $n=0$, by $i_1 \cdots i_n$ we denote the element $\emptyset$ of L. We say that the element $i_1 \cdots i_n$ of L is a *part* of the element $j_1 \cdots j_m$ and we write $i_1 \cdots i_n \leqq j_1 \cdots j_m$ if either $n=0$, or $n \leqq m$ and $i_t=j_t$ for every $t \leqq n$. The elements of L are also denoted by $\bar{i}, \bar{j}, \bar{i}_1$, etc. If $\bar{i}=i_1 \cdots i_n$ then by $\bar{i}0$ (respectively, $\bar{i}1$) we denote the element $i_1 \cdots i_n 0$ (respectively, $i_1 \cdots i_n 1$) of L.

We denote by Λ_n, $n=1, 2, \cdots$, the set of all ordered n-tuples $i_1 \cdots i_n$, where i_t, $t=1, \cdots, n$, is a positive integer. We set $\Lambda=\bigcup_{n=1}^{\infty} \Lambda_n$. The elements of Λ are denoted by $\bar{\alpha}$, $\bar{\beta}$, etc. Let $\bar{\alpha}=i_1 \cdots i_n$ and $\bar{\beta}=j_1 \cdots j_m$. We say that $\bar{\alpha}$ is a *part* of $\bar{\beta}$ and we write $\bar{\alpha} \leqq \bar{\beta}$ iff $1 \leqq n \leqq m$ and $i_t=j_t$ for every $t \leqq n$. Obviously, if $\bar{\alpha}$, $\bar{\beta} \in \Lambda_n$ and $\bar{\alpha} \leqq \bar{\beta}$ then $\bar{\alpha}=\bar{\beta}$. Also, for every $\bar{\alpha} \in \Lambda_n$ the set of all elements $\bar{\beta} \in \Lambda_{n+1}$ such that $\bar{\alpha} \leqq \bar{\beta}$, is a countable non-finite set.

We denote by C the Cantor ternary set. By $C_{\bar{i}}$, where $\bar{i}=i_1 \cdots i_n \in L$, $n \geqq 1$, we denote the set of all points of C for which the t^{th} digit in the ternary expansion, $t=1, \cdots, n$, coincides with 0 if $i_t=0$ and with 2 if $i_t=1$. Also, we set $C_{\emptyset}=C$. For every subset s of L_n, $n=0, 1, \cdots$, we set $C_s=\bigcup_{\bar{i} \in s} C_{\bar{i}}$. For every point a of C and for every integer $n \geqq 0$, by $\bar{i}(a, n)$ we denote the uniquely determined element $\bar{i} \in L_n$ for which $a \in C_{\bar{i}}$. For every subset F of C and for every integer $n \geqq 0$, we denote by $st(F, n)$ the union of all sets $C_{\bar{i}}$, $\bar{i} \in L_n$, such that $C_{\bar{i}} \cap F \neq \emptyset$. If $F=\{a\}$ we set $st(F, n)=st(a, n)$. Obviously, $st(a, n)=C_{\bar{i}(a, n)}$. If S is a subset of C, then the set $S \cap C_{\bar{i}}$ is denoted by $S_{\bar{i}}$.

Let D be a partition of a subset S of C, $\bar{i}$ an element of L_n, $n=0, 1, \cdots$. We set $D(1)=\{d \in D: d$ is not singletion$\}$, $D_{\bar{i}}=\{d \in D: d \cap C_{\bar{i}0} \neq \emptyset,\ d \cap C_{\bar{i}1} \neq \emptyset$ and $d \subseteq C_{\bar{i}0} \cup C_{\bar{i}1}\}$, $D_n=\bigcup_{\bar{i} \in L_n} D_{\bar{i}}$. It is easy to see that: (α) $D(1)=\bigcup_{n=0}^{\infty} D_n$, ($\beta$) $D_{\bar{i}} \cap D_{\bar{j}}=\emptyset$ if $\bar{i}, \bar{j} \in L$ and $\bar{i} \neq \bar{j}$ and (γ) $D_m \cap D_n=\emptyset$ if $m \neq n$.

A space X is called *rim-finite* (respectively, *rational*) iff X has a basis B of open sets such that the set $Bd(U)$ is finite (respectively, countable) for every $U \in B$.

We say that a space X has *rim-type* $\leqq \alpha$, where α is an ordinal and we write *rim-type*$(X) \leqq \alpha$ iff X has a basis B of open sets such that *type*$(Bd(U))$

$\leqq \alpha$, for every $U \in B$. If α is the least such ordinal, then we say that X has *rim-type* α, and we write *rim-type*$(X)=\alpha$.

In [G-I] (respectively, in [I_2] and [I_3]) the following definition is given: a space K has the *property of α-intersections* (respectively, the *property of finite intersections*) *with respect to a family Sp of spaces* iff the every $X \in Sp$ there exists a homeomorphism i_X of X in K such that if Y and Z are distinct elements of Sp, then the set $i_Y(Y) \cap i_Z(Z)$ has type $\leqq \alpha$ (respectively, the set $i_Y(Y) \cap i_Z(Z)$ is finite) (For the corresponding definitions of the present paper see Section 5.1).

1.2. Some known results. Let $\alpha > 0$ be an ordinal. We denote by $R(\alpha)$ the family of all spaces having *rim-type* $\leqq \alpha$. Natural subfamilies of $R(\alpha)$ are the family $R^{com}(\alpha)$ of all compact elements of $R(\alpha)$ and the family $R^{cont}(\alpha)$ of all elements of $R(\alpha)$ which are continua.

Another subfamily of $R(\alpha)$ is the family $R^{rim\text{-}com}(\alpha)$ defined as follows an element X of $R(\alpha)$ belongs to $R^{rim\text{-}com}(\alpha)$ iff X has a basis B of open sets such that for every $U \in B$, the set $Bd(U)$ is a compactum having type $\leqq \alpha$.

We denote by RF the family of all rim-finite spaces and by R the family of all rational spaces.

In [I-Z] some new subfamilies of $R(\alpha)$ are given. These families are denoted by $R_c^k(\alpha)$ and $R_{lc}^k(\alpha)$, $\alpha > 0$, $k = 0, 1, \cdots$. A space X belongs to $R_{lc}^k(\alpha)$ (respectively, to $R_c^k(\alpha)$) iff X has a basis $B = \{U_0, U_1, \cdots\}$ of open sets such that *type*$(Bd(U_i)) \leqq \alpha$ and *loc-com-type*$(Bd(U_i)) \leqq \alpha$ (respectively, *com-type*$(Bd(U_i)) \leqq \alpha$), for every $i = 0, 1, \cdots$.

It is easy to see that $R^{cont}(\alpha) \subseteqq R^{com}(\alpha) \subseteqq R^{rim\text{-}com}(\alpha) \subseteqq R_c^0(\alpha) \subseteqq \cdots \subseteqq R_c^k(\alpha) \subseteqq R_{lc}^k(\alpha) \subseteqq R_c^{k+1}(\alpha) \subseteqq \cdots \subseteqq R(\alpha)$.

We observe that if *type*$(M)=\alpha$, then by Lemma 1 of [I-T] it follows that M admits a compactification K having type $\leqq \gamma(\alpha)$. By the proof of this lemma it follows that if $\alpha > 0$ and *type*$(K)=\gamma(\alpha)$, theu K is the one-point compactification of some locally compact əxtension of M having type $\leqq \gamma(\alpha)-1$.

From the above it follows that $R_{lc}^{m^+(\alpha)-1}(\alpha) = R(\alpha)$ and hence, $R_{lc}^k(\alpha) = R_c^{k-1}(\alpha) = R(\alpha)$ if $k \geqq m^+(\alpha)-1$.

We recall some known results concerning the above mentioned families of spaces.

(1) Every element of RF has a compactification belonging to RF. (See [K], [R_1]).

(2) In the family RF there is no universal element. (See [N]).

(3) In the family $R(\alpha)$ there exists a universal element having the property

of finite intersections with respect to any subfamily of $R(\alpha)$ whose power is less than or equal to the continuum. (See [I_3]).

(4) Every element of $R^{rim\text{-}com}(\alpha)$ has a compactification belonging to $R^{com}(\alpha)$, (See [I_1]). Moreover, every element of $R^{rim\text{-}com}(\alpha)$ is topologically contained in an element of $R^{cont}(\alpha)$. (See [I_1]).

(5) In the family $R^{rim\text{-}com}(\alpha)$ there does not exist a universal element (See [I_4]). Hence, by (4), in the families $R^{cont}(\alpha)$ and $R^{com}(\alpha)$ there do not exist universal spaces.

(6) For the family $R^{com}(\alpha)$ there exists a containing space belong to the family $R^{cont}(\alpha+1)$. (This is a result of J. C. Mayer and E. D. Tymchatyn).

(7) For the family of all planar compacta having *rim-type*$\leqq\alpha$ there exists a containing planar locally connected continuum having *rim-type*$\leqq\alpha+1$. (See [M-T]).

(8) In the family $R^k_c(\alpha)$, where α is an isolated ordinal and $k=0, \cdots, m^+(\alpha)-1$, there is no universal element. (See [I-Z]).

(9) For a family Sp of rim-finite spaces there exists a containing rim-finite space (heving the property of finite intersections with respect to any subfamily of Sp whose the power is less than or equel to the continuum) if and only if Sp is a uniform family. (A family Sp of rim-finite spaces is called *uniform* iff for every $X\in Sp$ there exists an ordered basis $B(X)=\{U_0(X), U_1(X), \cdots\}$ having the properties: (α) $Bd(U_i(X))\cap Bd(U_j(X))=\emptyset$ if $i\neq j$ and (β) for every integer $k\geqq 0$ there exists an integer $n(k)\geqq 0$ (which is independent from the elements of Sp) such that for every $x, y\in\bigcup_{i=0}^k(Bd(U_i(X)))$, $x\neq y$, there exists an integer $j(x, y)$, $0\leqq j(x, y)\leqq n(k)$, for which either $x\in U_{j(x,y)}(X)$ and $y\in X\setminus Cl(U_{j(x,y)}(X))$, or $y\in U_{j(x,y)}(X)$ and $x\in X\setminus Cl(U_{j(x,y)}(X))$ (See [I_2]).

(10) In [G-I], for a given subfamily Sp of $R^{com}(\alpha)$, necessary and sufficient conditions are given for the existence of a containing space (having the property of α-intersections with respect to any subfamily of Sp whose power is less than or equal to the continuum) belonging to the family $R^{rim\text{-}com}(\alpha)$.

(11) In the family R of all rational spaces there exists a universal element having the property of finite intersections with respect to the subfamily of all rational continua. (See [I_5]).

1.3. Results. In the present paper we study the family $R^k_{lc}(\alpha)$, where $\alpha>0$ and $k=0, \cdots, m^+(\alpha)-1$. We construct a universal element K of this family as a subset of another space T. For the construction of these spaces we need in two "kinds" of countability.

In Section 2 starting with some properties of scattered spaces we prove the following theorem: every element of $R^k_{Ic}(\alpha)$ admits a compactification having *rim-type* $\leqq\alpha+k+1$. For the proof of this theorem, we construct for every $X\in R^k_{Ic}(\alpha)$ (See Lemma 2.4) an extension $\tilde{X}$ with a basis $B(\tilde{X})$ whose elements have boundaries with some special properties. These properties also provide us with the above mentioned two "kinds" of countability.

In Section 3 we consider a family A of pairs (S, D), where S is a subset of C and D is an upper semi-continuous partition of S such that $D_{\bar{i}}$, $\bar{i}\in L$, is homeomorphic to an element of a given family $\boldsymbol{M}$ of scattered compacta. The elements of A are called $\boldsymbol{M}$-representations. Using the $\boldsymbol{M}$-representations we construct a space T which will be used in Section 5. An important fact is the countability of the family $\boldsymbol{M}$ (this is the first "kind" of countability).

In $[I_3]$ we have considered a set of some specific subsets of a given scattered compactum M: a subset X of M is such a subset iff $M\setminus M^{(\beta(\alpha))}\subseteq X$. We have proved that if in the above set we consider the equivalence relation: $X_1\sim X_2$ iff there exists a homeomorphism f of X_1 onto X_2, then the number of equivalence classes is countable. In Section 4 of the present paper we improve this result by proving that if in the set of all pairs (X, M), where M is a compactum, $type(M)=\alpha$ and $M\setminus M^{(\beta(\alpha))}\subseteq X$, we consider the equivalence relation $(X_1, M_1)\sim(X_2, M_2)$ iff there exists a homeomorphism f of M_1 onto M_2 such that $f(X_1)=X_2$, then the number of equivalence classes is countable (this is the second "kind" of countability).

In Section 5 using the properties of the extension nentioned in Lemma 2.4 we give the notion of a c-extension of elements of the family $R^k_{Ic}(\alpha)$. For every element of this family we consider a fixed c-extension. By a standdard manner, we correspond to every such extension an $\boldsymbol{M}$-representation, where $\boldsymbol{M}$ is a countable set of scattered compacta. The space T constructed in Section 3 (for the above $\boldsymbol{M}$-representations) has *rim-type*$\leqq\alpha+k+1$ and it contains topologically the fixed c-extensions. Using the result of Section 4, the construction of the space T can be done in such a manner that a subset K of T has *type* $\leqq\alpha$ and contains topologically every element of $R^k_{Ic}(\alpha)$. Thus, the space T is a containing space for the family of fixed c-extensions and simultaneously the subset K is an universal element of $R^k_{Ic}(\alpha)$. The main result of this papers is Theorem 5.3.

We note the following corollaries of the main results: In the family $R^k_{Ic}(\alpha)$ there exists a universal element having the property of α^k_{Ic}-intersections (See Definitions 5.1.) with respect to any subfamily of $R^k_{Ic}(\alpha)$ the power of which is less than or equal to the continuum.

Also, for the family $R^k_{C}(\alpha)$, there exists a containing space belonging to the family $R^k_{lc}(\alpha)$ and, hence, there exists a containing continuum having *rim-type* $\leqq\alpha-k+1$. In particular, for $k=0$ (since $R^{com}(\alpha)\subseteqq R^0_c(\alpha)$) we have: There exists a continuum having *rim-type*$\leqq\alpha+1$ which is containing space for all compacta having *rim-type*$\leqq\alpha$. (This is a result of J.C. Mayer and E.E. Tymcharyn).

2. Extensions of elements of $R^k_{lc}(\alpha)$.

2.1. LEMMA. *Let M be a scattered space having type $\alpha=\beta(\alpha)+m(\alpha)>0$. Let X be a zero-dimensional metric compactification of M. Then, there is a compactification K of M for which the natural projection π of X onto K exists and such that:*

(1) *$type(K)=com\text{-}type(M)$ (and, hence, by Lemma* 1 *of* [I-T], *$type(K)\leqq\gamma(\alpha)$).*

(2) *$type(M\cup(K\setminus K^{(\beta(\alpha))}))=\alpha$.*

(3) *$loc\text{-}com\text{-}type(M)=loc\text{-}com\text{-}type(M\cup(K\setminus K^{(\beta(\alpha))}))$ and*

(4) *if $K=\{z_1, z_2, \cdots\}$, then $\lim_{i\to+\infty}(diam(\pi^{-1}(z_i)))=0$.*

PROOF. We prove the lemma by induction on the ordinal *com-type*(M). The proof can be done in such a manner that besides properties (1)-(4) of the lemma the following properties will be also true:

(5) for a given $\varepsilon>0$, $diam(\pi^{-1}(z))<\varepsilon$ for every $z\in K$, and

(6) for every $a\in M$, $type(a, K)=com\text{-}type(a, M)$

Let *com-type*$(M)=1$. We set $K=M$. Then, K is a compactification of M having properties (1)-(6).

Suppose that for every space M for which $1\leqq com\text{-}type(M)<\gamma$ there exists a compactification K of M having properties (1)-(6). Since for every scattered space M, *com-type*(M) is an isolated ordinal, we may suppose that γ is also an isolated ordinal.

Let M be a space such that *com-type*$(M)=\gamma$ and $\varepsilon>0$ be a number. Suppose that $type(M)=\alpha$. By Lemma 1 of [I-T] it follows that $\beta(\alpha)=\beta(\gamma)$.

First we suppose that $max(M)$ is infinite. By Lemma 2.4 of [I-Z] it follows that $com\text{-}type(a, M)=\gamma-1$, for every $a\in max(M)$.

Let $F=Cl(max(M))\setminus max(M)$. (The closure is considered in the space X). Let $F_1, \cdots, F_n$ be open and closed non-empty subsets of F such that (α) $F=F_1\cup\cdots\cup F_n$, (β) $F_i\cap F_j=\emptyset$ if $i\neq j$, and (γ) $diam(F_i)<\varepsilon$ for every $i=1, \cdots, n$.

There exist open and closed subsets U_{ij}, $i=1, \cdots, n$, $j=1, 2, \cdots$, of X such that: (α) $U_{11}\cup U_{21}\cup\cdots\cup U_{n1}=X$, ($\beta$) $U_{i(j+1)}\subseteqq U_{ij}$, ($\gamma$) $(U_{ij}\setminus U_{i(j+1)})\cap max(M)\neq\emptyset$, ($\delta$) $U_{i1}\cap U_{j1}=\emptyset$, if $i\neq j$, and (ε)$\bigcap_{j=1}^{\infty}U_{ij}=F_i$.

Let $M_{ij}=(U_{ij}\setminus U_{i(j+1)})\cap M$, $i=1, \cdots, n$, $j=2, 2, \cdots$. Obviously, $max(M_{ij})=M_{ij}\cap max(M)$ and, hence, the set $max(M_{ij})$ is finite and $com\text{-}type(a, M_{ij})=\gamma-1$ for every $a\in max(M_{ij})$. By Lemma 2.4 of [I-Z], $com\text{-}type(M_{ij})=\gamma-1$.

Hence, by induction, there is a compactification K_{ij} of M_{ij}, $i=1, \cdots, n$, $j=1, 2, \cdots$, for which the natural projection π_{ij} of $U_{ij}\setminus U_{i(j+1)}$ onto K_{ij} exists and such that properties (1)-(6) are true, where in place of ε in property (5) we take the number ε/j.

Let $K=(\bigcup_{i,j}K_{ij})\cup\{F_1, \cdots, F_n\}$. We topologize K as follows: a subset V of K is an open subset iff V has the following properties: (α) the set $V\cap K_{ij}$, $i=1, \cdots, n$, $j=1, 2, \cdots$, is an open subset of K_{ij}, and (β) if $F_i\in V$, then V contains all but finitely many of the sets K_{ij}, $j=1, 2, \cdots$.

Let π be the map of X onto K defined as follows: if $x\in U_{ij}\setminus U_{i(j+1)}$, then $\pi(x)=\pi_{ij}(x)$ and if $x\in F_i$, $i=1, \cdots, n$, then $\pi(x)=F_i$.

It is easy to see that K is a compactification of M and π the natural projection of X onto K.

Since K_{ij} is an open and closed subset of K and $type(K_{ij})\leqq\gamma-1$ we have $type(F_i, K)=\gamma$ and, hence, $type(K)=com\text{-}type(M)=\gamma$, that is, property (1) is satisfied.

By induction, $type(M_{ij}\cup(K_{ij}\setminus K_{ij}^{(\beta(\alpha))}))\leqq\alpha$. Hence, since $M\cup(K\setminus K^{(\beta(\alpha))})=\bigcup_{i,j}(M_{ij}\cup(K_{ij}\setminus K_{ij}^{(\beta(\alpha))}))$ we have $type(M\cup(K\setminus K^{(\beta(\alpha))}))=\alpha$, that is, property (2) is satisfied.

Since the subset $K\setminus\{F_1, \cdots, F_n\}$ is a locally compact extension of $M\cup(K\setminus K^{(\beta(\alpha))})$ and $type(K\setminus\{F_1, \cdots, F_n\})=\gamma-1$ we have $loc\text{-}com\text{-}type(M\cup(K\setminus K^{(\beta(\alpha))}))\leqq\gamma-1$. Since the set $max(M)$ is infinite and $com\text{-}type(M)=\gamma$, by Lemma 2.4 of [I-Z] it follows that $loc\text{-}com\text{-}type(M)=\gamma-1$, that is, property (3) is true.

Properties (4) and (5) follow by the construction of K.

For every $x\in M_{ij}$ we have $type(x, K_{ij})=type(x, K)=com\text{-}type(x, M)$. Hence, property (6) is also true.

Now, we suppose that $max(M)$ is finite. Then, by Lemma 2.4 of [I-Z], $com\text{-}type(a, M)=\gamma$, for every $a\in max(M)$. Let $max(M)=\{a_1, \cdots, a_n\}$ and let U_{ij}, $i=1, \cdots, n$, $j=1, 2, \cdots$, be open and closed subsets of X such that: (α) $U_{11}\cup\cdots\cup U_{n1}=X$, (β) $U_{i(j+1)}\subsetneqq U_{ij}$, (γ) $U_{ij}\setminus U_{i(j+1)}\neq\emptyset$, (δ) $U_{i1}\cap U_{j1}=\emptyset$, if $i\neq j$, and (ε) $\bigcap_{j=1}^{\infty}U_{ij}=\{a_i\}$.

Let $M_{ij}=(U_{ij}\setminus U_{i(j+1)})\cap M$. Then, either $com\text{-}type(M_{ij})\leqq\gamma-1$, or $com\text{-}type(M_{ij})=\gamma$ and the set $max(M_{ij})$ is infinite. Hence, by induction, there is a compactification K_{ij} of M_{ij} (for which the natural projection π_{ij} of $U_{ij}\setminus U_{i(j+1)}$

onto K_{ij} exists) having properties (1)-(6).

Let K and π be the compactification of M and the natural projection of X onto K, respectively, constructed from K_{ij} in the same manner as in case, where the set $max(M)$ is infinite (replacing the set $\{F_1, \cdots, F_n\}$ by the set $max(M)=\{a_1, \cdots, a_n\}$ and the subset F_i, in the definition of π, by the subset $\{a_i\}$ of X).

By construction, $type(K_{ij})\leqq\gamma$. On the other hand, for a given i, there exists an integer j_0 such that $type(K_{ij})\leqq\gamma-1$ for every $j\geqq j_0$. (See Section 2.2.4 of [I-Z]). Hence, $type(a_i, K)=\gamma$. Thus, $type(K)=com\text{-}type(M)=\gamma$. Hence, property (1) is satisfied.

Since the subset K_{ij} of K is an open subset and since $type(a_i, K)=\gamma$, property (6) is also satisfied.

For the proof of property (2) it is sufficient to prove that $(M\cup(K\setminus K^{(\beta(\alpha))}))^{(\beta(\alpha))}=M^{(\beta(\alpha))}$. Obviously, $M^{(\beta(\alpha))}\subseteqq(M\cup(K\setminus K^{(\beta(\alpha))}))$. Let $x\in(M\cup(K\setminus K^{(\beta(\alpha))}))^{(\beta(\alpha))}$. Then, it is clear that $x\notin K\setminus K^{(\beta(\alpha))}$. Hence, $x\in M$. If $x\in M\setminus M^{(\beta(\alpha))}$, then $com\text{-}type(x, M)<\beta(\alpha)$ and, therefore, $type(x, K)<\beta(\alpha)$, that is, $x\in K\setminus K^{(\beta(\alpha))}$ which is impossible. Hence, $x\in M^{(\beta(\alpha))}$ and property (2) is satisfied.

Since the set $max(M)$ is finite, by Lemma 2.4 of [I-Z] it follows that $loc\text{-}com\text{-}type(M)=com\text{-}type(M)=type(K)$. Hence, $loc\text{-}com\text{-}type(M)\cup(K\setminus K^{(\beta(\alpha))}))=type(K)$ and property (3) is satisfied.

Since for a fixed i, $\lim_{j\to 0}(diam(U_{ij}\setminus U_{i(j+1)}))=0$, properties (4) and (5) follow by the construction of K.

2.2. LEMMA. *Let M be a locally finite union of closed subset $M_1, M_2, \cdots$ such that $loc\text{-}com\text{-}type(M_i)\leqq\alpha$, $i=1, 2, \cdots$. Then, $loc\text{-}com\text{-}type(M)\leqq\alpha$.*

PROOF. Let $a\in M$. There exist an open neighbourhood U of a in M and a set $\{n_1, \cdots, n_t\}$ of integers such that $U=(U\cap M_{n_1})\cup\cdots\cup(U\cap M_{n_t})$. Since, $loc\text{-}com\text{-}type(M_{n_i})\leqq\alpha$ we have $loc\text{-}com\text{-}type(U\cap M_{n_t})\leqq\alpha$, $i=1, \cdots, t$.

By Theorem 2.5 of [I-Z] it follows that $loc\text{-}com\text{-}type(U)\leqq\alpha$. Hence, by Lemma 2.4 of [I-Z], $com\text{-}type(a, U)=com\text{-}type(a, M)\leqq\alpha$. By the same lemma we have $loc\text{-}com\text{-}type(M)\leqq\alpha$.

2.2.1. COROLLARY. *Let $X\in R^k_{lc}(\alpha)$ (See the Introduction). Then, every pair of disjoint closed subsets of X can be separated by a subset M such that $type(M)\leqq\alpha$ and $loc\text{-}com\text{-}type(M)\leqq\alpha+k$.*

The proof follows by Lemma 2.2 and Lemma 4 of [I-T]. This corollary is used in the proof of the following Lemma 2.3.

2.3. LEMMA. *Let* $X \in R^k_{lc}(\alpha)$ *and* $B=\{U_0, U_1, \cdots\}$ *be a basis of open sets of* X *such that for every* i, $type(Bd(U_i)) \leqq \alpha$ *and* $loc\text{-}com\text{-}type(Bd(U_i)) \leqq \alpha+k$. *Let* F *be the family of all pairs* $A_m=(U_{i_m}, U_{j_m})$ *such that* $Cl(U_{i_m}) \subseteqq U_{j_m}$ *and* U_{i_m}, $U_{j_m} \in B$. *Let* D *denote the set of triadic rationals in the open interval* (0, 1). *Then, there exists a sequence* (f_m) *of continus functions* $f_m: X \to [0, 1]$ *such that for integers* $m, r, m \neq r$ *and* $d \in D$:

(1) $f_m(Cl(U_{i_m}))=\{0\}$,

(2) $f_m(X \setminus U_{j_m})=\{1\}$,

(3) $type(f_m^{-1}(d)) \leqq \alpha$ *and* $loc\text{-}com\text{-}type(f_m^{-1}(d)) \leqq \alpha+k$,

(4) $Bd(f_m^{-1}([0, d)))=Bd(f_m^{-1}((d, 1]))=f_m^{-1}(d)$,

(5) $f_r(f_m^{-1}(d)) \cap D=\emptyset$, *and*

(6) $f_r(f_m^{-1}(d))$ *is a closed subset of* $[0, 1]$ *of dimension* $\leqq 0$.

This lemma, except condition 3, is the same as Lemma 7 of [I-T] and it is proven similarly.

2.4. LEMMA. *Let* $X \in R^k_{lc}(\alpha)$. *There exist an extension* $\tilde{X}$ *of* X *and a basis* $B(\tilde{X})=\{V_0, V_1, \cdots\}$ *of open sets of* $\tilde{X}$ *such that:*

(1) *the set* $Bd(V_i)$, $i=0, 1, \cdots$, *is a compactum,*

(2) $V_i=Int(Cl(V_i))$, $i=0, 1, \cdots$,

(3) $Bd(V_i) \cap Bd(V_j)=\emptyset$ *if* $i \neq j$,

(4) $type(Bd(V_i)) \leqq \alpha+k+1$,

(5) $type((Bd(V_i) \cap X) \cup (Bd(V_i) \setminus (Bd(V_i))^{(\beta(\alpha))})) \leqq \alpha$ *and*

(6) $loc\text{-}com\text{-}type((Bd(V_i) \cap X) \cup (Bd(V_i) \setminus (Bd(V_i))^{(\beta(\alpha))})) \leqq \alpha+k$.

The proof is similar to the proof of theorem 8 of [I-T]. The extension $\tilde{X}$ is constructed in the same manner as the space Z is constructed in the proof of Theorem 8 of [I-T]. Instead of Theorem 3 of [I-T] which was used in the proof of Theorem 8 of [I-T] we have use Lemma 2.1.

2.5. THEOREM. *Let* $X \in R^k_{lc}(\alpha)$. *Then,* X *admits a compacification having rim-type* $\leqq \alpha+k+1$.

This theorem is proved using properties (1)-(4) of extension $\tilde{X}$ of X of Lemma 2.4 and Theorem 2 of [I_1].

3. Construction of specific spaces.

3.1. DEFINITIONS AND NOTATIONS. Let M be a scattered space. A finite cover ω of M is called a *decomposition* iff every element of ω is an open and closed subset of M and the intersection of any two distinct elements of ω is empty.

A decomposition ω is a *subdivision* of a decomposition ω' of M iff every element of ω is contained in an element of ω'.

A sequence ω^n, $n\in N$, of decompositions of M is called a *decreasing sequence of decompositions* iff (α) the decomposition ω^{n+1}, $n\in N$, is a subdivision of the decomposition ω^n and (β) the set of all elements of all ω^n, $n\in N$, is a basis of open sets of M.

In what follows by $\boldsymbol{M}$ we denote a countable set of scattered compacta. We suppose that two distinct elements of $\boldsymbol{M}$ are not homeomorphic.

Also, we suppose that for every $M\in\boldsymbol{M}$ there exists a fixed decreasing sequence of decompositions of M. The n^{th} decomposition of this sequence is denoted by M^n, $n\in N$.

Let $x\in M\in\boldsymbol{M}$ and $n\in N$. We denote by $F(n, x)$ the element F of M^n for which $x\in F$.

A pair $g=(S, D)$ is called an *$\boldsymbol{M}$-representation* iff: (α) S is a subset of C, (β) D is an upper semi-continuous partition of S, (γ) every element of $D(1)$ consists of exactly two points, and (δ) for every $q\in N$, D_q is homeomorphic to an element of $\boldsymbol{M}$.

In Section 3, we denote by A a family of $\boldsymbol{M}$-representations the power of which is less than or equal to the continuum. We suppose that for distinct elements $g=(S, D)$ and $f=(S', D')$ of A it may happen that $S=S'$ and $D=D'$.

For every element $g=(S, D)$ of A and for every $q\in N$ by $M_q(g)$ we denote the element of $\boldsymbol{M}$ which is homeomorphic to D_q and by $\phi_q(g)$ a fixed homeomorphism of $M_q(g)$ onto D_q.

Let A' be a subfamilly of A such that for some $q\in N$, $M_q(g)=M_q(f)$ for any elements g, f of A'. In this case the element $M_q(g)$ of $\boldsymbol{M}$ is also denoted by $M_q(A')$ and we shall say that *the element $M_q(A')$ of $\boldsymbol{M}$ is then determined.*

For any subfamilly A' of A and for any subset C' of C we denoted by $C' \,\bar{\times}\, A'$ the subset of $C'\times A'$ consisting of all elements (a, g) of $C'\times A'$ such that if $g=(S, D)$, then $a\in S$.

A *decomposition* Ω of A is a countable set of subfamilies of A such that: (α) the intersection of any two distinct elements of Ω is empty and (β) the union of all elements of Ω is A.

A decomposition Ω is a *subdivision* of a decomposition Ω' of A iff every element of Ω is contained in an element of Ω'.

A sequence Ω^n, $n\in N$, of decompositions of A is called *a decreasing sequence af decompositions* iff: (α) Ω^{n+1} is a subdivision of Ω^n, $n\in N$, and (β) if g and

f are distinct elements of A, then there exists an integer n such that g and f belong to distinct elements of Ω^n.

Since the power of A is less than or equal to the continuum, the existence of decreasing sequence of decompositions of A is easily proved.

In what follows, we suppose that there exists a fixed such sequence of A denoted by Ω^n, $n\in N$. Moreover, without loss of generality, we may suppose that for every $E\in\Omega^n$ and for every q, $0\leqq q\leqq n$, the element $M_q(E)$ is determined.

3.2. LEMMA. *For every integer $m\in N$ there exist:*

(1) *A decomposition $A^m=\{A_r^m : r\in I(m)\}$ of A which is a subdivision of Ω^m (hence, for every $r\in I(m)$ and for every integer q, $0\leqq q\leqq m$, the element $M_q(A_r^m)$ of $\boldsymbol{M}$ is determined). In what follows, we denote by r an arbitrary element of $I(m)$ and by q an integer such that $0\leqq q\leqq m$.*

(2) *An integer $n(q, A_r^m)\geqq m$ (denoted also by $n(q, m, r)$).*

(3) *An integer $n(A_r^m)>m$ (denoted also by $n(m, r)$).*

(4) *A subset $s(F)$ of $L_{n(m,r)}$ for every $F\in(M_q(A_r^m))^{n(q,m,r)}$ (denoted also by $s(q, m, r, F)$).*

(5) *A subset $U(F)$ of $C\times A$ for every $F\in(M_q(A_r^m))^{n(q,m,r)}$ (denoted also by $U(q, m, r, F)$) such that:*

(6) *If $m\geqq 1$, then A^m is a subdivision of A^{m-1} (hence, the sequence A^0, A^1, $\cdots$ is a decreasing sequence of decompositions of A).*

(7) *If $m\geqq 1$, $t\in I(m-1)$ and $A_r^m\subseteq A_t^{m-1}$, then $n(m, r)>n(m-1, t)$.*

(8) *If $t\in I(q)$ and $A_r^m\subseteq A_t^q$, then $n(q, m, r)=n(q, q, t)+m-q$.*

(9) *If $m\geqq 1$, $t\in I(m-1)$, f, $g\in A_r^m\subseteq A_t^{m-1}$ and $x\in F\in(M_m(A_r^m))^{n(m,m,r)}$, then $st(\psi_m(g)(x), n(m-1, t))=st((\psi_m(f)(F))^*, n(m-1, t))$.*

(10) *If $m\geqq 1$, $q<m$, $t\in I(m-1)$, $g=(S, D)\in A_r^m\subseteq A_t^{m-1}$, $d\in D$, $F\in (M_q(g))^{n(q,m,r)}$, $Q\in(M_q(g))^{n(q,m,r)-1}$, $F\subseteq Q$* and *$d\cap st((\psi_q(g)(F))^*, n(m, r))\neq\emptyset$, then $d\subseteq st((\psi_q(g)(Q))^*, n(m-1, t))$.*

(11) *If $g\in A_r^m$ and $F\in(M_q(A_r^m))^{n(q,m,r)}$, then $st((\psi_q(g)(F))^*, n(m, r))=C_{s(F)}$.*

(12) *$U(F)=C_{s(F)}\times A_r^m$ for every $F\in(M_q(A_r^m))^{n(q,m,r)}$.*

(13) *If $F\in(M_k(A_r^m))^{n(k,m,r)}$ and $Q\in(M_q(A_r^m))^{n(q,m,r)}$, where $0\leqq k<q$, then $U(F)\cap U(Q)=\emptyset$.*

(14) *If F, $Q\in(M_q(A_r^m))^{n(q,m,r)}$ and $F\neq Q$, then $U(F)\cap U(Q)=\emptyset$.*

PROOF. We prove the lemma by induction on integer m.

Let $m=0$. Let $E\in\Omega^0$. For every $g\in E$ there exists an integer $n(g)>0$ such that if F, $Q\in(M_0(g))^0$, then $st((\psi_0(g)(F))^*, n(g)(\cap st((\psi_0(g)(Q))^*, n(g))=\emptyset$.

We observe that if $f, g \in Q$, then $M_0(f)=M_0(g)$.

Now, we define the decomposition A^0 of A as follows: two elements g and f of A belong to the same element of A^0 iff there exists an element $E \in \Omega^0$ such that: (α) $g, f \in E$, (β) $n(g)=n(f)$ and (γ) $st((\phi_0(g)(F))^*, n(g))=st((\phi_0(f)(F))^*, n(f))$ for every $F \in (M_0(g))^0=(M_0(f))^0$.

Obviously, A^0 is a countable set and by the construction, A^0 is a subdivision of Ω^0. Let $A^0=\{A_r^0 : r \in I(0)\}$.

For every $r \in I(0)$ we set $n(0, A_r^0)=0$ and $n(A_r^0)=n(g)$, where $g \in A_r^0$. Obviously, the integer $n(A_r^0)$ is independent from $g \in A_r^0$.

For every $F \in (M_0(A_r^0))^0$ we denote by $s(F)$ the set of all elements $\bar{i}$ of $L_{n(0,r)}$ for which $C_{\bar{i}} \subseteq st((\phi_0(g)(F))^*, n(g))$, where $g \in A_r^0$. Obviously, the set $s(F)$ is independent from $g \in A_r^0$.

Finally, we set $U(F)=C_{s(F)} \times A_r^0$ for every $F \in (M_0(A_r^0))^0$. It is easy to see that properties (8), (11), (12) and (14) of the lemma are satisfied.

Suppose that the lemma is proved for every m, $0 \leqq m < p$. We prove the lemma for $m=p$.

Let $E \in \Omega^p$, $t \in I(p-1)$ and $g=(S, D) \in E \cap A_t^{p-1}$. Since the map $\phi_p(g)$ is continuous, for every $x \in M_p(g)$ there exists an open neighbourhood $O(x)$ of x in $M_p(g)$ such that for every $y \in O(x)$ we have $st(\phi_p(g)(x), n(p-1, t))= st(\phi_p(g)(y), n(p-1, t))$. (For example, we can suppose that $O(x)= (\phi_p(g))^{-1}(O(\phi_p(g)(x)))$, where $O(\phi_p(g)(x))$ is the set of all elements of D_p which are contained in the open set $st(\phi_p(g)(x)$ '$n(p-1, t))$ of C). The set of all such neighbourhoods $O(x)$ is an open cover of $M_p(g)$. Hence, since $M_p(g)$ is a compactum there exists an integer $n_0(g) \geqq 0$ such that every element of $(M_p(g)^{n_0(g)}$ is contained in the neighbourhood $O(x)$ for some x.

There exists an integer $n_1(g) \geqq 0$ such that $st((\phi_k(g)(F))^*, n_1(g)) \cap st((\phi_q(g)(Q))^*, n_1(g))=\emptyset$ for every $F \in (M_k(g))^{n(k, p-1, t)+1}$ and for every $Q \in (M_q(g))^{n(q, p-1, t)+1}$, where $0 \leqq k \leqq p-1$, $0 \leqq q \leqq p-1$ and either $k \neq q$ or $k=q$ and $F \neq Q$.

Also, since D is an upper semi-continuous partition of S, there exists an integer $n_2(g) \geqq 0$ such that if $0 \leqq q \leqq p-1$, $d \in D$, $F \in (M_q(g))^{n(q, p-1, t)+1}$, $Q \in (M_q(g))^{n(q, p-1, t)+1}$, $F \subseteq Q$ and $d \cap st((\phi_q(g)(F))^*, n_2(g)) \neq \emptyset$, then $d \subseteq st((\phi_q(g)(Q))^*, n(p-1, t))$.

There exists an integer $n_3(g) \geqq 0$ such that if F and Q are distinct elements of $(M_p(g))^{n_0(g)}$, then $st((\phi_p(g)(F))^*, n_3(g)) \cap st((\phi_p(g)(Q))^*, n_3(g))=\emptyset$.

Finally, there exists an integer $n_4(g) \geqq 0$ such that if $0 \leqq q \leqq p-1$, $F \in (M_q(g))^{n(q, p-1, t)+1}$, $Q \in (M_p(g))^{n_0(g)}$, then $st((\phi_q(g)(F))^*, n_4(g)) \cap st((\phi_p(g)(Q))^*, n_4(g))=\emptyset$.

Let $n(g)=max\{n_1(g), n_2(g), n_3(g), n_4(g), p+1, n(p-1, t)+1\}$.

We now define the decomposition A^p. Let $g, f \in A$. The elements g and f belong to the same element of A^p iff there exist an element E of Ω^p and an element $t \in I(p-1)$ such that: (α) $g, f \in E \cap A_t^{p-1}$ (hence, $M_q(g)=M_q(f)$ for every q, $0 \leqq q \leqq p$), (β) $n(g)=n(f)$, (γ) $n_0(g)=n_0(f)$, (δ) if $0 \leqq q \leqq p-1$ and $F \in (M_q(g))^{n(q,p-1,t)+1}=(M_q(f))^{n(q,p-1,t)+1}$, then $st((\psi_q(g)(F))^*, n(g))=st((\psi_q(f)(F))^*, n(f))$, and ($\varepsilon$) if $F \in (M_p(g))^{n_0(g)}=(M_p(f))^{n_0(f)}$, then $st((\psi_p(g)(F))^*, n(g))= st((\psi_p(f)(F))^*, n(f))$.

It is easy to see that the set A^p is countable. Let $A^p=\{A_r^p : r \in I(p)\}$.

Property (6) of the lemma follows by the definition of the decomposition A^p.

Let $r \in I(p)$. We define the integers $n(p, r)$ and $n(q, p, r)$ for $0 \leqq q \leqq p$ setting $n(p, r)=n(g)$, $n(p, p, r)=n_0(g)$, where $g \in A_r^p$ and $n(q, p, r)=n(q, p-1, t)+1$ if $0 \leqq q \leqq p-1$, where $t \in I(p-1)$ such that $A_r^p \subseteq A_t^{p-1}$.

Property (7) of the lemma follows by the definition of the number $n(g)$. Also, if $t \in I(p-1)$, $q \leqq p-1$ and $e \in I(q)$ such that $A_r^m \subseteq A_t^{m-1} \subseteq A_e^q$, then we have $n(q, p, r)=n(q, p-1, t)+1=n(q, q, e)+p-1-q+1=n(q, q, e)+p-q$, that is, property (8) of the lemma is satisfied.

Property (9) of the lemma follows by the definition of the integer $n_0(g)$ (considering that $n(p, p, r)=n_0(g)$) and by property (ε) of the definition of the set A^p (from which it follows that $st((\psi_p(g)(F))^*, n(p-1, t))=st((\psi_p(f)(F))^*, n(p-1, t))$).

Property (10) of the lemma follows by the definition of the integers $n_2(g)$ and $n(g)$ (considering that $n(q, p, r)=n(q, p-1, t)+1$).

The set $s(F)$, where $F \in (M_q(A_r^p))^{n(q,p,r)}$ is defined as follows: an element $\bar{i}$ of $L_{n(p,r)}$ belongs to $s(F)$ iff $C_{\bar{i}} \subseteq st((\psi_q(g)(F))^*, n(p, r))$, where $g \in A_r^p$. By properties (δ) and (ε) of the definition of the decomposition A^p it follows that $s(F)$ is independent from $g \in A_r^p$.

Property (11) of the lemma follows immediately from the above definition of the set $s(F)$.

The set $U(F)$, where $F \in (M_q(A_r^p))^{n(q,p,r)}$, is defined setting $U(F)=C_{s(F)} \times A_r^p$. Then, property (12) of the lemma is clear.

Finally, properties (13) and (14) of the lemma follows by the definition of the integers $n_1(g)$, $n_3(g)$, $n_4(g)$ and $n(g)$ and the definition of the sets $s(F)$ and $U(F)$.

3.3. NOTATIONS. For every $q \in N$ and $g \in A$ we denote by $r(q, g)$ the elements $t \in I(q)$ for which $g \in A_t^q$.

Let $m \in N$ and $r \in I(m)$. We denote by $s(m, r)$ the union of all sets $s(q, m, r, F)$, where $0 \leqq q \leqq m$ and $F \in (M_q(A_r^m))^{n(q,m,r)}$. Obviously, $s(m, r) \subseteq L_{n(m,r)}$.

Let $m\in N$, $r\in I(m)$ and $x\in M_m(A_r^m)$. Obviously, if $(a, g)\in C\times A_r^m$, then $g\in A_r^m$ and $M_m(A_r^m)=M_m(g)$. We denote by $d(x, m, r)$ the set of all elements $(a, g)\in C\times A_r^m$ for which $\phi_m(g)(x)=a$. We denote by $T(1)$ the set of all subsets of $C\times A$ of the form $d(x, m, r)$. By T we denote the union of the set $T(1)$ and the set of all singletons $\{(a, g)\}$, where (a, g) belongs to $C\times A$ and does not belong to any $d(x, m, r)\in T(1)$.

Let $d(x, m, r)$ be a fixed element of $T(1)$ and let $k\in N$. We denote by $U(d(x, m, r), k)$ the union of all sets of the form $U(m, m+k, t, F)$, where $t\in I(m+k)$ such that $A_t^{m+k}\subseteq A_r^m$ and $x\in F\in(M_m(A_r^{m+k}))^{n(m, m+k, t)}$.

Since $M_m(A_t^{m+k})=M_m(A_r^m)$ and by property (8) of Lemma 3.2, $n(m, m+k, t)$ $=n(m, m, r)+k$ we have $(M_m(A_t^{m+k}))^{n(m, m+k, t)}=(M_m(A_r^m))^{n(m, m, r)+k}$. This means that F is independent from the elements t of $I(m+k)$ for which $A_t^{m+k}\subseteq A_r^m$.

We observe that for every $y\in F$ we have $U(d(x, m, r), k)=U(d(y, m, r), k)$.

We denote by $\hat{U}$ the set of all sets of the form $U(d, k)$, where $d=d(x, m, r)$ $\in T(1)$ and $k\in N$.

Let $m\in N$, $r\in I(m)$ and $\bar{i}\in L_{m(m, r)}$ such that $\bar{i}\notin s(m, r)$. Then, we set $V(\bar{i}, m, r)=C_{\bar{i}}\times A_r^m$. We denote by $\hat{V}$ the set of all sets of the form $V(\bar{i}, m, r)$.

REMARKS. It is not difficult to prove that:

(1) For every $d(x, m, r)\in T(1)$, $d(x, m, r)\subseteq C\times A_r^m$.

(2) If $g\in A_r^m$ and $d(x, m, r)\in T(1)$, then $d(x, m, r)\cap(C\times\{g\})=\phi_m(g)(x)\times\{g\}$ $\neq\emptyset$.

(3) For every $d\in T(1)$ and $k\in N$, $d\subseteq U(d, k)$.

(4) For every $d(x, m, r)\in T(1)$ and $k\in N$, $U(d(x, m, r), k)\subseteq C\times A_r^m$.

(5) Foɹ every $d\in T(1)$ and $k\in N$, $U(d, k+1)\subseteq U(d, k)$.

(6) If $x\in F\in(M_m(A_r^m))^{n(m, m, r)}$, then $U(d(x, m, r), 0)=U(m, m, r, F)$.

(7) If $t\in I(m+k)$, $A_t^{m+k}\subseteq A_r^m$ and $x\in F\in(M_m(A_t^{m+k}))^{n(m, m+k, t)}$, then $U(d(x, m, r), k)\cap(C\times A_t^{m+k})=U(m, m+k, t, F)$.

(8) If $V(\bar{i}, m, r)\in\hat{V}$ and $d(x, q, t)\in T(1)$, where $0\leqq q\leqq m$, then $V(\bar{i}, m, r)\cap d(x, q, t)=\emptyset$.

(9) If $d_1, d_2\in T(1)$ and $d_1\neq d_2$, then $d_1\cap d_2=\emptyset$.

(10) The union of all elements of T is the set $C\times A$.

3.5. LEMMA. *Let $d=d(x, m, r)\in T(1)$ and $U=U(d_1, n_1)\in\hat{U}$, where $d_1=d(y, m_1, r_1)\in T(1)$. The following are true:*

(1) *If $d\subseteq U$, then there exists an integer $n\geqq 0$ such that $U(d, n)\subseteq U$.*

(2) *If $d\cap U=\emptyset$, then there exists an integer $n\geqq 0$ such that $U(d, n)\cap U=\emptyset$.*

(3) *If $d\cap U\neq\emptyset$ and $d\cap((C\times A)\setminus U)\neq\emptyset$, then there exists an open and closed*

neighbourhood $O(x)$ of x in $M_m(A^m_r)$ such that $d(z, m, r)\cap U\neq\emptyset$ and $d(z, m, r)\cap((C\times A)\setminus U)\neq\emptyset$ for every $z\in O(x)$.

PROOF. (1) By properties (1)-(4) of Remarks 3.4 it follows that $A^m_r\subseteqq A^{m_1}_{r_1}$.

First we suppose that $m\leqq p$, where $p=m_1+n_1$. Let t be an arbitrary element of $I(p)$ such that $A^p_t\subseteqq A^m_r\cap A^{m_1}_{r_1}$ and let $F=F(n(m, p, t), x)$ and $F_1=F(n(m_1, p, t), y)$.

Suppose that either $m\neq m_1$ or $m=m_1$ and $F\neq F_1$. By properties (13) and (14) of Lemma 3.2 we have $U(m, p, t, F)\cap U(m_1, p, t, F_1)=\emptyset$.

Obviously, $d\cap(C\times A^p_t)\neq\emptyset$ (See property (1) of Remarks 3.4) and since $d\subseteqq U$ we have $d\cap(C\times A^p_t)\subseteqq U\cap(C\times A^p_t)$.

On the other hand, $U\cap(C\times A^p_t)=U(m_1, p, t, F_1)$ (See property (7) of Remarks 3.4) and $d\cap(C\times A^p_t)\subseteqq U(m, p, t, F)$ (See properties (6) and (7) of Remarks 3.4). From this follows that $(d\cap(C\times A^p_t))\cap(U\cap(C\times A^p_t))=\emptyset$ which is a contradiction.

Hence, $m=m_1$ and $F=F_1$. Setting $n=n_1$ we have that $U(d, n)=U(d_1, n_1)$, that is, the integer $n=n_1$ is the required integer.

Now, let $m_1+n_1=p<m$. Let $e\in I(m-1)$ and $t\in I(p)$ such that $A^m_r\subseteqq A^{m-1}_e\subseteqq A^p_t\subseteqq A^{m_1}_{r_1}$ and let $F=F(n(m, m, r), x)$ and $F_1=F(n(m_1, p, t), y)$.

We have $U(d_1, n_1)\cap(C\times A^p_t)=U(m_1, p, t, F_1)$. Since $d\subseteqq C\times A^m_r\subseteqq C\times A^p_t$ we have that $d\subseteqq U(m_1, p, t, F_1)=C_s\times A^p_t$, where $s=s(F_1)$. Hence, $st(\phi_m(g)(x), n(p, t))\subseteqq C_s$ for every $g\in A^m_r$.

Since $n(m-1, e)\geqq n(p, t)$ (See property (7) of Lemma 3.2) we have that $st(\phi_m(g)(x), n(m-1, e))\subseteqq st(\phi_m(g)(x), n(p, t))$. By proyerty (9) of Lemma 3.2 it follows that $st((\phi_m(g)(F))^*, n(m-1, e))\subseteqq C_s$. By property (11) of Lemma 3.2 we have that $C_{s(F)}\subseteqq C_s$. Hence, by property (12) of Lemma 3.2, $U(m, m, r, F)=C_{s(F)}\times A^m_r\subseteqq C_s\times A^p_t=U(m_1, p, t, F_1)\subseteqq U$. Obviously, $U(m, m, r, F)=U(d, 0)$ (See property (6) of Remarks 3.4). Hence, the integer $n=0$ is the required integer.

(2) If $A^m_r\cap A^{m_1}_{r_1}=\emptyset$, then by properties (1)-(4) of Remarks 3.4 it follows that for every $n\in N$, $U(d, n)\cap U(d_1, n_1)=\emptyset$. Hence, we can suppose that $A^m_r\cap A^{m_1}_{r_1}\neq\emptyset$.

Let $m\leqq p$, where $p=m_1+n_1$ and let t, F and F_1 be the same as in the corresponding part of case (1).

If $m=m_1$ and $F=F_1$, then $r=r_1$ and $d\subseteqq U$ which is a contradiction. Hence, either $m\neq m_1$, or $m=m_1$ and $F\neq F_1$.

In both cases, by properties (13) and (14) of Lemma 3.2 we have that $U(m, p, t, F)\cap U(m_1, p, t, F_1)=\emptyset$. Since $U(d, p-m)\cap(C\times A^p_t)=U(m, p, t, F)$ and $U(d_1, n_1)\cap(C\times A^p_t)=U(m_1, p, t, F_1)$ and since t is an arbitrary element of $I(p)$ for which $A^p_t\subseteqq A^m_r\cap A^{m_1}_{r_1}$ we have that $U(d, p-m)\cap U(d_1, n_1)=\emptyset$, that is, the integer $n=p-m$ is the required integer.

Now, let $p<m$, hence, $A_r^m \subseteqq A_{r_1}^{m_1}$ and let e, t, F and F_1 be the same as in the corresponding part of case (1).

We have $U(d_1, n_1) \cap (C \times A_t^p) = U(m_1, p, t, F_1) = C_s \times A_t^p$, where $s=s(F_1)$. Hence, $(C_s \times A_t^p) \cap d = \emptyset$. This means that for every $g \in A_r^m$, $st(\psi_m(g)(x), n(p, t)) \cap C_s = \emptyset$. Since $n(m-1, e) \geqq n(p, t)$ (See property (7) of Lemma 3.2) we have $st(\psi_m(g)(x), n(m-1, p)) \cap C_s = \emptyset$.

By property (9) of Lemma 3.2 it follows that $st((\psi_m(g)(F))^*, n(m-1, e)) \cap C_s = \emptyset$. Since $n(m, r) > n(m-1, e)$ we have that $st((\psi_m(g)(F))^*, n(m, r)) \cap C_s = \emptyset$, that is, $C_{s(F)} \cap C_s = \emptyset$.

Thus, $(C_{s(F)} \times A_r^m) \cap (C_s \times A_t^p) = \emptyset$, that is, $U(m, m, r, F) \cap U(m_1, p, t, F_1) = \emptyset$. Hence, $U(m, m, r, F) \cap U(d_1, n_1) = \emptyset$, that is, $U(d, 0) \cap U(d_1, n_1) = \emptyset$ and $n=0$ is the required integer.

(3) It is easy to see that $A_r^m \cap A_{r_1}^{m_1} \neq \emptyset$. Let $m \leqq p$, where $p=m_1+n_1$ and let $t \in I(p)$ such that $A_t^p \subseteqq A_r^m$ and $A_t^p \subseteqq A_{r_1}^{m_1}$. Let F and F_1 be the same as in the corresponding part of case (1). As in that case we prove that if $m=m_1$ and $F=F_1$, then $d \subseteqq U$ and if either $m \neq m_1$ or $m=m_1$ and $F \neq F_1$, then $d \cap U = \emptyset$, which is a contradiction.

Hence $p<m$. Then. $A_r^m \subseteqq A_{r_1}^{m_1}$. Let e, t, F and F_1 be same as in the corresponding part of case (1).

We have $U \cap (C \times A_t^p) = U(m_1, p, t, F_1)$. Since $d \subseteqq C \times A_r^m \subseteqq C \times A_t^p$ we have $d \cap U(m_1, p, t, F_1) \neq \emptyset$ and $d \cap ((C \times A) \setminus U(m_1, p, t, F_1)) \neq \emptyset$. Moreover, if $(a, g) \in d \cap ((C \times A) \setminus U(m_1, p, t, F_1))$, then $(a, g) \notin U$.

There exist elements g_1 and g_2 of A_r^m such that $\psi_m(g_1)(x) \cap C_s \neq \emptyset$ and $\psi_m(g_2)(x) \cap (C \setminus C_s) \neq \emptyset$, where $s=s(F_1)$. Since $n(m-1, e) \geqq n(p, t)$ there exist elements $\bar{i}_1$ and $\bar{i}_2$ of $C_{n(m-1, e)}$ such that $C_{\bar{i}_1} \subseteqq C_s$, $C_{\bar{i}_2} \subseteqq C \setminus C_s$, $\psi_m(g_1)(x) \cap C_{\bar{i}_1} \neq \emptyset$ and $\psi_m(g_2)(x) \cap C_{\bar{i}_2} \neq \emptyset$.

By property (9) of Lemma 3.2 it follows that for every $z \in F$ we have $\psi_m(g_1)(z) \cap C_{\bar{i}_1} \neq \emptyset$ and $\psi_m(g_2)(z) \cap C_{\bar{i}_2} \neq \emptyset$. This means that $d(z, m, r) \cap U(m_1, p, t, F_1) \neq \emptyset$ and $d(z, m, r) \cap ((C \times A) \setminus U(m_1, p, t, F_1)) \neq \emptyset$, that is, $d(z, m, r) \cap U \neq \emptyset$ and $d(z, m, r) \cap ((C \times A) \setminus U) \neq \emptyset$. Hence, the neighbourhood $O(x)=F$ is the required neighbourhood of x in $M_m(A_r^m)$.

3.6. LEMMA. *Let $d=d(x, m, r) \in T(1)$ and $V=V(\bar{i}, p, t) \in \hat{V}$. The following are true:*

(1) *If $d \subseteqq V$, then there exists an integer $n \geqq 0$ such that $U(d, n) \subseteqq V$.*

(2) *If $d \cap V = \emptyset$, then there exists an integer $n \geqq 0$ such that $U(d, n) \cap V = \emptyset$.*

(3) *If $d \cap V \neq \emptyset$ and $d \cap ((C \times A) \setminus V) \neq \emptyset$ then there exists an open and closed*

neighbourhood $O(x)$ *of* x *in* $M_m(A_r^m)$ *such that* $d(z, m, r)\cap V\neq\emptyset$ *and* $d(z, m, r)\cap((C\times A)\setminus V)\neq\emptyset$ *for every* $z\in O(x)$.

PROOF. (1) By properties (1) and (8) of Remarks 3.4 it follows that $p<m$ and $A_r^m\subseteq A_t^p$. Hence $n(m, r)>n(p, t)$. Let $F=F(n(m, m, r), x)$.

Since $d\subseteq V$ and $n(m, r)>n(p, t)$ we have that $\phi_m(g)(x)\subseteq C_{\bar{i}}$ for every $g\in A_r^m$. Hence, by property (9) of Lemma 3.2 it follows that $(\phi_m(g)(F))^*\subseteq C_{\bar{i}}$.

By property (11) of Lemma 3.2 and since $n(m, r)>n(p, t)$ we have $C_{s(F)}\subseteq C_{\bar{i}}$. Since $A_r^m\subseteq A_t^p$ we have $C_{s(F)}\times A_r^m\subseteq C_{\bar{i}}\times A_t^p$. Hence, $U(m, m, r, F)=U(d, 0)\subseteq V(\bar{i}, p, t)$. Thus, the integer $n=0$ is the required integer.

(2) If $A_r^m\cap A_t^p=\emptyset$, then for any integer $n\in N$, $U(d, n)\cap V=\emptyset$. Hence, we can suppose that $A_r^m\cap A_t^p\neq\emptyset$.

Let $m\leqq p$. Then, $A_t^p\subseteq A_r^m$. Let $F=F(n(m, p, t), x)$. By the definition of the elements of $\hat{V}$ it follows that $U(m, p, t, F)\cap(C_{\bar{i}}\times A_t^p)=\emptyset$. Setting $n=m_2-m$ we have $U(d, n)\cap(C\times A_t^p)=U(m, p, t, F)$. Hence, $U(d, n)\cap V(\bar{i}, p, t)=\emptyset$, that is, the integer $n=m_2-m$ is the required integer.

Now, let $p<m$. Then, $A_r^m\subseteq A_t^p$. Let $e\in I(m-1)$ such that $A_r^m\subseteq A_e^{m-1}$ and $F=F(n(m, m, r), x)$.

We have $U(d, 0)=U(m, m, r, F)=C_{s(F)}\times A_r^m$ (See property (12) of Lemma 3.2). Hence, $U(d, 0)\cap V\neq\emptyset$ if and only if $C_{s(F)}\cap C_{\bar{i}}\neq\emptyset$.

If $g\in A_r^m$, then $st((\phi_m(g)(F))^*, n(m, r))=C_{s(F)}$ (See property (11) of Lemma 3.2). Since $d\cap V=\emptyset$ it follows that $st(\phi_m(g)(x), n(p, t))\cap C_{\bar{i}}=\emptyset$. Since $n(m-1, e)\geqq n(p, t)$, we have $st(\phi_m(g)(x), n(m-1, e))\subseteq st(\phi_m(g)(x), n(p, t))$ and, hence, $st(\phi_m(g)(x), n(m-1, e))\cap C_{\bar{i}}=\emptyset$.

By property (9) of Lemma 3.2 it follows that $st(\phi_m(g)(x), n(m-1, e))=st((\phi_m(g)(F))^*, n(m-1, e))$. Since $n(m, r)>n(m-1, e)$ we have $st((\phi_m(g)(F))^*, n(m, r))\subseteq st((\phi_m(g)(F))^*, n(m-1, e))$ and, hence, $st((\phi_m(g)(F))^*, n(m, r))\cap C_{\bar{i}}=\emptyset$, that is, the integer $n=0$ is the required integer.

(3) As in case (1) we have $p<m$ and $A_r^m\subseteq A_t^p$. Let $e\in I(m-1)$ such that $A_r^m\subseteq A_e^{m-1}$ and let $F=F(n(m, m, r), x)$.

Since $d\cap V\neq\emptyset$ there exists $g_1\in A_r^m$ such that $\phi_m(g_1)(x)\cap C_{\bar{i}}\neq\emptyset$. Also, since $d\cap((C\times A)\setminus V)\neq\emptyset$ there exists $g_2\in A_r^m$ such that $\phi_m(g_2)(x)\cap(C\setminus C_{\bar{i}})\neq\emptyset$. Since $n(m-1, e)\geqq n(p, t)$ there exist $\bar{i}_1, \bar{i}_2\in L_{n(m-1,e)}$ such that $C_{\bar{i}_1}\subseteq C_{\bar{i}}$, $C_{\bar{i}_2}\subseteq C\setminus C_{\bar{i}}$, $\phi_m(g_1)(x)\cap C_{\bar{i}_1}\neq\emptyset$ and $\phi_m(g_2)(x)\cap C_{\bar{i}_2}\neq\emptyset$.

By property (9) of Lemma 3.2, for every $g\in A_r^m$ and for every $z\in F$ we have $\phi_m(g)(z)\cap C_{\bar{i}_1}\neq\emptyset$ and $\phi_m(g)(z)\cap C_{\bar{i}_2}\neq\emptyset$, and, hence, $\phi_m(g)(z)\cap C_{\bar{i}}\neq\emptyset$ and $\phi_m(g)(z)\cap(C\setminus C_{\bar{i}})\neq\emptyset$, that is, $d(z, m, r)\cap V\neq\emptyset$ and $d(z, m, r)\cap((C\times A)\cap V)\neq\emptyset$. Thus, the neighbourhood $O(x)=F$ is the required neighbourhood of x in $M_m(A_r^m)$.

3.7. LEMMA. *Let* $d=\{(a, g)\}$, *where* $g=(S, D)$, $V, V_1\in\hat{V}$ *and* $U, U_1\in\hat{U}$. *The following are true:*

(1) *If* $d\subseteqq C_{\bar{i}}\times A_r^m$, *then there exists an element* W *of* $\hat{U}\cup\hat{V}$ *such that* $d\subseteqq W\subseteqq C_{\bar{i}}\times A_r^m$.

(2) *If* $V\cap V_1\neq\emptyset$, *then either* $V\subseteqq V_1$ *or* $V_1\subseteqq V$.

(3) *If* $d\subseteqq V\cap U$, *then there exists an element* W *of* $\hat{U}\cup\hat{V}$ *such that* $d\subseteqq W\subseteqq V\cap U$.

(4) *If* $d\subseteqq U\cap U_1$, *then there exists an element* W *of* $\hat{U}\cap\hat{V}$ *such that* $d\subseteqq W\subseteqq U\cap U_1$.

(5) *If* $d\cap V=\emptyset$, *then there exists an element* W *of* $\hat{U}\cup\hat{V}$ *such that* $d\subseteqq W$ *and* $W\cap V=\emptyset$.

(6) *If* $d\cap U=\emptyset$, *then there exists an element* W *of* $\hat{U}\cup\hat{V}$ *such that* $d\subseteqq W$ *and* $W\cap U=\emptyset$.

PROOF. Let $\bar{i}\in L_n$ and let k be an integer such that $k-1\geqq max\{n, m\}$.

There exists an integer $p\geqq k$ such that $st(a, n(p, t))\cap st((D_q)^*, n(p, t))=\emptyset$ for every $q\leqq k$, where $t=r(p, g)$.

Let $\bar{j}\in L_{n(p,t)}$ and $a\in C_{\bar{j}}$. Suppose that $\bar{j}\notin s(p, t)$. Then, the set $W=C_{\bar{j}}\times A_t^p$ belongs to $\hat{V}$. Obviously, we have $\{(a, g)\}\subseteqq W$, $C_{\bar{j}}\subseteqq C_{\bar{i}}$ and $A_t^p\subseteqq A_r^m$. Hence, $W\subseteqq V$, that is, W is the required element of $\hat{U}\cup\hat{V}$. Suppose that $\bar{j}\in s(p, t)$, that is, $\bar{j}\in s(q, p, t, F)$ for some q, $0\leqq q\leqq p$, and some $F\in(M_q(A_t^p))^{n(q,p,t)}$. Hence, $C_{\bar{j}}\subseteqq st((\phi_q(g)(F))^*, n(p, t))$ (See property (11) of Lemma 3.2). This means that $st(a, n(p, t))\cap st((D_q)^*, n(p, t))\neq\emptyset$ and, hence, $k<q$.

Let $x\in F$ and $\phi_q(g)(x)\cap C_{\bar{j}}\neq\emptyset$. Since $q>n$ we have that $\phi_q(g)(x)\subseteqq C_{\bar{i}}$. Let $Q=F(n(q, q, e), x)$, where $e=r(q, g)$. Since $n(q-1, r(q-1, g))>n$ we have that $st(\phi_q(g)(x), n(q-1, r(q-1, g)))\subseteqq C_{\bar{i}}$ and, hence $st(\phi_q(g)(Q))^*, n(q-1, r(q-1, g)))\subseteqq C_{\bar{i}}$ (See property (9) of Lemma 3.2). Since $n(q, e)>r(p-1, g))$ we have $st((\phi_q(g)(Q))^*, n(q, e))=C_{s(Q)}\subseteqq C_{\bar{i}}$.

By properties (11) and (12) of Lemma 3.2 it follows that $U(q, q, e, Q)=C_{s(Q)}\times A_e^q\subseteqq C_{\bar{i}}\times A_e^q\subseteqq V$.

Since $\{(a, g)\}\subseteqq U(q, q, e, Q)=U(d(x, q, e), 0)\in\hat{U}$, the set $W=U(q, q, e, Q)$ is the required element of $\hat{U}\cup\hat{V}$.

(2) Let $V=V(\bar{i}, m, r)$ and $V_1=V(\bar{j}, p, t)$. Since $V\cap V_1\neq\emptyset$ we have $A_r^m\cap A_t^p\neq\emptyset$ and $C_{\bar{i}}\cap C_{\bar{j}}\neq\emptyset$. Let $m\leqq p$. Then, $A_t^p\subseteqq A_r^m$ and since $n(p, t)\geqq n(m, r)$, $C_{\bar{j}}\subseteqq C_{\bar{i}}$. Hence, $V_1\subseteqq V$. Similarly, if $p\leqq m$, then $V\subseteqq V_1$.

(3) Let $U=U(d(x, m, r), n)$ and $V=V(\bar{i}, p, t)$. We have $\{(a, g)\}\subseteqq U(m, q, e, F)=C_{s(F)}\times A_e^q\subseteqq U$, where $q=m+n$, $e=r(q, g)$ and $F=F(n(m, q, e), x)$.

Let $k=max\{p, q\}$ and $n_1=max\{n(p, t), n(q, e)\}$. Let s be a subset of all elements $\bar{j}$ of L_{n_1} for which $C_{\bar{j}}\subseteq C_{\bar{i}}\cap C_{s(F)}$. Then, $C_s=C_{\bar{i}}\cap C_{s(F)}$. Also, we have $A_t^p\cap A_e^q=A_{r(k,g)}^k$. Then, $d\subseteq(C_{\bar{i}}\times A_t^p)\cap(C_{s(F)}\times A_e^q)=C_s\times A_{r(k,g)}^k\subseteq V\cap U$. Hence, the proof of this case follows from case (1).

(4) Let $U=(U(d(x, m, r), n)$ and $U_1=U(d(x_1, m_1, r_1), n_1)$. As in case (3) we have $d\subseteq C_{s(F)}\times A_e^q\subseteq U$, where $q=m+n$, $e=r(q, g)$ and $F=F(n(m, q, e), x)$. Similarly, $d\subseteq C_{s(F_1)}\times A_{e_1}^{q_1}\subseteq U_1$, where $q_1=m_1+n_1$, $e_1=r(q_1, g)$ and $F_1=F(n(m_1, q_1, e_1), x)$.

Let $p=max\{q, q_1\}$ and $k=max\{n(q, g), n(q_1, g)\}$. There exists a subset s of L_k such that $C_s=C_{s(F)}\cap C_{s(F_1)}$. Hence, $d\subseteq(C_{s(F)}\times A_e^q)\cap(C_{s(F_1)}\times A_{e_1}^{q_1})=C_s\times A_t^p\subseteq U\cap U_1$, where $t=r(p, g)$. The rest of the proof of this case follows from case (1).

(5) Let $V=V(\bar{i}, m, r)$ and let $a\in C_{\bar{j}}$, where $\bar{j}\in L_{n(m,r)}$. Since $d\cap V=\emptyset$ we have that either $C_{\bar{i}}\cap C_{\bar{j}}=\emptyset$ or $A_r^m\cap A_{r(m,g)}^m=\emptyset$. Hence, $(C_{\bar{j}}\times A_{r(m,g)}^m)\cap(C_{\bar{i}}\times A_r^m)=\emptyset$. Since $\{(a, g)\}\subseteq C_{\bar{j}}\times A_{r(m,g)}^m$, the existence of the set W follows from case (1).

(6) Let $U=U(d(x, m, r), n)$. Let $\bar{i}$ be an element of L_k, where $k=n(m+n, r(m+n, g))$, such that $a\in C_{\bar{i}}$. Then, it is easy to see that $(C_{\bar{i}}\times A_{r(m+n,g)}^{m+n})\cap U=\emptyset$. Hence, the proof of this case also follows from case (1).

3.8. LEMMA. *Let $d_1, d_2\in T$ and $d_1\neq d_2$. Then, there exist elements W_1 and W_2 of $\hat{U}\cup\hat{V}$ such that $d_1\subseteq W_1$, $d_2\subseteq W_2$ and $W_1\cap W_2=\emptyset$.*

PROOF. We consider the cases:

(1) $d_1=\{(a_1, g_1)\}$ and $d_2=\{(a_2, g_2)\}$,

(2) $d_1=\{(a, g)\}$ and $d_2=d(x, m, r)\in T(1)$, and

(3) $d_1=d(x_1, m_1, r_1)\in T(1)$ and $d_2=d(x_2, m_2, r_2)\in T(1)$.

In the first case either $a_1=a_2$ or $a_1=a_2$ and $g_1\neq g_2$. If $a_1\neq a_2$, then there exist an integer n and distinct elements $\bar{i}$ and $\bar{j}$ of L_n such that $a_1\in C_{\bar{i}}$ and $a_2\in C_{\bar{j}}$. Then, we set $V_1=C_{\bar{i}}\times A_{r(0,g_1)}^0$ and $V_2=C_{\bar{i}}\times A_{r(0,g_2)}^0$.

If $a_1=a_2$ and $g_1\neq g_2$, then there exists an integer m such that $r(m, g_1)\neq r(m, g_2)$. Then, we set $V_1=C_{\emptyset}\times A_{r(m,g_1)}^m$ and $V_2=C_{\emptyset}\times A_{r(m,g_2)}^m$.

In both subcases we have $d_1\subseteq V_1$, $d_2\subseteq V_2$ and $V_1\cap V_2=\emptyset$. By case (1) of Lemma 3.7 there exist elements W_1 and W_2 of $\hat{U}\cup\hat{V}$ such that $d_1\subseteq W_1\subseteq V_1$ and $d_2\subseteq W_2\subseteq V_2$. Hence, $W_1\cap W_2=\emptyset$.

In the second case if $g\notin A_r^m$, then there exists an element W_1 of $\hat{U}\cup\hat{V}$ such that $d_1\subseteq W_1\subseteq C_{\emptyset}\times A_{r(m,g)}^m$. Let $W_2=U(d(x, m, r), 0)$. Then, $W_1\cap W_2=\emptyset$.

Let $g\in A_r^m$. Then, $a\notin\phi_m(g)(x)$. There exists an integer $p\geqq m$ such that $st(a, n)\cap st((D_m)^*, n)=\emptyset$, where $n=n(p, r(p, g))$. Let $\bar{i}\in L_n$ such that $a\in C_{\bar{i}}$.

Then, $\bar{i}\notin s(m, p, e, F)=s(F)$, where $e=r(p, g)$ and $F=F(n(m, p, e), x)$ (See property (11) of Lemma 3.2).

Let $W_2=U(d(x, m, r), p-m)$. We have $W_2\cap(C_g\times A_e^p)=U(m, p, e, F)$. Since $U(m, p, e, F)=C_{s(F)}\times A_e^p$ and since $\bar{i}\in s(F)$ we have of $d\notin W_2$.

By property (6) of Lemma 3.7 it follows that there exists an element W_1 of $\hat{U}\cup\hat{V}$ such that $d\subseteqq W_1$ and $W_1\cap W_2$.

Finally in the third case we consider the following subcases: (α) $m_1=m_2$ and $r_1\neq r_2$, (β) $m_1=m_2$ and $r_1=r_2$. and (γ) $m_1\neq m_2$.

In the first subcase we set $W_1=U(d(x_1, m_1, r_1), 0)$ and $W_2=U(d(x_2, m_2, r_2), 0)$. Obviously, $d_1\subseteqq W_1$, $d_2\subseteqq W_2$ and $W_1\cap W_2=\emptyset$.

In the second subcase let $n_1\geqq n(m_1, m_1, r_1)$ be an integer such that there exist two distinct elements F_1 and F_2 of $(M_{m_1}(A_{r_1}^{m_1}))^{n_1}$ for which $x_1\in F_1$ and $x_2\in F_2$. Let $n=n_1-n(m_1, m_1, r_1)$. We set $W_1=U(d(x_1, m_1, r_1), n)$ and $W_2=U(d(x_2, m_2, r_2), n)$ and we prove that $W_1\cap W_2=\emptyset$.

Indeed, if $W_1\cap W_2\neq 0$, then there exists an element $r\in I(m_1+n)$ such that $A_r^{m_1+n}\subseteqq A_{r_1}^{m_1}$ and $(W_1\cap(C_g\times A_r^{m_1+n}))\cap(W_2\cap(C_g\times A_r^{m_1+n}))\neq\emptyset$. We have $W_1\cap(C_g\times A_r^{m_1+n})=U(m_1, m_1+n, r, F_1)$ and $W_2\cap(C_g\times A_r^{m_1+n})=U(m_2, m_2+n, r, F_2)$. Hence, $U(m_1, t_1, m_1+n, F_1)\cap U(m_2, m_2+n, r, F_2)\neq\emptyset$. By property (14) of Lemma 3.2 this is a contradiction.

In the third subcase, without loss of generality, we can suppose that $m_1<m_2$. Then, either $A_{r_2}^{m_2}\subseteqq A_{r_1}^{m_1}$, or $A_{r_2}^{m_2}\cap A_{r_1}^{m_1}=\emptyset$. If $A_{r_2}^{m_2}\subseteqq A_{r_1}^{m_1}$, then we set $W_1=U(d(x_1, m_1, r_1), m_2-m_1)$ and $W_2=U(d(x_2, m_2, r_2), 0)$. Obviously, we have $W_1\cap W_2=U(m_1, m_2, r_2, F_1)\cap U(m_2, m_2, r_2, F_2)=\emptyset$, where $F_1=F(n(m_1, m_2, r_2), x_1)$ and $F_2=F(n(m_2, m_2, r_2), x_2)$.

If $A_{r_2}^{m_2}\cap A_{r_1}^{m_1}=\emptyset$, then it is sufficient to put $W_1=U(d(x_1, m_1, r_1), 0)$ and $W_2=U(d(x_2, m_2, r_2), 0)$.

3.9. LEMMA. *Let $d\in T$ and $d\subseteqq W\in\hat{U}\cup\hat{V}$. There exists an element W_1 of $\hat{U}\cup\hat{V}$ such that $d\subseteqq W_1\subseteqq W$ and every element of $T(1)$ intersecting W_1, is contained in W.*

PROOF. First we suppose that $d=d(x, m, r)$. By property (1) of Lemma 3.5 and property (1) of Lemma 3.6 if follows that there exists an integer $n\geqq 0$ such that $U(d(x, m, r), n)\subseteqq W$.

We prove that the set $W_1=U(d(x, m, r), n+1)$ is the required element of $\hat{U}\cup\hat{V}$. Indeed, let $d_1=d(x_1, m_1, r_1)\in T(1)$ and $(a, g)\in d_1\cap W_1$. We have $U(d(x, m, r), n+1)\cap(C_g\times A_t^p)=U(m, p, t, F)$, where $p=n+m+1$, $t=r(m+n+1, g)$ and $F=F(n(m, p, t), x)$.

If $m_1<p$, then we can consider the set $U(m_1, p, t, F_1)$, where $F_1=F(n(m_1, p, t), x_1)$. Since $(a, g)\in U(m, p, t, F)\cap U(m_1, p, t, F_1)$ by properties (13) and (14) of Lemma 3.2 it follows that $m=m_1$ and $F=F_1$. In this case, by the definition of the elements of the set $\hat{U}$ it follows that $d_1\subseteq U(d(x, m, r), n+1)\subseteq U(d(x, m, r), n)$.

Hence, we can suppose that $m+n+1<m_1$. We have $(a, g)\in U(m, p, t, F)=C_{s(F)}\times A_t^p$. Hence, $a\in C_{s(F)}$.

Let $a\in C_{\bar{i}}$ and $\bar{i}\in L_k$, where $k=n(m_1-1, r(m_1-1, g))$. Since $a\in C_{s(F)}$ and $k\geqq n(p, t)$ we have $C_{\bar{i}}\subseteq C_{s(F)}$.

By property (9) of Lemma 3.2 it follows that if $g_1=(S_1, D_1)\in A^{m}_{r(m-1,g)}$, then $\phi_{m1}(g_1)(x_1)\cap C_{\bar{i}}\neq\emptyset$ (we observe that $a\in\phi_{m_1}(g)(x_1)$), that is $\phi_{m_2}(g_1)(x_1)\cap st((\phi_m(g_1)(F))^*, n(p, t))\neq 0$. By property (10) of Lemma 3.2 it follows that $\phi_m(g_1)(x_1)\subseteq st((\phi_m(g_1)(Q))^*, n(m+n, r(m+n, g))=C_{s(Q)}$, where $Q=F(n(m, m+n, r(m+n, g)), x)$. This means that $d_1\subseteq C_{s(Q)}\times A^{m+n}_{r(m+n,g)}=U(m, m+n, r(m+n, g))\subseteq U(d(x, m, r), n)$.

Now, we suppose that $d=\{(a, g)\}$, where $g=(S, D)$. It is easy to see that there exists an integer $m\geqq 0$ such that $(a, g)\in C_{\bar{i}}\times A^m_{r(m,g)}\subseteq W$, where $\bar{i}\in L_{n(m,r(m,g))}$. Let q_0 be an integer such that $q_0-1>n(m, r(m, g))$. Since D is an upper semi-continuous partition of S there exists an integer $p\geqq q_0$ such that $st(a, n(p, t))\cap st((D_q)^*, n(p, t))=\emptyset$, for every $q\leqq q_0$, where $t=r(p, g)$.

Let s be the subset of $L_{n(p,t)}$ for which $a\in C_s$ and either $s=\{\bar{j}\}$ and $\bar{j}\notin s(p, t)$ or $s=s(q, p, t, F)=s(F)$ for some q, $0\leqq q\leqq p$, and some $F=F(n(q, p, t), M_q(g))$.

We set $W_1=C_s\times A_t^p\in\hat{V}$ and we prove that $W_1\subseteq C_{\bar{i}}\times A^m_{r(m,g)}$. This is clear if $s=\{\bar{j}\}$. Suppose that $s=S(F)$. Then, $st(a, n(p, t))\cap st((D_q)^*, n(p, t))\neq\emptyset$ and, hence, $q_0<q$.

Let $x\in F$ and $\phi_q(g)(x)\cap st(a, n(p, t))\neq\emptyset$. Since $q>n(m, r(m, g))$ and $st(a, n(p, t))\subseteq C_{\bar{i}}$ we have that $\phi_q(g)(x)\subseteq C_{\bar{i}}$.

Let $Q=F(n(q, q, r(q, g)), x)$. Since $n(q-1, r(q-1, g))>n(m, r(m, g))$ by property (9) of Lemma 3.2 it follows that $(\phi_q(g)(Q))^*\subseteq C_{\bar{i}}$ and hence, $st((\phi_q(g)(Q))^*, n(q, r(q, g)))=C_{s(Q)}\subseteq C_{\bar{i}}$.

By properties (11) and (12) of Lemma 3.2 it follows that $U(q, q, r(q, g), Q)=C_{S(Q)}\times A^q_{r(q,g)}\subseteq C_{\bar{j}}\times A^m_{r(m,g)}$. Since $U(q, p, t, F)=U(q, q, r(q, g), Q)$ we have $W_1\subseteq C_{\bar{i}}\times A^m_{r(m,g)}$.

Now, we prove that if $d_1\in T(1)$ and $d_1\cap W_1\neq\emptyset$, then $d_1\subseteq C_{\bar{i}}\times A^m_{r(m,g)}$. Indeed, let $d_1=d(x_1, m_1, t_1)$ and $(a_1, g_1)\in d_1\cap W_1$.

If $m_1\leqq p$, then we can consider the set $U(m_1, p, t, F_1)=U(F_1)$, where $F_1=F(n(m_1, p, t), x_1)$. Obviously, $d_1\cap W_1\subseteq U(F_1)\cap W_1$. It $s=\{\bar{j}\}$ and $\bar{j}\notin s(p, t)$, then

$U(F_1)\cap W_1=\emptyset$ which is contradiction. Hence, $s=s(F)$ and since $U(m_1, p, t, F_1)$ $\cap U(q, p, t, F)\neq\emptyset$ by properties (13) and (14) of Lemma 3.2 it follows that $m_1=q$ and $F=F_1$. Hence, $d_1\subseteqq U(F)=W_1\subseteqq C_{\bar{i}}\times A^m_{r(m,g)}$.

Thus we can suppose that $p<m_1$. Obviously, $A^{m_1}_{r(m_1,g_1)}\subseteqq A^p_t$. Since $a_1\in C_s$ and $n(m_1-1, r(m_1-1), g_1))\geqq n(p, t)$ by property (9) of Lemma 3.2 it follows that if g_0 is an arbitrary element of $A^{m_1}_{r(m_1,g_1)}$, then $\phi_{m_1}(g_0)(x_1)\cap C_s\neq\emptyset$. Since $m_1>n(m, r(m, g))$ we have that $\phi_{m_1}(g_0)(x_1)\subseteqq C_{\bar{i}}$, that is, $d_1\subseteqq C_{\bar{i}}\times A^m_{r(m,g)}$.

3.10. Definitions and notations. For every $U=U(d, n)\in\hat{U}$ (respectively, $V=V(\bar{i}, m, r)\in\hat{V}$) we denote by $O(U)$ or by $O(d, n)$ (respectively, by $O(V)$ or by $O(\bar{i}, m, r)$) the set of all elements $d\in T$ such that $d\subseteqq U$ (respectively, $d\subseteqq V$).

We denote by $\mathcal{U}$ (respectivety, by $\mathcal{V}$) the set of all sets of the form $O(U)$, $U\in\hat{U}$ (respectively, $O(V)$, $V\in\hat{V}$). Also, we set $\boldsymbol{B}=\mathcal{U}\cup\mathcal{V}$.

Let $m\in N$, $r\in I(m)$ and F be a subset of $M_m(A^m_r)$. We denote by $d(F)$ the subset of T consisting of all elements $d(x, m, r)$, where $x\in F$.

By $d(m, r)$ we denote the map of $M_m(A^m_r)$ onto $d(M_n(A^m_r))$ defined as follows: $d(m, r)(x)=d(x, m, r)$. Obviously, the map $d(m, r)$ is one-to-one.

We say that a pair (S, D), where S is a subset of C and D is an upper semi-continuous partition of C, has the *dense property* iff for every $k=0, 1, \cdots$ and for every $a\in d\in D_k$ the point a is o limit point of the set $S\setminus(D_k)^*$.

3.11. Theorem. *The set $\boldsymbol{B}$ is a countable basis of open sets for a topology τ on the set T. The space T (that is, the set T with topology τ) is a Hausdorff regular space. The boundary of every element of $\boldsymbol{B}$ is a countable free union of subsets of T which are homeomorphic to closed subsets of elements of M. Moreover, if every element of the family A has the dense property, then the boundary of every element of $\boldsymbol{B}$ is a countable free union of subsets of T which are homeomorphic to simultaneously open and closed subsets of elements of $\boldsymbol{M}$.*

Proof. If $m, n\in N$, $r\in I(m)$, $F\in(M_m(A^m_r))^k$, where $k=n(m, m, r)+n$, and $x, y\in F$, then $U(d(x, m, r), n)=U(d(y, m, r), n)$. From this and since for every $m\in N$ the set A^m is countable it follows that the set $\hat{U}$, as well as, the set $\hat{V}$ are countable. Hence, $\boldsymbol{B}$ is a countable set.

It is easy to see that the union of all elements of $\boldsymbol{B}$ is the set T. Hence in order to prove that $\boldsymbol{B}$ is a basis of open sets for a topology on the set T it is sufficient to prove that if $d\in T$, $W_1, W_2\in\hat{U}\cup\hat{V}$ and $d\in O(W_1)\cap O(W_2)$, then there exists an element W of $\hat{U}\cup\hat{V}$ such that $d\in O(W)\subseteqq O(W_1)\cap O(W_2)$, that is, $d\subseteqq W\subseteqq W_1\cap W_2$. This follows immediately from the properties (1) of Lemma

3.5, (1) of Lemma 3.6, (5) of Remarks 3.4 and from properties (2), (3) and (4) of Lemma 3.7.

Let τ be the topology on T for which $\boldsymbol{B}$ is a basis of open sets. By Lemma 3.8 it follows that the space T is a Hausdorff space.

We observe that by properties (2) of Lemma 3.5, (2) of Lemma 3.6 and by (5) and (6) of Lemma 3.7 it follows that in the space T the boundary of every element of $\boldsymbol{B}$ is contained in the subset $T(1)$ of T. Hence, by Lemma 3.9 it follows that the space T is regular.

Let $m \in N$ and $r \in I(m)$. We prove that the map $d(m, r)$ of $M_m(A_r^m)$ onto $d(M_m(A_r^m))$ is a homeomorphism. Indeed, by properties (1) of Lemma 3.5, (1) of Lemma 3.6 and (5) of Remarks 3.4 it follows that the set $\{U(d(x, m, r), n), n \in N\}$ is a basis of open neighbourhoods of $d(x, m, r)$ (in the space T).

On the other hand, the set $\{F(n(m, m, r)+n, x): n \in N\}$ is a basis of open neighbourhoods of x in $M_m(A_r^m)$ (See Definitions and notations 3.1).

Also, by the construction of elements of $\hat{U}$ it follows that an element $d(y, m, r)$ of $d(M_m)A_r^m))$ belongs to $U(d(x, m, r), n)$ if and only if $y \in F(n(m, m, r)+n, x)$. From this it follows that the map $d(m, r)$ is a homeomorphism.

Let $m \in N$ and $r \in I(m)$. Let $V = C_s \times A_r^m$, where s is a subset of $L_{n(m,r)}$ such that either $s = \{\bar{i}\}$ and $\bar{i} \notin s(m, r)$ or $s = s(F)$ for some element F of $M_q(A_r^m)^{n(q,m,r)}$, $0 \leqq q \leqq m$. We grove that for every $p > n(m, r)$ and $t \in I(p)$ is $y \in M_p(A_t^p)$ and $d(y, p, t) \cap V \neq \emptyset$ (hence, $A_t^p \subseteqq A_r^m$), then $d(y, p, t) \subseteqq V$.

Indeed, let $(a, g) \in d(y, p, t) \cap V$. Let $a \in C_{\bar{j}}$, where $\bar{j} \in L_{n(p-1, r(p-1, g))}$. Since $n(p-1, r(p-1, g)) > p-1 \geqq n(m, r)$ we have that $C_{\bar{j}} \subseteqq C_s$. By property (9) of Lemma 3.2 it follows that $\phi_p(g_1)(y) \cap C_{\bar{j}} \neq \emptyset$ for every $g_1 \in A_t^p$. Since $p > n(m, r)$ we have that $\phi_p(g_1)(y) \subseteqq C_s$ and, hence, since $A_t^p \subseteqq A_r^m$ we have that $d(y, p, t) \subseteqq C_s \times A_r^m = V$.

Now, let $s = \{\bar{i}\}$ and $\bar{i} \notin s(m, r)$, that is, $V = V(\bar{i}, m, r) \in \hat{V}$. Then, by property (8) of Remarks 3.4 and by Lemma 3.6 (properties (1) and (2)) it follows that the boundary $Bd(O(V))$ of the element $O(V)$ of $\boldsymbol{B}$ is contained in the set $B(k, m, r)$, where $k = n(m, r)$, which is the union of all sets of the form $(M_q(A_e^q))$, where $m < q \leqq k$ and $e \in I(q)$ such that $A_e^q \subseteqq A_r^m$.

We prove that the set $B(k, m, r)$ is the free union of the corresponding sets $d(M_q(A_e^q))$. For this it is sufficient to prove that for every q, $m \leqq q \leqq k$, and for every $e \in I(q)$ for which $A_e^q \subseteqq A_r^m$, there exists and open subset $H(q, e, m, r)$ $H(q, e)$ of T such that $B(k, m, r) \cap H(q, e) = d(M_q(A_e^q))$.

For every $F \in (M_q(A_e^q))^{n(q,q,e)+k-q}$ by $x(F)$ we denote a point of F. We set $H(q, e) = \bigcup_F O(d(x(F), q, e), k-q)$. Obviously, $H(q, e)$ is an open subset of T.

Also, it is easy to see that $d(M_q(A_e^q)) \subseteq Q(k, m, r) \cap H(q, e)$.

Let $d(y, q_1, e_1) \in B(k, m, r) \cap H(q, e)$. We prove that $d(y, q_1, e_1) \in d(M_q(A_e^q))$. Indeed since $d(y, q_1, e_1) \in B(k, m, r)$ we have $m < q_1 \leqq k$ and $A_{e_1}^{q_1} \subseteq A_r^m$. There exists an element F of $(M_q(A_e^q))^{n(q,q,e)+k-q}$ such that $d(y, q_1, e_1) \cap U(d(x(F), q, e), k-q) \neq \emptyset$. Let (a, g) belongs to this intersection. Consider the sets $U(q_1, k, r(k, g), F_1) = U(F_1)$ and $U(q, k, r(k, g), F) = U(F)$, where $F_1 = F(n(q_1, k, r(k, g)), y)$. Since $(a, g) \in U(F) \cap U(F_1)$ by properties (13) and (14) of Lemma 3.2 it follows that $q = q_1$ and $F = F_1$, that is, $d(y, q_1, e_1) \in d(M_q(A_e^q))$.

Thus, $B(k, m, r) \cap H(q, e) = d(M_q(A_e^q))$ and hence, the boundary of the set $O(\bar{i}, m, r)$ is a countable free union of subsets of T which are homeomorphic to closed subsets of elements of $\boldsymbol{M}$.

Suppose now that $U = U(d(x_1, m_1, r_1), n_1)$ be an arbitrary element of $\hat{U}$. Let $m = m_1 + n_1$. We prove that the boundary $Bd(O(U))$ of the set $O(U)$ is contained in the union of all sets of the form $B(n(m, r), m, r)$, where $r \in I(m)$ and $A_r^m \subseteq A_{r_1}^{m_1}$.

Indeed, let $d(y, p, t) \in Bd(O(U))$ and let $(a, g) \in d(y, p, t) \cap U$. There exist an integer q, $0 \leqq q \leqq m$, an element $r \in I(m)$ and an element $F \in (M_q(A_r^m))^{n(q,m,r)}$ such that $(a, g) \in U(q, m, r, F) = U(F)$. If $p \leqq m$, then we can consider the set $U(p, m, r, Q) = U(Q)$, where $Q = F(n(p, m, r), y)$. (We observe that $r(m, g) = r$). Then, $(a, g) \in U(F) \cap U(Q)$ and, hence, $p = q$ and $F = Q$, that is, $d(y, p, t) \subseteq U$, which is a contradiction. Hence, $m < p$.

On the other hand, since $U(F) = C_{s(F)} \times A_r^m$, $d(y, p, t) \cap U \neq \emptyset$ and $d(y, p, t) \nsubseteq U$ by the preceding it follows that $p \leqq n(m, r)$. Hence, $d(y, p, t) \in B(n(m, r), m, r)$.

Let $k = n(m, r)$. For a fixed $r \in I(m)$ as we already proved the set $B(k, m, r)$ is the free union of the corresponding sets $d(M_q(A_e^q))$. Since the union of all elements of $H(q, e, m, r)$ is contained in the set $C \times A_r^m$ we have that the union of sets $B(k, m, r)$ for all $r \in I(m)$ for which $A_r^m \subseteq A_{r_1}^{m_1}$ is also free.

Hence, the boundary of the set $O(d(x_1, m_1, r_1), m_1)$ is a countable free union of subsets of T which are homeomorphic to closed subset of elements of $\boldsymbol{M}$.

Finally, suppose that every element of the family A has the dense property. In this case we prove that if $O(W) \in \boldsymbol{B}$ and $d = d(x, m, r) \in T(1)$ such that $d(x, m, r) \cap W \neq \emptyset$ and $d(x, m, r) \cap ((C \times A) \setminus W) \neq \emptyset$, then $d \in Bd(O(W))$.

Indeed, obviously, $d \notin O(W)$. Let $g \in A_r^m$ such that $(\phi_m(g)(x) \times \{g_1\}) \cap W \neq \emptyset$. Let $O(U)$ be an arbitrary neighbourhood of d in T. We prove that $O(U) \cap O(W)$. $\neq \emptyset$. We can suppose that $U = U(d(x, m, r), n)$ for some integer $n \in N$.

Let $\phi_m(g)(x) = \{a, b\} \in D(1)$. We can suppose that $(a, g) \in W$ and that there exists an integer q such that $(a, g) \in V = C_s \times A_{r(q,g)}^q \subseteq U \cap W$, where s is a subset of $L_{n(q, r(q, g))}$ and either $s=\{\bar{i}\}$ and $\bar{i}\in s(q, r(q, g))$ or $s=s(F)$ for some element F of $(M_k(A^q_{r(q, g)}))^{n_1}$, where $n_1=n(k, q, r(q, g))$ and $0\leqq k\leqq m$. Let $V\cap(C\times\{g\})=O\times\{g\}$. Then, O is an open neighbourhood of a in C.

Since g has the dense property there exists a point $c\in O\cap(S\setminus(D_m)^*)$ such that either $c\in S\setminus(D(1))^*$ or $c\in d_1\in D_p$ and $p>n(q, r(q, g))$. In the first case, $\{(c, g)\}\in O(V)\subseteqq O(U)\cap O(W)$, and hence $O(U)\cap O(W)\neq\emptyset$.

In the second case, let $y\in M_p(A^p_{r(p, g)})$ such that $c\in\psi_p(g)(y)$. As we proved above, $d(y, p, r(p, g))\subseteqq V$. Hence, $d(y, p, r(p, g))\in O(V)\subseteqq O(U)\cap O(W)$ and $O(U)\cap O(W)\neq\emptyset$. Thus, $d\in Bd(O(W))$.

By properties (3) of Lemma 3.5 and (3) of Lemma 3.6 it follows that the boundary of every element of $\boldsymbol{B}$ is a countable free union of subsets of T which are homeomorphic to simultaneously open and closed subsets of elements of $\boldsymbol{M}$.

4. Some properties of scattered spaces.

Definitions and notations. Let $\alpha=\beta+m$ be an ordinal, where $\beta=\beta(\alpha)$ and $m=m(\alpha)>0$.

We denote by $Tr(\alpha)$ the set of all triads $\tau=(a, X, M)$ such that: (α) M is a compactum having type α, (β) $M^{(\alpha-1)}=\{a\}$, and (γ) X is a subset of M for which $M\setminus M^{(\beta)}\subseteqq X$. We observe that if U is an open and closed neighbourhood of a in M, then the triad $(a, X\cap U, U)=\tau(U)$ is an element of $Tr(\alpha)$.

Let $\tau_1=(a_1, X_1, M_1)$ and $\tau_2=(a_2, X_2, M_2)$ be two elements of $T_r(\alpha)$. We say that τ_1 and τ_2 are *equivalent* and we write $\tau_1\sim\tau_2$ iff there exist: (α) an open and closed neighbourhood U of a_1 in M_1, (β) an open and closed neighbourhood V of a_2 in M_2, and (γ) a homeomorphism f of U onto V such that $f(U\cap X_1)=V\cap X_2$ (Obviously, in this case $f(a_1)=f(a_2)$).

It is easy to prove that the relation "$\sim$" on the set $Tr(\alpha)$ is an equivalent relation. We denote by $ETr(\alpha)$ the set of all equivalence classes of this relation. For every $\tau\in T_r(\alpha)$ we denote by $e(\tau)$ the equivalence class of $ETr(\alpha)$ which contains the element τ.

Let $\tau=(a, X, M)\in Tr(\alpha)$. An open and closed neighbourhood U of a in M is called *standard* iff tor every $\tau_1=(a_1, X_1, M_1)\in e(\tau)$ there exists an open and closed neighbourhood V of a_1 in M_1 and a homeomorphism f of U onto V such that $f(U\cap X)=V\cap X_1$. In this case we say that the element τ *has a standard neighbourhood*. It is clear that it an element of an equivalence class of $ETr(\alpha)$ has a standard neighbourhood, then every element of this class has also a standard neighbourhood.

The element τ is called *standard* iff the neighbourhood $U=M$ of a is standard. Obviously, if U is a standard neighbourhood of a in M, then $\tau(U)$ is a standard element of $e(\tau)$.

It is easy to prove that an open and closed ueighbourhood U of a in M is standard if and only if for every neighbourhood W of a in M there exist an open and closed neighbourhood V of a in M, which is contained in W and a homeomorphism f of U onto V such that $f(U\cap X)=V\cap X$.

We denote by $P(\alpha)$ the set of all pairs $\zeta=(X, M)$ such that M is a compactum having type α and X is a subset of M for which $M\setminus M^{(\beta)}\subseteqq X$.

We say that the pairs $\zeta_1=(X_1, M_1)$ and $\zeta_2=(X_2, M_2)$ of $P(\alpha)$ are *equivalent* and we write $\zeta_1\sim\zeta_2$ iff there exists a homeomorphism f of M_1 onto M_2 such that $f(X_1)=X_2$.

It is clear that the relation "$\sim$" on the set $P(\alpha)$ is an equivalent relation. We denote by $EP(\alpha)$ the set of all equivalent classes of this relation and for every $\zeta\in P(\alpha)$ by $e(\zeta)$ the equivalence class of $EP(\alpha)$ which contains the element ζ.

4.2. LEMMA. *For every isolated ordinal α the set $ETr(\alpha)$ is finite and every element of this set contains a standard element of $Tr(\alpha)$.*

PROOF. Let $\alpha=\beta-m$, where $\beta=\beta(\alpha)$ and $m=m(\alpha)>0$. We prove the lemma by induction on integer m.

Let $m=1$. Let $\tau_1=(a_1, X_1, M_1)\in Tr(\alpha)$ and $\tau_2=(a_2, X_2, M_2)\in Tr(\alpha)$ such that $X_1=M_1$ and $X_2=M\setminus M^{(\beta)}=M\setminus\{a_2\}$.

Let $\tau=(a, X, M)$ be an element of $Tr(\alpha)$. Then, $M^{(\beta)}=M^{(\alpha-1)}=\{a\}$ and, hence, either $X=M$ or $X=M\setminus M^{(\beta)}=M\setminus\{a\}$. By [M-S] it follows that there exist a homeomorphism f_1 of M_1 onto M and a homeomorphism f_2 of M_2 onto M. We have that if $X=M$, then $f_1(X_1)=X$ and if $X=M\setminus M^{(\beta)}$, then $f_2(X_2)=X$. Hence, either $e(\tau)=e(\tau_1)$ or $e(\tau)=e(\tau_2)$, that is, $ETr(\alpha)=\{e(\tau_1), e(\tau_2)\}$. Also, by the above it follows that the elements τ_1 and τ_2 are standard.

Now, we suppose that the lemma is proved for every m for which $1\leqq m<n$ and we prove it for $m=n$.

Let $ETr(\alpha_1)=\{e^1(\alpha-1), \cdots, e^t(\alpha-1)\}$. For every $k=1, \cdots, t$ we denote by $\tau^k(\alpha-1)=(c^k, X^k, M^k)$ a fixed standard element of $e^k(\alpha-1)$.

Let $\tau_j=(a_j, X_j, M_j)$, $j=1, 2$, be two arbitrary elements of $Tr(\alpha)$. Whithout loss of generality we can suppose that the spaces M_1 and M_2 are metric.

Let $M_j^{(\alpha-2)}\setminus M_j^{(\alpha-1)}=\{b_{j1}, b_{j2}, \cdots\}$, $j=1, 2, \cdots$. Every element of these sets is isolated (in the corresponding relative topology). Let W_{ji}^0 be an open and closed neighbourhood of b_{ji} in M_j such that $W^0_{ji}\cap M_j^{(\alpha-2)}=\{b_{ji}\}$. Then the triad $\tau_{ji}=(b_{ji}, X_j\cap W^0_{ji}, W^0_{ji})$ is an element of $Tr(\alpha)$ and the element $e(\tau_{ji})$ of $ETr(\alpha)$ is independent from the neighbourhood W^0_{ji}, that is, if W'_{ji} is another such neighbourhood of b_{ji} in M_j and $\tau'_{ji}=(b_{ji}, X_j\cap W'_{ji}, W'_{ji})$, then $e(\tau_{ji})=e(\tau'_{ji})$. We denote by e_{ji} the element $e(\tau_{ji})$.

There exists an open and closed neighbourhood W_{ji} of b_{ji} in M_j, $j=1, 2$, $i=1, 2, \cdots$, such that: (α) $W_{ji}\cap M_j^{(\alpha-2)}=\{b_{ji}\}$, ($\beta$) $W_{ji_1}\cap W_{ji_2}=\emptyset$ if $i_1\neq i_2$, (γ) $\lim_{i\to\infty}(diam(W_{ji}))=0$, ($\delta$) $a_j\in(M_j\setminus W_j)^{(\alpha-2)}$, where $W_j=W_{j1}\cup W_{j2}\cup\cdots$ and (ε) if $e_{ji}=e^{k(ji)}(\alpha-1)$, then there exists a homeomorphism f_{ji} of $M^{k(ji)}$ onto W_{ij} such that $f_{ji}(X^{k(ji)})=X_j\cap W_{ji}$. We observe that by the properties of the sets W_{ji} it follows that W_j, $j=1, 2, \cdots$, is an open subset of M_j such that $Cl(W_j)\setminus W_j=\{a_j\}$.

Let V_j be an open and closed neighbourhood of a_j in $M_j\setminus W_j$ such that $(V_j)^{(\alpha-2)}=\{a_j\}$. Then, the triad $\tau^j=(a_j, X_j\cap V_j, V_j)$ is an element of $Tr(\alpha-1)$. We can suppose that if $e(\tau^j)=e^{k(j)}(a-1)$, then there exists a homeomorphism f_j of $M^{k(f)}$ onto Λ_j such that $f_j(X^{k(j)})=X_j\cap V_j$.

There exists an open and closed neighbourhood U_j, $j=1, 2$, of a_j in M_j such that: (α) $U_j\cap(M_j\setminus W_j)=V_j$, ($\beta$) if for some integer $i=1, 2, \cdots$, $W_{ji}\cap U_j\neq\emptyset$, then $W_{ji}\subseteqq U_j$, and (γ) if for some integer i, $W_{ji}\subseteqq U_j$, then there exists an increasing sequence of integers $i_1, i_2, \cdots$ for which $W_{ji_q}\subseteqq U_j$ and $e_{ji}=e_{ji_q}$, $q=1, 2, \cdots$.

Now, we prove that $\tau_1\sim\tau_2$ if the following conditions are true: (α) $e(\tau^1)=e(\tau^2)$ and (β) if for some integer $k\in\{1, \cdots, t\}$ there exists an integer $i(1)\geqq 1$ such that $W_{1i(1)}\subseteqq U_1$ and $e_{1i(1)}=e^k(\alpha-1)$, then there exists an integer $i(2)\geqq 1$ such that $W_{2i(2)}\subseteqq U_2$ and $e_{2i(2)}=e^k(\alpha-1)$.

Indeed, it is not difficult to prove that between the set $U_1\cap(M_1^{(\alpha-1)}\setminus M_1^{(\alpha-1)}$ and the set $U_2\cap(M_2^{(\alpha-2)}\setminus M_2^{(\alpha-1)})$ there exists an one-to-one correspondence such that if b_{1p} corresponds to b_{2q}, then $e_{1p}=e_{2q}$.

We construct a homeomorphism f of U_1 onto U_2 as follows: on the set V_1 we set $f=f_2\circ f_1^{-1}$. Let $W_{1p}\subseteqq U_1$. Then, $b_{1p}\in U_1$ and if b_{1p} corresponds to b_{2q}, then on the set W_{1p} we set $f=f_{2q}\circ f_{1p}^{-1}$. Obviously, f is a homeomorphism of U_1 onto U_2 such that $f(X_1\cap U_1)=X_2\cap U_2$. Hence, $\tau_1\sim\tau_2$.

From the above it follows that the number of equivalence classes of the set $Tr(\alpha)$ is finite, that is, the set $ETr(\alpha)$ is finite.

In order to complete the lemma it is sufficient to prove that every element of $ETr(\alpha)$ contains a standard element of $Tr(\alpha)$. For this, since τ_1 is an abitrary element of $Tr(\alpha)$, it is sufficient to prove that $\tau_1(U_1)$ is a standard element.

Let W be an arbitrary neighbourhood of a in M_1. Let V be an open and closed neighbourhood of a_1 in $M_1 \setminus W_1$ such that: (α) $V \subseteqq W$ and (β) there exists a homeomorphism f_V of $M^{k(1)}$ onto V for which $f_V(X^{k(1)}) = X_1 \cap V$.

There exists a neighbourhood U' of a_1 in M_1 such that: (α) $U' \subseteqq W$, (β) $U' \cap (M_1 \setminus W_1) = V$ and (γ) if for some integer i, $W_{1i} \cap U' \neq \emptyset$, then $W_{1i} \subseteqq U'$.

A homeomorphism f' of U_1 onto U' for which $f'(X_1 \cap U_1) = X_1 \cap U'$ can be constructed in the same manner as we constructed the homeomorphism f of U_1 onto U_2. Hence, $\tau(U_1)$ is a standard element.

4.3. THEOREM. *For every isolated orainal α the set $EP(\alpha)$ is countable.*

PROOF. Let $\alpha = \beta + m$, where $\beta = \beta(\alpha)$ and $m = m(\alpha) \geqq 1$. We prove the theorem by induction on integer m.

Let $m=1$. For every $i = 1, 2, \cdots$ we denote by M_i a compactum such that $|M_i^{(\alpha-1)}| = |M_i^{(\beta)}| = i$. Hence, if X_1 and X_2 are two subsets of M_k for which $M \setminus M^{(\beta)} \subseteqq X_1 \cap X_2$, then $X_1 = X_2$ iff $X_1 \cap M^{(\alpha-1)} = X_2 \cap M^{(\alpha-1)}$. Therefore, the number of such set is finite. Let $X_{i1}, \cdots, X_{it(i)}$ be these sets and let $\zeta_{ij} = (X_{ij}, M_i)$, $i = 1, 2, \cdots$, $j = 1, \cdots, t(i)$.

Let $\zeta = (X, M)$ be an arbitrary element of $P(\alpha)$ and let $|M^{(\alpha-1)}| = i$. Then, by [M-S] there exists a homeomorphism f of M_i onto M. There exists an integer j, $1 \leqq j \leqq t(i)$, such that $X_{ij} = f^{-1}(X)$. Hence, $f(X_{ij}) = X$, that is, $\zeta \sim \zeta_{ij}$. From this it follows that the set $EP(\alpha)$ is countable.

We suppose that the theorem is proved for every m for which $1 \leqq m < n$ and we prove the theorem for $m = n$.

Let $\tau^1 = (c_1, X^1, M^1), \cdots, \tau^2 = (c^p, X^p, M^p)$ be standard elements of $Tr(\alpha-1)$ such that $ETr(\alpha-1) = \{e(\tau^1), \cdots, e(\tau^p)\}$. Also, let $\zeta(1) = (X(1), M(1))$, $\zeta(2) = (X(2), M(2))$, $\cdots$ be elements of $P(\alpha-1)$ such that $EP(\alpha-1) = \{e(\zeta(1)), e(\zeta(2)), \cdots\}$.

Now, let $\zeta_j = (X_j, M_j)$, $j = 1, 2$, be two arbitrary elements of the set $P(\alpha)$, such that $|M_j^{(\alpha-1)}| = \{a_{j1}, \cdots, a_{jt}\}$. Without loss of generality we can suppose that the spaces M_1 and M_2 are metric. There exists en open and closed subset U_{ji} of M_j, $j = 1, 2$, $t = 1, \cdots, i$, such that: (α) $U_{ji_1} \cap U_{ji_2} = \emptyset$ if $i_1 \neq i_2$, (β) $U_{j1} \cup \cdots \cup U_{ji} = M_j$, and ($\gamma$) $a_{ji} \in U_{ji}$.

Let $U_{ji} \cap (M_j^{(\alpha-2)} \setminus M_j^{(\alpha-1)}) = \{b_{ji}^1, b_{ji}^2, \cdots\}$. Let $(W_{ji}^k)^0$ be an arbitrary neighbourhood of b_{ji}^k in M_j, $k = 1, 2, \cdots$, such that: (α) $(W_{ji}^k)^0 \subseteqq U_{ji}$ and (β) $(W_{ji}^k)^0 \cap M_j^{(\alpha-2)} = \{b_j^k\}$. We denote by e_{ji}^k the element $e(\tau_{ji}^k)$ of $ETr(\alpha-1)$, where $\tau_{ji}^k = (b_{ji}^k, X_j \cap (W_{ji}^k)^0, (W_{ji}^k)^0)$. Obviously, the element e_{ji}^k is independent from the neighbourhood $(W_{ji}^k)^0$.

For every $j=1, 2$, $i=1, \cdots, t$, $k=1, 2, \cdots$, let W_{ji}^k be an open and closed neighbourhood of b_{ji}^k in M_j such that: $W_{ji}^k \subseteq U_{ji}$, ($\beta$) $W_{ji}^k \cap M_j^{(\alpha-2)} = \{b_{ji}^k\}$, ($\gamma$) $W_{ji}^{k_1} \cap W_{ji}^{k_2} = \emptyset$, if $k_1 \neq k_2$, (δ) $\lim_{k\to\infty}(diam(W_{ji}^k))=0$, ($\varepsilon$) the set $(U_{ji} \setminus W_{ji})^{(\alpha-2)}$, where $W_{ji} = W_{ji}^1 \cup W_{ji}^2 \cup \cdots$ contains at least two distinct points and the point a_{ji} belongs to this set, and (ζ) if $e_{ji}^k = e(\tau^{r(kji)})$, then there exists a homeomorphism f_{ji}^k of $M^{r(kji)}$ onto W_{ji}^k such that $f_{ji}^k(X^{r(kji)}) = X_j \cap W_{ji}^k$. Obviously, W_{ji} is an open subset of M_j such that $Cl(W_{ji}) \setminus W_{ji} = \{a_{ji}\}$.

Let V_{ji} be an open and closed neighbourhood of a_{ji} in $M_{ji} \setminus W_{ji}$ such that $V_{ji} \subseteq U_{ji}$ and $(V_{ji})^{(\alpha-2)} = \{a_{ji}\}$. The triad $\tau_{ji} = (a_{ji}, X_j \cap V_{ji}, V_{ji})$ is an element of $Tr(\alpha-1)$. We suppose that if $e(\tau_{ji}) = e(\tau^{r(ji)})$, then there exists a homeomorphism f_{ji} of $M^{r(ji)}$ onto V_{ji} such that $f_{ji}(X^{r(ji)}) = X_j \cap V_{ji}$.

We observe that the set $H_{ji} = U_{ji} \setminus (W_{ji} \cup V_{ji})$ is an open and closed subset of M_j and by property (ε) of the sets W_{ji}^k it follows that $(H_{ji})^{(\alpha-2)} \neq \emptyset$. Hence, the pair $\zeta_{ji} = (X_j \cap H_{ji}, H_{ji})$ is an element of $P(\alpha-1)$.

If $e(\zeta_{ji}) = e(\zeta(q(ji)))$, then by g_{ji} we denote a homeomorphism of $M(q(ji))$ onto H_{ji} such that $g_{ji}(X(q(ji))) = X_j \cap H_{ji}$.

Now, we prove that $\zeta_1 \sim \zeta_2$ if the following conditions are true: (α) for a given element $e(\tau^r)$ of $ETr(\alpha-1)$ and for a fixed integer i, the number of elements b_{1i}^k of the set $\{b_{1i}^1, b_{1i}^2, \cdots\}$ for which $e(\tau^r) = e_{1i}^k$ is the same with the number of the elements b_{2i}^k of the set $\{b_{2i}^1, b_{2i}^2, \cdots\}$ for which $e_{2i}^k = e(\tau^r)$, (β) for every integer $i=6, \cdots, t$, $e(\tau_{1i}) = e(\tau_{2i})$, and ($\gamma$) for every integer $i=1, \cdots, t$, $e(\zeta_{1j}) = e(\zeta_{2i})$.

Indeed, by the above condition (α) it follows that for every integer i, betweed the elements of the set $\{b_{1i}^1, b_{1i}^2, \cdots\}$ and the elements of the set $\{b_{2i}^1, b_{2i}^2, \cdots\}$ there exists an one-to-one correspondence such that if b_{1i}^k corresponds to b_{2i}^τ, then $e_{1i}^k = e_{2i}^\tau$.

We construct a homeomorphism f of M_1 onto M_2 as follows: for every integer i, on the set V_{1i} we set $f = f_{2i} \circ f_{1i}^{-1}$ and on the set H_{1i} we set $f = g_{2i} \circ g_{1i}^{-1}$. If the point b_{1i}^k corresponds to b_{2i}^τ, then on the set W_{1i}^k we set $f = f_{2i}^\tau \circ (f_{1i}^k)^{-1}$. It is easy to prove that f is a homeomorphism of M_1 onto M_2 such that $f(X_1) = X_2$.

From the above it follows that the set $EP(\alpha)$ is countable.

4.3.1. REMARK. From Theorem 4.3 it follows Lemma 2 of Section I.3 of [I_3], that is, for a given isolated ordinal α the set of all (mutually non-homeomorphic) spaces X for which there exists a compactum K having type α, such that $X \subseteq K$ and $K \setminus K^{\beta(\alpha)} \subseteq X$, is countable.

Also, from Lemma 4.2 it follows Lemma 1 of Section I.2 of [I_3].

5. Universal spaces.

5.1. DEFINITIONS. Let $\alpha>0$ be an ordinal and $k\in N$ such that $0\leqq k\leqq m^{+}(\alpha)-1$. Let $X\in R_{lc}^{k}(\alpha)$. An extension $\tilde{X}$ of X is called a *c-extension* (respectively, *lc-extension*) iff $\tilde{X}$ has a basis $B(\tilde{X})=\{V_0, V_1, \cdots\}$ of open sets such that:

(1) the set $Bd(V_i)$, $i=0, 1, \cdots$, is a compactum (respectively, a locally compact subset of $\tilde{X}$),

(2) $type(Bd)V_i))\leqq\alpha+k+1$,

(3) $type((Bd(V_i)\cap X)\cup(Bd(V_i)\setminus(Bd(V_i))^{(\beta(\alpha))}))\leqq\alpha$,

(4) $loc\text{-}com\text{-}type((Bd(V_i)\cap X)\cup(Bd(V_i)\setminus(Bd(V_i))^{(\beta(\alpha))}))\leqq\alpha+k$.

We observe that by Lemma 2.4 for every element $X\in R_{lc}^{k}(\alpha)$ there exists a c-extension of X. Also, if $\tilde{X}$ is a c-extension of X, then using the method of the proof of Lemma 1 of $[I_1]$ we can construct a basis $B(\tilde{X})=\{V_0, V_1, \cdots\}$ of open sets of $\tilde{X}$ having properties (1)-(6) of Lemma 2.4.

Let K be a space, Sp be a family of spaces, $(Sp)_1$ be a subfamily of Sp and let $\mathcal{P}$ be a property of topological spaces. We say that the space K *has the property of $\mathcal{P}$-intersections with respect to subfamily $(Sp)_1$ of Sp* iff for every $X\in Sp$ there exists a homeomorphism i_X of X into K such that if Y and Z are distinct elements of Sp and $Y\in(Sp)_1$, then the set $i_Y(Y)\cap i_Z)Z)$ has property $\mathcal{P}$.

For every $X\in Sp$ let i_X be a homeomorphism of X into K. We say that the space K *has the property of $\mathcal{P}$-intersections with respect to subfamily $\{i_X: X\in(Sp)_1\}$ of all homeomorphisms i_X* iff for every $Y\in(Sp)_1$ and for every $Z\in Sp$, the set $i_Y(Y)\cap i_Z(Z)$ has the property $\mathcal{P}$.

In particular, if $\mathcal{P}$ means that the corresponding intersection (α) is finite, (β) has *type* $\leqq\alpha$, (γ) is compact and has *tyye* $\leqq\alpha$, (δ) has *type* $\leqq\alpha$ and *comfact type* $\leqq\alpha+k$, and (ε) has *type* $\leqq\alpha$ and *locally compact type* $\leqq\alpha+k$, then instead of phrase "$\mathcal{P}$-intersections" we will use, respectively, the words: (α) "finite intersections", (β) "α-intersections", (γ) "compact α-intersections", (δ) "α_c^k-intersections", and (ε) "α_{lc}^k-intersections".

We observe that the notion of "the property of finite intersections" given in $[I_3]$ is different from that of the present paper, because in $[I_3]$ we suppose that both spaces Y and Z belong to the corresponding subfamily. But, it is not difficult to see that the universal space T for the family $R(\alpha)$ constructed in $[I_3]$ has the property of finite intersections (in sense of the present paper) with respect to a given subfamily of $R(\alpha)$ whose cardinality is less than on equal to the continuum.

The same is true with the notion of "the property of α-intersections" (in actually, with the notion of "the property of compact α-intersections") given in [G-I].

5.2. REPRESENTATIONS. For every $X\in R^k_{Ic}(\alpha)$ let $\tilde{X}$ be a c-extension of X and $B(\tilde{X})=\{V_0(\tilde{X}), V_1(\tilde{X}), \cdots\}$ be an ordered basis of open sets of $\tilde{X}$ having properties (1)-(6) of Lemma 2.4.

We recall the contruction (with respect to the ordered basis $B(\tilde{X})$) of the subset $S(\tilde{X})$ of C, the upper semi-continuous partition $D(\tilde{X})$ of $S(\tilde{X})$, the map $q(\tilde{X})$ of $S(\tilde{X})$ onto $\hat{X}$ and the homeomorphism $i(\tilde{X})$ of $D(\tilde{X})$ onto $\tilde{X}$ given in Sections I.5 and I.8 of $[I_1]$.

For every $i=0, 1, \cdots$, we set $V_i^0(\tilde{X})=Cl(V_i(\tilde{X}))$ and $V_i^1(\tilde{X})=\tilde{X}\setminus V_i(\tilde{X})$. For every $\bar{i}=i_1\cdots i_n\in L_n$, we set $\tilde{X}_{\emptyset}=C$ if $n=0$ and $\tilde{X}_{\bar{i}}=V_0^{i_1}(\tilde{X})\cap\cdots\cap V_{n-1}^{i_n}(\tilde{X})$ if $n\geqq 1$. The point $a\in C$ belongs to $S(\tilde{X})$ if and only if $\tilde{X}_{\bar{i}(a,0)}\cap\tilde{X}_{\bar{i}(a,1)}\cap\cdots\neq\emptyset$. The last set is a singleton for every point a of $S(\tilde{X})$. We define the $q(\tilde{X})$ of $S(\tilde{X})$ onto $\tilde{X}$ setting $q(\tilde{X})(a)=x$, where $a\in S(\tilde{X})$ and $\{x\}=\tilde{X}_{\bar{i}(a,0)}\cap\tilde{X}_{\bar{i}(a,1)}\cap\cdots$. Finally, we set $D(\tilde{X})=\{(q(\tilde{X}))^{-1}(x): x\in\tilde{X}\}$ and define $i(\tilde{X})$ setting $i(\tilde{X})((q(\tilde{X}))^{-1}(x))=x$.

5.2.1. LEMMA. *For every $X\in R^k_{Ic}(\alpha)$, the pair $(S(\tilde{X}), D(\tilde{X}))$ has the dense property.*

PROOF. Let $n\in N$ and $a\in d\in(D(\tilde{X}))_n$. There exist elements $x\in Bd(V_n(\tilde{X})$ and $b\in C$ such that $d=\{a, b\}=(q(\tilde{X}))^{-1}(x)$. Let $x_1, x_2, \cdots$ be a sequence of points of $\tilde{X}$ snch that $\lim_{i\to\infty} x_i=x$, $x_i\in V_n(\tilde{X})$ if $a<b$ and $x_i\in\tilde{X}\setminus Cl(V_n(\tilde{X}))$ if $b<a$, $i=1, 2, \cdots$. If $n\geqq 1$ we can suppose that $x_i\notin Cl(V_0(\tilde{X})\cup\cdots\cup V_{n-1}(\tilde{X}))$.

By the construction of the sets $\tilde{X}_{\bar{i}}$ it follows that there exists an element $\bar{i}$ of L_n such that $a\in C_{\bar{i}_0}$ and $b\in C_{\bar{i}1}$ if $a<b$ and $a\in C_{\bar{i}1}$ and $b\in C_{\bar{i}0}$ if $b<a$. Also, for every $i=1, 2, \cdots$, we have that the set $(q(\tilde{X}))^{-1}(x_i)$ is contained in that of the sets $C_{\bar{i}0}$ and $C_{\bar{i}1}$ which contains the point a.

Since $D(\tilde{X})$ is an upper semi-continuous parition of $S(\tilde{X})$ we have $\lim_{i\to\infty} d_i=d$. where $d_i=(q(\tilde{X}))^{-1}(x_i)$, $i=1, 2, \cdots$. Hence, if $a_i\in d_i$, then $\lim_{i\to\infty} a_i=a$, that is, the point a is a limit point of the set $S(\tilde{X})\setminus((D(\tilde{X}))_n)^*$. This means that the pair $(S(\tilde{X}), D(\tilde{X}))$ has the dense property.

5.2.2. THE FAMILY A OF REPRESENTATIONS. Let R_1 be a subfamily of $R^k_{Ic}(\alpha)$ the cardinality of which is less than or equal to the continuum and let $R_2=R^k_{Ic}(\alpha)\setminus R_1$.

For every $X\in R_2$ we set $\hat{S}(X)=C$ and we denote by $\hat{D}(X)$ the set which is the union of the set $D(\tilde{X})$ and all singletons $\{x\}$, where $x\in C\setminus(\bigcup_{n=0}^{\infty}((D(\tilde{X}))_n)^*)$. It is easy to see that $\hat{D}(X)$ is an upper semi-continuous partition of $\hat{S}(X)$ and the quotient space $D(\tilde{X})$ is homeomorphic to a subset of the quotient space $\hat{D}(X)$.

Let A_2 be the family of all pair $(\hat{S}(X), \hat{D}(X))$, $X\in R_2$. It is easy to see that the cardinality of A_2 is less than or equal to the continuum.

For every $X\in R_1$ we set $\hat{S}(X)=S(\tilde{X})$ and $\hat{D}(X)=D(\tilde{X})$. Let A_1 be the set of all pairs $(\hat{S}(X), \hat{D}(X))$, $X\in R_1$. If X and Y are distinct elements of R_1, then $(\hat{S}(X), \hat{D}(X))$ and $(\hat{S}(Y), \hat{D}(Y))$ are considered as distinct elements of A_1, while it is possible $\hat{S}(X)=\hat{S}(Y)$ and $\hat{D}(X)=\hat{D}(Y)$.

Let A be the free union of A_1 and A_2. (Hence, if $g_1\in A_1$ and $g_2\in A_2$, then g_1 and g_2 are distinct elements of A). Obviously, the cardinality of A is less than or equall to the continuum.

By Lemma 5.2.1 it follows that every element of A has the dense property.

In the present section we denote by $\boldsymbol{M}$ the set of all scattered compacta M such that either $type(M)\leqq\beta(\alpha)$ or $type(M)=\beta(\alpha)+n$, where $n=1, 2, \cdots$. We suppose that distinct elements of $\boldsymbol{M}$ are not homeomorphic.

Let $EP(\beta(\alpha))=EP(\beta(\alpha)+1)\cup EP(\beta(\alpha)+2)\cup\cdots$. By Theorem 4.3 the set $EP(\beta(\alpha))$ is countable. Let $e\in EP(\beta(\alpha))$. We denote by $M(e)$ the element M of $\boldsymbol{M}$ (if there exists such element) for which for some subset F of M, $(F, M)\in e$. Obviously, if there exists the element $M(e)$, then it is uniquely determined, while the subset F of $M(e)$ for whch $(F, M(e))\in e$, in general, is not unique. We denote by $F(e)$ a fixed subset of M such that $(F(e), M(e))\in e$.

For every $X\in R_{lc}^k(\alpha)$ and $q\in N$ by the construction of the pair $(\hat{S}(X), \hat{D}(X))$ it follows that $(\hat{D}(X))_q=(D(\tilde{X}))_q$. Since $(D(\tilde{X}))_q$ is homeomorphic to $Bd(V_q(\tilde{X}))$ (See the proof of Lemma 11 of [I_3]) by properties (1) and (4) of Lemma 2.4 it follows that the pair $g(X)=(\hat{S}(X), \hat{D}(X))$ is an $\boldsymbol{M}$-representation. By $M_q(g(X))$ we denote the element of $\boldsymbol{M}$ which is homeomorphic to $(\hat{D}(X))_q$. If $type((\hat{D}(X))_q)\leqq\beta(\alpha)$, then by $\phi_q(g(X))$ we denote a fixed homeomorphism of $M_q(g(X))$ onto $(\hat{D}(X))_q$.

Suppose that $type((\hat{D}(X))_q)=\beta(\alpha)+n$. Let $F_q(\tilde{X})=(Bd(V_q(\tilde{X}))\cap X)\cup(Bd(V_q(\tilde{X}))\setminus(Bd(V_q(\tilde{X}))^{(\beta(\alpha))})$. Then, the pair $(F_q(\tilde{X}), Bd(V_q(\tilde{X})))$ belongs to an element e of $EP(\beta(\alpha))$ and, hence, there exists the pair $(F(e), M(e))$. By $\phi_q(g(X))$ we denote a fixed homeomorphism of $M_q(g(X))=M(e)$ onto $(\hat{D}(X))_q$ for which $\phi_q(g(X))(F(e))=(i(\tilde{X}))^{-1}(F_q(\tilde{X}))$. (We observe that by the construction of the homeomorphism $i(\tilde{X})$ it follows that $i(\tilde{X})(D(\tilde{X}))_q)=Bd(V_q(\tilde{X}))$).

We suppose that for every $M\in\boldsymbol{M}$ there exists a fixed decreasing sequence of decompositions of M.

Also we suppose that there exists a fied decreasing sequence of decompositions of A such that if E is an element of q^{th} decompositions, then the element $M_q(E)$ of $\boldsymbol{M}$ is determined (for notations see Section 3.1). Moreover, since the set $EP(\beta(\alpha))$ is countable, we can suppose that if $type(M_q(E))=\beta(\alpha)+n$ and $(\hat{S}(X), \hat{D}(X))$ and $(\hat{S}(Y), \hat{D}(Y))$ are two elements of E, then the pairs $(F_q(\tilde{X}), Bd(V_q(\tilde{X})))$ and $(F_q(\tilde{Y}), Bd(V_q(\tilde{Y})))$ belong to the same element of $EP(\beta(\alpha))$.

5.3. THEOREM. *Let R_1 be a subfamily of $R^k_{kc}(\alpha)$ the cardinality of which is less than or equal to the continuum. For every element $X\in R^k_{ic}(\alpha)$ let $\tilde{X}$ be a c-extension of X. Then, there exist:*

(1) *an element $K\in R^k_{kc}(\alpha)$,*

(2) *a space T which is an lc-extension of K,*

(3) *a homeomorphism i_X of X into K for every $X\in R^k_{ic}(\alpha)$, and*

(4) *a homeomorphism $j_{\tilde{X}}$ of $\tilde{X}$ into T, for every $X\in R^k_{ic}(\alpha)$, which is an extension of i_X, that is, $j_{\tilde{X}}|_X=i_X$, such that:*

(5) *the space K has the property of α^k_{ic}-intersections with respect to the subfamily $\{i_X: X\in R_1\}$ of all homeomorphisms i_X, $X\in R^k_{ic}(\alpha)$.*

(6) *the space T has the property of compact $(\alpha+k+1)$-intersections with respect to subfamily $\{j_{\tilde{X}}: X\in R_1\}$ of all homeomorphisms $j_{\tilde{X}}$, $X\in R^k_{ic}(\alpha)$. Moreover,*

(7) *the set $j_{\tilde{X}}(\tilde{X})$ is a closed subset of T, for every $X\in R_1$.*

PROOF. We use all notions and notations of Sections 5.2 and 5.2.2. Let T be a space of Theorem 3.11 constructed for the family A of $\boldsymbol{M}$-representations of Section 5.2.2.

Now we define the subspace K of T as follows: every element d of T of the form $\{(a, g)\}$, where $(a, g)\in C\times C$, belongs to K. Let $d\in T(1)$. Then, there exist an integer $m\in N$, an element r of $I(m)$ and an element x of $M_m(A^m_r)$ such that $d=d(x, m, r)$. If $type(M_m(A^m_r))<\beta(\alpha)$, then we consider that $d\in K$. Let $type(M_m(A^m_r))=\beta(\alpha)+n$. By the properties of the fixed decreasing sequence of decompositions of A it follows that there exists an element e of $EP(\beta(\alpha))$ such that for every $X\in R^k_{ic}(\alpha)$ for which $g(X)=(\hat{S}(X), \hat{D}(X))\in A^m_r$ we have $(F_m(\tilde{X}), Bd(V_m(\tilde{X})))\in e$. Hence, $M_m(A^m_r)=M_m(g(X))=M(e)$ and $F(e)=(\psi_m(g(X)))^{-1}(F_m(\tilde{X}))$. We consider that $d\in K$ iff $x\in F(e)$.

By the definition of the set $F_m(\tilde{X})$ and properties of a c-extension of X (see Section 5.1) it follows that: (α) $(d(M_m(A^m_r))\setminus(d(M_m(A^m_r)))^{(\beta(\alpha))})\subseteq d(M_m(A^m_r))$

$\cap K$, (β) $type(d(M_m(A_r^m))\cap K)\leqq\alpha$, ($\gamma$) $type(d(M_m(A_r^m)))\leqq\alpha+k+1$, ($\delta$) $loc\text{-}com\text{-}type(d(M_m(A_r^m))\cap K)\leqq\alpha+k$.

We observe that the above properties (α)-(δ) are true if we replace the set $d(M_m(A_r^m))$ by an open and closed subset of it. Hence, these properties are also true if we replace the set $d(M_m)A_r^m))$ by a set which is a free union of simultaneously open and closets of sets $d(M_m(A_r^m))$, $m\in N$, $r\in I(m)$.

Consider the basis $\boldsymbol{B}$ of the space T. Let $O(W)\in\boldsymbol{B}$. By Theorem 5.3 the set $Bd(O(W))$ is a free union of simultaneously open and closed subsets of sets $d(M_m(A_r^m))$. Hence, properties (α)-(δ) are true if we replace the set $d(M_m(A_r^m))$ by the set $Bd(O(W))$. From the it follows that $K\in R_{lc}^k(\alpha)$. Since the set $Bd(O(W))$ is a locally compact subset of T we also have that the space T is an lc-extension of the space K.

Let $T(\tilde{X})$ be the subset of T consisting of all elements z of T for which $z\cap(C\times\{g(X)\})\neq\emptyset$. We observe that for every $z\in T(\tilde{X})$ there exists an element $d\in\hat{D}(X)$ such that $z\cap(C\times\{g(X)\})=d\times\{g(X)\}$. Also, for every $d\in\hat{D}(X)$ there exists an element $z\in T(\tilde{X})$ such that the above relation is true. Hence, setting $j_{\tilde{X}}(d)=z$ we have an one-to-one map of $\hat{D}(X)$ onto $T(\tilde{X})$. It is easy to verify, that $j_{\tilde{X}}((\hat{D}(X))_q)=d(M_q(A^q_{\tau(q,g(X))}))$, for every $q\in N$.

We prove that $j_{\tilde{X}}$ is a homeomorphism. Let $j_{\tilde{X}}(d)=z$. Let $z\in O(W)\in\boldsymbol{B}$. Since the space T is regular there exists an element $O(W_1)$ of B such that $z\in O(W_1)\subseteq Cl(O(W_1))\subseteq O(W)$. By the construction of the element of the set $\hat{U}\cup\hat{V}$, there exists an open subset V of $\hat{S}(X)$ such that $d\subseteq V$ and $V\times\{g(X)\}\subseteq W_1$. Let U be the set of all elements d' of $\hat{D}(X)$ for which $d'\subseteq V$. Then, U is an open subset of $\hat{D}(X)$ containing d. If $d'\in U$, then $j_{\tilde{X}}(d')\cap W_1\neq\emptyset$ and, hence, $j_{\tilde{X}}(d')\in O(W)$, that is, $j_{\tilde{X}}(U)\subseteq O(W)$. Thus, $\zeta_{\tilde{X}}$ is a continuous map. Let U be an open subset of $\hat{D}(X)$ containing d. Let $V=(\hat{p}(X))^{-1}(U)$, where $\hat{p}(X)$ is the natural projection of $\hat{S}(X)$ onto $\hat{D}(X)$. There exists an element W of $\hat{U}\cap\hat{V}$ such that $W\cap C\times\{g(X)\})\subseteq V\times\{g(X)\}$ and $z\subseteq W$. Hence, $z\in O)W)$. If $z'\in O(W)\cap T(\tilde{X})$, then $z\subseteq W$ and therefore $z'\cap(C\times\{g(X)\})\subseteq V\times\{g(X)\}$, that is, if $d'=(j_{\tilde{X}})^{-1}(z')$, then $d'\subseteq V$. This means that $d'\in U$. Hence, $(j_{\tilde{X}})^{-1}(O(W)\cap T(\tilde{X}))\subseteq U$ and the map $(j_{\tilde{X}})^{-1}$ is continuous. Thus, $(j_{\tilde{X}})^{-1}$ is a homeomorphism of $\hat{D}(X)$ onto $T(\tilde{X})$.

Since $D(\tilde{X})$ is a subset of $\hat{D}(X)$ we can consider the restriction $j_{\tilde{X}}|_{D(\tilde{X})}$ of $j_{\tilde{X}}$ onto $D(\tilde{X})$. We set $j_{\tilde{X}}=(j_{\tilde{X}}|_{D(\tilde{X})})\circ(i(\tilde{X}))^{-1}$. Obviously, the map $j_{\tilde{X}}$ is a homeomorphism of $\tilde{X}$ into a subset of $T(\tilde{X})$.

If $X\in R_1$, then $D(\tilde{X})=\hat{D}(X)$ and, hence, $j_{\tilde{X}}=j_{\tilde{X}}\circ(i(\tilde{X}))^{-1}$, that is, the map $j_{\tilde{X}}$ is a homeomorphism of $\tilde{X}$ onto $T(\tilde{X})$.

Set $i_X=j_{\tilde{X}}|_X$. Hence, the map i_X is a homeomorphism of X into $T(\tilde{X})$.

Let X and Y be distinct elements $R^k_{ic}(\alpha)$ such that $X \in R_1$. There exists an integer $m \in N$ such that $r(q, g(X)) = r(q, g(Y))$ for every $0 \leqq q < m$ and $r(m, g(X)) \neq r(m, g(Y))$. It is clear that an element z of T belongs to $T(\tilde{X}) \cap T(\tilde{Y})$ if and only if $d \in d(M_q(A^q_{r(q,g(X))}))$ for some q, $0 \leqq q < m$. Hence, the subset $T(X) \cap T(Y)$ of T is a compact subset having $type \leqq \alpha + k + 1$.

Since $(D)\tilde{Y}))_q = (\hat{D}(Y))_q$ for every $q \in N$, we have $j_{\tilde{Y}}((\hat{D}(Y))_q) \subseteq j_{\tilde{Y}}(\tilde{Y})$. Hence $T(\tilde{X}) \cap T(\tilde{Y}) = j_{\tilde{X}}(\tilde{X}) \cap j_{\tilde{Y}}(\tilde{Y})$, that is, property (6) of the theorem is true.

Since for every q, $0 \leqq q < m$, there exists an element $e \in EP(\beta(\alpha))$ such that $K \cap d(M_q(A^q_{r(q,g(X))})) = d(F(e))$ it follows that the set $i_X(X) \cap i_Y(Y)$ has $type \leqq \alpha$, and *locally compact type* $\leqq \alpha + k$, that is, property (5) of the theorem is true.

Hence, in order to complete the proof of the theorem it is sufficient to prove property (7). For this, since $j_{\tilde{X}}(\tilde{X}) = T(\tilde{X})$ if $X \in R_1$, it is sufficient to prove that the set $T(\tilde{X})$ is a closed subset of T.

Let $z \in T \setminus T(\tilde{X})$. If z has the ferm $d(y, m, r)$ for some $m \in N$, $r \in I(m)$ and $y \in M_m(A^m_r)$, then $g(X) \notin A^m_r$. Hence, $z \in O(U)$ and $O(U) \cap T(\hat{X}) = \emptyset$, where $U = U(d(y, m, r), 0)$.

Let $z = \{(a, g)\}$. There exists an integer $m \in N$ and distinct elements τ and τ_1 of $I(m)$ such that $g \in A^m_\tau$ and $g(X) \in A^m_{\tau}$. Then, $z \subseteq C_g \times A^m_\tau$. By Lemma 3.7 case (1), there exists an element W of the set $\hat{U} \cup \hat{V}$ such that $z \subseteq W \subseteq C_g \times A^m_\tau$. Hence, $z \in O(W)$ and $O(W) \cap T(\tilde{X}) = \emptyset$.

Thus, in both cases, the element z has an open neighbourhood which do not intersect the subspace $T(\tilde{X})$. Hence, $T(\tilde{X})$ is closed.

5.4. COROLLARIES. (1) *In the family* $R^k_{ic}(\alpha)$ *there exists a universal element having the property of* α^k_{ic}*-intersections with respect to any subfamily of* $R^k_{ic}(\alpha)$ *the cardinality of which is less than or equal to the continuum.*

(2) *For the family* $R^k_c(\alpha)$ *there exists a containing space belaining to* $R^k_{ic}(\alpha)$.

(3) *For the family* $R^k_c(\alpha)$ *there exists a containing continuum having type* $\leqq \alpha + k + 1$ *and the property of* α^{k+1}_c*-intersections with respect to a fixed subfamily of* $R^k_c(\alpha)$ *the cardinality of which is less than or equal to the continuum.*

This corollary follows from Theorem 5.3 (*See property* (6)), *Theorem* 2.5 *and Theorem* 3 *of* [I_1].

In particular, if $k=0$ *and since* $R^{com}(\alpha) \subseteq R^0_c(\alpha)$ *we have:*

There exists a continuum having rim-type $\leqq \alpha + 1$ *which is a containing space for all compacta having rim-type* $\leqq \alpha$.

(4) *In the family* $R(\alpha)$ (*that is, in the family* $R^k_{ic}(\alpha)$, *where* $k = m^+(\alpha) - 1$) *there exists a universal element* (*See* [I_8]).

5.5. SOME PROBLEMS. (1) Does there exist a universal element of the family $R_{lc}^{k}(\alpha)$, where $\alpha>0$ and $k=0, \cdots, m^{+}(\alpha)-1$, having the property of $\mathcal{P}$-intersections with respect to a given subfamily of $R_{lc}^{k}(\alpha)$ the cardinality of which is less than or equal to the continuum if "$\mathcal{P}$-intersections" means (α) finite intersections, (β) compact α-intersections, (γ) α_{lc}^{n}-intersections, where $n=0, \cdots, k-1$ and (δ) α_{c}^{n}-intersections, where $n=0, \cdots, k$?

(2) Let K be a universal element of the family $R_{lc}^{k}(\alpha)$, where $\alpha=0, \cdots, m^{+}(\alpha)$, and let R_1 be a fixed subfamily of $R_{lc}^{k}(\alpha)$ the cardinality of which is less than or equal to the continuum. Does the space K have the property of (α) finite intersections, (β) compact α-intersections, (γ) α-intersections, (δ) α_{lc}^{n}-intersections, where $n=0, \cdots,$ and (ε) α_{c}^{n}-intersections, where $n=0, \cdots$, with respect to the subfamily R_1?

(3) Are the results and problems of the present paper true if we replace all corresponding famillies of spaces by their *plant part*? (Plane part of a family A is the subfamily consisting of all elements of A admitting an embedding in the plane).

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Department of Mathematics
University of Patras
Patras, Greece