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ON REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE II

Dedicated to Professor Mikio NAKAMURA on his retirement from Kumamoto University, College of Medical Science

By

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§ 0. Introduction

Let $P_n(C)$ be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4, and let M be a real hypersurface of $P_n(C)$. M has an almost contact metric structure (ϕ, ξ, η, g) induced from the complex structure J of $P_n(C)$ (see, § 1). We denote by A, R, S, the shape operator, the curvature tensor and the Ricci tensor of type (1, 1) on M, respectively. Many differential geometers have studied M (cf. [1], [6], [9], [11] and [12]) by using the structure (ϕ, ξ, η, g).

Typical examples of real hypersurfaces in $P_n(C)$ are homogeneous ones. R. Takagi ([10]) showed that all homogeneous real hypersurfaces in $P_n(C)$ are realized as the tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or rank 2. Namely, he showed the following:

THEOREM T ([10]). *Let M be a homogeneous real hypersurface of $P_n(C)$. Then M is a tube of radius r over one of the following Kaehler submanifolds:*

(A_1) *hyperplane $P_{n-1}(C)$, where $0<r<\pi/2$,*

(A_2) *totally geodesic $P_k(C)$ $(1\leqq k\leqq n-2)$, where $0<r<\pi/2$,*

(B) *complex quadric Q_{n-1}, where $0<r<\pi/4$,*

(C) *$P_1(C)\times P_{(n-1)/2}(C)$, where $0<r<\pi/4$ and $n(\geqq 5)$ is odd,*

(D) *complex Grassmann $G_{2,5}(C)$, where $0<r<\pi/4$ and $n=9$,*

(E) *Hermitian symmetric space $SO(10)/U(5)$, where $0<r<\pi/4$ and $n=15$.*

Due to his classification, we find the number of distinct constant principal curvatures of a homogeneous real hypersurface is 2, 3 or 5. Here note that the vector ξ of any homogeneous real hypersurface M (which is a tube of radius r) is a principal curvature vector with principal curvature $\alpha=2\cot 2r$ with multiplicity 1 (for further details, see [11]).

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Now it is well-known that there does not exist a real hypersurface M with parallel second fundamental form A. So the authors introduced the notion of η-parallel second fundamental form (cf. [6]), which is defined by $g((\nabla_X A)Y, Z)=0$ for any vector fields X, Y and Z orthogonal to the structure vector field ξ, where g and ∇ denote the induced Riemannian metric and the induced Riemannian connection, respectively. In this paper, we consider another condition on the derivative of A.

The main purpose of this paper is to classify real hypersurfaces M which satisfy $\nabla_\xi A=0$ (, that is, the second fundamental form A is parallel in the direction of ξ) in $P_n(C)$. Our main result is as follows:

THEOREM. *Let M be a real hypersurface in $P_n(C)$. If $\nabla_\xi A=0$, then M is locally congruent to one of the following:*

(i) *a non-homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a certain Kaehler submanifold $\tilde{N}$ in $P_n(C)$,*

(ii) *a homogeneous real hypersurface which lies on a tube of radius r over a totally geodesic $P_k(C)$ $(1\leqq k\leqq n-1)$, where $0<r<\pi/2$.*

We here remark that there exist many real hypersurfaces M's which are of case (i) in our Theorem (for details, see the Proof of Theorem).

Now it is known that there does not exist a real hypersurface M with parallel Ricci tensor S in $P_n(C)$, $n\geqq 3$ (cf. [2] and [3]). As an immediate consequence of this result, $P_n(C)$ $(n\geqq 3)$ does not admit a locally symmetric real hypersurface M. Motivated by their results and our Theorem, we investigate real hypersurfaces M in $P_n(C)$ by using the conditions "$\nabla_\xi S=0$" and "$\nabla_\xi R=0$".

We have the following:

PROPOSITION A. *Let M be a real hypersurface with constant mean curvature in $P_n(C)$. Suppose that ξ is a principal curvature vector and the corresponding principal curvature is non-zero. If $\nabla_\xi S=0$, then M is a tube of radius r over one of the following Kaehler submanifolds:*

(A_1) *hyperplane $P_{n-1}(C)$, where $0<r<\pi/2$ and $r\neq\pi/4$,*

(A_2) *totally geodesic $P_k(C)$ $(1\leqq k\leqq n-2)$, where $0<r<\pi/2$ and $r\neq\pi/4$,*

(B) *complex quadric Q_{n-1}, where $0<r<\pi/4$ and $\cot^2 2r=n-2$,*

(C) *$P_1(C)\times P_{(n-1)/2}(C)$, where $0<r<\pi/4$, $\cot^2 2r=1/(n-2)$ and $n(\geqq 5)$ is odd,*

(D) *complex Grassmann $G_{2,5}(C)$, where $0<r<\pi/4$, $\cot^2 2r=3/5$ and $n=9$,*

(E) *Hermitian symmetric space $SO(10)/U(5)$, where $0<r<\pi/4$, $\cot^2 2r=5/9$ and $n=15$.*

PROPOSITION B. *Let M be a real hypersurface in $P_n(C)$. Suppose that ξ is a principal curvature vector and the corresponding principal curvature is non-zero. If $\nabla_\xi R=0$, then M is a tube of radius r over one of the following Kaehler submanifolds:*

(A_1) *hyperplane $P_{n-1}(C)$, where $0<r<\pi/2$ and $r\neq\pi/4$,*

(A_2) *totally geodesic $P_k(C)$ $(1\leqq k\leqq n-2)$, where $0<r<\pi/2$ and $r\neq\pi/4$.*

§ 1. Preliminaries

Let M be an orientable real hypersurface of $P_n(C)$ and let N be a unit normal vector field on M. The Riemannian connections $\tilde{\nabla}$ in $P_n(C)$ and ∇ in M are related by the following formulas for arbitrary vector fields X and Y on M:

$$\tilde{\nabla}_X Y=\nabla_X Y+g(AX, Y)N, \tag{1.1}$$

$$\tilde{\nabla}_X N=-AX, \tag{1.2}$$

where g denotes the Riemannian metric of M induced from the Fubini-Study metric G of $P_n(C)$ and A is the shape operator of M in $P_n(C)$. An eigenvector X of the shape operator A is called a *principal curvature vector*. Also an eigenvalue λ of A is called a *principal curvature*. In what follows, we denote by V_λ the eigenspace of A associated with eigenvalue λ. It is known that M has an almost contact metric structure induced from the complex structure J on $P_n(C)$, that is, we define a tensor field ϕ of type (1, 1), a vector field ξ and a 1-form η on M by $g(\phi X, Y)=G(JX, Y)$ and $g(\xi, X)=\eta(X)=G(JX, N)$. Then we have

$$\phi^2 X=-X+\eta(X)\xi, \quad g(\xi, \xi)=1, \quad \phi\xi=0. \tag{1.3}$$

It follows from (1.1) that

$$(\nabla_X\phi)Y=\eta(Y)AX-g(AX, Y)\xi, \tag{1.4}$$

$$\nabla_X\xi=\phi AX. \tag{1.5}$$

Let $\tilde{R}$ and R be the curvature tensors of $P_n(C)$ and M, respectively. Since the curvature tensor $\tilde{R}$ has a nice form, we have the following Gauss and Codazzi equations:

$$\begin{aligned} g(R(X, Y)Z, W)=&g(Y, Z)g(X, W)-g(X, Z)g(Y, W)+g(\phi Y, Z)g(\phi X, W)\\ &-g(\phi X, Z)g(\phi Y, W)-2g(\phi X, Y)g(\phi Z, W)\\ &+g(AY, Z)g(AX, W)-g(AX, Z)g(AY, W), \end{aligned} \tag{1.6}$$

$$(\nabla_X A)Y-(\nabla_Y A)X=\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X, Y)\xi. \tag{1.7}$$

From (1.3), (1.5), (1.6) and (1.7) we get

$$SX=(2n+1)X-3\eta(X)\xi+hAX-A^2X, \tag{1.8}$$

$$(\nabla_X S)Y=-3\{g(\phi AX, Y)\xi+\eta(Y)\phi AX\}+(Xh)AY +(hI-A)(\nabla_X A)Y-(\nabla_X A)AY, \tag{1.9}$$

where $h=\text{trace } A$, S is the Ricci tensor of type (1, 1) on M and I is the identity map.

In the following, we use the same terminology and notations as above unless otherwise stated. Now we prepare without proof the following in order to prove our results:

THEOREM K ([4]). *Let M be a real hypersurface of $P_n(C)$. Then M has constant principal curvatures and ξ is a principal curvature vector if and only if M is locally congruent to a homogeneous real hypersurface.*

PROPOSITION 1 ([8]). *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

PROPOSITION 2 ([8]). *Assume that ξ is a principal curvature vector and the corresponding principal curvature is α. If $AX=rX$ for $X\perp\xi$, then we have $A\phi X=((\alpha r+2)/(2r-\alpha))\phi X$.*

PROPOSITION 3 ([8]). *Let M be a real hypersurface of $P_n(C)$. Then the following are equivalent:*

(i) *M is locally congruent to one of homogeneous ones of type* (A_1) *and* (A_2).

(ii) *$g((\nabla_X A)Y, Z)=-\eta(Y)g(\phi X, Z)-\eta(Z)g(\phi X, Y)$ for any vector fields X, Y and Z on M.*

PROPOSITION 4 ([7]). *Let M be a real hypersurface of $P_n(C)$. Suppose that ξ is a principal curvature vector and the corresponding principal curvature is nonzero. If $\nabla_\xi A=0$, then M is a tube of radius r over one of the following Kaehler submanifolds:*

(A_1) *hyperplane $P_{n-1}(C)$, where $0<r<\pi/2$ and $r\neq\pi/4$,*

(A_2) *totally geodesic $P_k(C)$ $(1\leqq k\leqq n-2)$, where $0<r<\pi/2$ and $r\neq\pi/4$.*

PROPOSITION 5 ([7]). *Let M be a real hypersurface of $P_n(C)$. Then "$A\xi=0$" implies "$\nabla_\xi A=0$".*

PROPOSITION 6 ([1]). *Let M be a connected orientable real hypersurface (with unit normal vector N) in $P_n(C)$ on which ξ is a principal curvature vector with*

principal curvature $\alpha=2\cot 2r$. *Then the following hold:*

(i) *M lies on a tube (in the direction of $\eta=\gamma'(r)$, where $\gamma(r)=\exp_x(rN)$ and x is a base point of the normal vector N) of radius r over a certain Kaehler submanifold $\tilde{N}$ in $P_n(C)$.*

(ii) *Let $\cot\theta$ be a principal curvature of the shape operator A_η at $y=\gamma(r)$ of the Kaehler submanifold $\tilde{N}$. Then the real hypersurface M has a principal curvature $\cot(\theta-r)$ at $x=\gamma(0)$.*

Finally we prove the following which is the main tool for the proof of our Theorem.

PROPOSITION 7. *Let M be a real hypersurface of $P_n(C)$. Then "$\nabla_\xi A=0$" implies "ξ is a principal curvature vector of M".*

PROOF. First we shall show that $A\xi$ is principal. From (1.5), (1.7) we find that the condition $\nabla_\xi A=0$ is equivalent to

$$(\nabla_X A)\xi=-\phi X \qquad (\text{for any } X\in TM), \tag{1.10}$$

that is,

$$\nabla_X(A\xi)=-\phi X+A\phi AX \qquad (\text{for any } X\in TM). \tag{1.11}$$

The equation (1.11) yields

$$\nabla_Y(\nabla_X(A\xi))-\nabla_{\nabla_Y X}(A\xi)=-\nabla_Y(\phi X)+\nabla_Y(A\phi AX)+\phi\nabla_Y X-A\phi A\nabla_Y X.$$

which, together with (1.4), shows

$$\begin{aligned}\nabla_Y(\nabla_X(A\xi))-\nabla_{\nabla_Y X}(A\xi)=&-\eta(X)AY+g(AY, X)\xi+(\nabla_Y A)\phi AX\\&+A\{\eta(AX)AY-g(AY, AX)\xi\}+A\phi(\nabla_Y A)X.\end{aligned} \tag{1.12}$$

Exchanging X and Y in (1.12), we have

$$\begin{aligned}\nabla_X(\nabla_Y(A\xi))-\nabla_{\nabla_X Y}(A\xi)=&-\eta(Y)AX+g(AX, Y)\xi+(\nabla_X A)\phi AY\\&+A\{\eta(AY)AX-g(AX, AY)\xi\}+A\phi(\nabla_X A)Y.\end{aligned} \tag{1.13}$$

It follows from (1.3), (1.7), (1.12) and (1.13) that

$$R(X, Y)A\xi=(\nabla_X A)\phi AY-(\nabla_Y A)\phi AX+\eta(AY)A^2X-\eta(AX)A^2Y. \tag{1.14}$$

On the other hand the Gauss equation (1.6) tells us that

$$\begin{aligned}R(X, Y)A\xi=&g(Y, A\xi)X-g(X, A\xi)Y+g(\phi Y, A\xi)\phi X-g(\phi X, A\xi)\phi Y\\&-2g(\phi X, Y)\phi A\xi+g(AY, A\xi)AX-g(AX, A\xi)AY.\end{aligned} \tag{1.15}$$

Then from (1.14) and (1.15), for any $X, Y(\in TM)$ we get

$$(\nabla_X A)\phi AY-(\nabla_Y A)\phi AX+\eta(AY)A^2X-\eta(AX)A^2Y$$
$$=\eta(AY)X-\eta(AX)Y+\eta(A\phi Y)\phi X-\eta(A\phi X)\phi Y-2g(\phi X, Y)\phi A\xi$$
$$+\eta(A^2Y)AX-\eta(A^2X)AY.$$

This, combined with (1.3), yields

$$g((\nabla_X A)\xi, \phi AY)-g((\nabla_Y A)\xi, \phi AX)+\eta(AY)\eta(A^2X)-\eta(AX)\eta(A^2Y) \tag{1.16}$$
$$=\eta(X)\eta(AY)-\eta(Y)\eta(AX)+\eta(AX)\eta(A^2Y)-\eta(AY)\eta(A^2X).$$

Here, from (1.10) we find the following:

$$g((\nabla_X A)\xi, \phi AY)=-g(X, AY)+\eta(X)\eta(AY). \tag{1.17}$$

$$g((\nabla_Y A)\xi, \phi AX)=-g(Y, AX)+\eta(Y)\eta(AX). \tag{1.18}$$

Therefore from (1.16), (1.17) and (1.18) we see

$$\eta(AX)\eta(A^2Y)=\eta(AY)\eta(A^2X) \qquad \text{(for any } X, Y\in TM)$$

so that

$$g(A\xi, X)A^2\xi=g(A^2\xi, X)A\xi,$$

which shows that $A\xi$ is principal. So we can set

$$A^2\xi=\alpha A\xi. \tag{1.19}$$

Next we shall show that ξ is principal by making use of (1.19). Now we find the following:

$$\nabla_X(A^2\xi)=(\nabla_X A)A\xi+A(\nabla_X A)\xi+A^2\phi AX, \tag{1.20}$$

$$\nabla_X(\alpha A\xi)=(X\alpha)A\xi+\alpha(\nabla_X A)\xi+\alpha A\phi AX. \tag{1.21}$$

Then from (1.19), (1.20) and (1.21) for any $X, Y(\in TM)$ we obtain

$$g(A\xi, (\nabla_X A)Y)+g((\nabla_X A)\xi, AY)+g(A^2\phi AX, Y) \tag{1.22}$$
$$=(X\alpha)g(A\xi, Y)+\alpha g((\nabla_X A)\xi, Y)+\alpha g(A\phi AX, Y).$$

Exchanging X and Y in (1.22), we have

$$g(A\xi, (\nabla_Y A)X)+g((\nabla_Y A)\xi, AX)+g(A^2\phi AY, X) \tag{1.23}$$
$$=(Y\alpha)g(A\xi, X)+\alpha g((\nabla_Y A)\xi, X)+\alpha g(A\phi AY, X).$$

The equations (1.7), (1.22) and (1.23) assert that

$$g((\nabla_X A)\xi, AY)-g((\nabla_Y A)\xi, AX) \tag{1.24}$$
$$=(X\alpha)\eta(AY)-(Y\alpha)\eta(AX)-2\alpha g(\phi X, Y)+2\alpha g(A\phi AX, Y)-\eta(X)\eta(A\phi Y)$$
$$+\eta(Y)\eta(A\phi X)+2\eta(A\xi)g(\phi X, Y)-g(A\phi AX, AY)-g(\phi A^2X, AY).$$

By virtue of (1.10) and (1.24) we find

$$(1.25)\quad -g(\phi X, AY)+g(\phi Y, AX)$$
$$=(X\alpha)\eta(AY)-(Y\alpha)\eta(AX)+2\alpha g(A\phi AX, Y)-2\alpha g(\phi X, Y)-\eta(X)\eta(A\phi Y)$$
$$+\eta(Y)\eta(A\phi X)+2\eta(A\xi)g(\phi X, Y)-g(A\phi AX, AY)-g(\phi A^2X, AY).$$

Setting $X=\xi$ in (1.25), for any $(Y\in TM)$ we see

$$(1.26)\quad \eta(A\xi)(Y\alpha)=(\xi\alpha)\eta(AY)+\alpha g(A\phi A\xi, Y)-2\eta(A\phi Y)-g(A\phi A\xi, AY).$$

Now we put $\mu=g(A\xi, \xi)$. In the following, we may assume that $\mu\neq 0$.

(Suppose that $\mu=0$. Since $A\xi$ is principal, there exists a principal curvature vector X (with principal curvature λ) orthogonal to $A\xi$. So the following holds. $0=g(X, A\xi)=g(AX, \xi)=\lambda g(X, \xi)$, which implies that $g(X, \xi)=0$ in the case of $\lambda\neq 0$. And hence $\xi\in\mathrm{span}\{A\xi\}+\mathrm{Ker}\,A$. Therefore the hypothesis $\mu=0$ shows that $\xi\in\mathrm{Ker}\,A$, so that ξ is principal.)

Multiplying (1.25) by $\mu(\neq 0)$, we get

$$-\mu g((\phi A+A\phi)X, Y)=\mu(X\alpha)\eta(AY)-\mu(Y\alpha)\eta(AX)+2\alpha\mu g(A\phi AX, Y)$$
$$-2\alpha\mu g(\phi X, Y)-\mu\eta(X)\eta(A\phi Y)+\mu\eta(Y)\eta(A\phi X)$$
$$+2\mu^2 g(\phi X, Y)-\mu g(A\phi AX, AY)-\mu g(\phi A^2X, AY).$$

This, together with (1.26), yields

$$(1.27)\quad -\mu(\phi A+A\phi)X=\{\alpha g(A\phi A\xi, X)+2g(\phi A\xi, X)-g(A^2\phi A\xi, X)\}A\xi$$
$$-g(A\xi, X)(\alpha A\phi A\xi+2\phi A\xi-A^2\phi A\xi)+2\alpha\mu A\phi AX$$
$$-2\alpha\mu\phi X+\mu\eta(X)\phi A\xi-\mu g(\phi A\xi, X)\xi+2\mu^2\phi X$$
$$-\mu A^2\phi AX-\mu A\phi A^2X.$$

Now, putting $X=A\xi$ in (1.27), we can see that

$$(1.28)\quad A\phi A\xi=3(\alpha-\mu)\phi A\xi.$$

It follows from (1.26) and (1.28) that

$$\mu\,\mathrm{grad}\,\alpha=(\xi\alpha)A\xi+\alpha A\phi A\xi+2\phi A\xi-A^2\phi A\xi$$
$$=(\xi\alpha)A\xi+3\alpha(\alpha-\mu)\phi A\xi+2\phi A\xi-9(\alpha-\mu)^2\phi A\xi.$$

Namely, setting $\beta=\xi\alpha$, we obtain

$$(1.29)\quad \mu\,\mathrm{grad}\,\alpha=\beta A\xi+\{2-3(\mu-\alpha)(3\mu-2\alpha)\}\phi A\xi.$$

On the other hand, from (1.5), (1.11) and (1.28) we have

$$(1.30)\quad X\mu=6(\mu-\alpha)g(\phi A\xi, X)$$

so that

$$(1.31)\quad \mathrm{grad}\,\mu=6(\mu-\alpha)\phi A\xi.$$

Differentiating (1.31) with respect to X, we see that

$$\nabla_X(\text{grad}\,\mu)=6((X\mu)-(X\alpha))\phi A\xi+6(\mu-\alpha)\nabla_X(\phi A\xi). \tag{1.32}$$

Multiplying (1.32) by μ, from (1.26), (1.28) and (1.30) we find

$$\begin{aligned}\mu\nabla_X(\text{grad}\,\mu)=&[36\mu(\mu-\alpha)g(\phi A\xi, X)-6\beta g(A\xi, X)\\&-6\{2-3(\mu-\alpha)(3\mu-2\alpha)\}g(\phi A\xi, X)]\phi A\xi\\&+6\mu(\mu-\alpha)\{(\nabla_X\phi)A\xi+\phi\nabla_X(A\xi)\}.\end{aligned}$$

This, combined with (1.3), (1.4) and (1.11), yields

$$\begin{aligned}\mu g(\nabla_X(\text{grad}\,\mu), Y)=&-6\beta g(A\xi, X)g(\phi A\xi, Y)\\&+6\{3(\mu-\alpha)(5\mu-2\alpha)-2\}g(\phi A\xi, X)g(\phi A\xi, Y)\\&+6\mu(\mu-\alpha)\{\mu g(AY, X)-\alpha\eta(AX)\eta(Y)+g(X, Y)\\&-\eta(X)\eta(Y)+g(\phi A\phi AX, Y)\}.\end{aligned} \tag{1.33}$$

Since $g(\nabla_X(\text{grad}\,\mu), Y)=g(\nabla_Y(\text{grad}\,\mu), X)$, the equation (1.33) implies

$$\begin{aligned}&6\beta\{g(A\xi, Y)g(\phi A\xi, X)-g(A\xi, X)g(\phi A\xi, Y)\}\\&+6\mu\alpha(\mu-\alpha)\{\eta(AY)\eta(X)-\eta(AX)\eta(Y)\}\\&+6\mu(\mu-\alpha)\{g(\phi A\phi AX-A\phi A\phi X, Y)\}=0.\end{aligned} \tag{1.34}$$

Here, from (1.3) and (1.28) the following holds:

$$\phi A\phi A\xi=3(\mu-\alpha)(A\xi-\mu\xi). \tag{1.35}$$

Setting $Y=\xi$ in (1.34), from (1.35) we have

$$\beta\phi A\xi+(\mu-\alpha)(3\mu-2\alpha)(-A\xi+\mu\xi)=0. \tag{1.36}$$

Since $g(A\xi-\mu\xi, A\xi)=\alpha\mu-\mu^2$, the equation (1.36) implies

$$(\mu-\alpha)^2(3\mu-2\alpha)\mu=0. \tag{1.37}$$

Now we suppose that $\alpha\neq\mu$ (, which is equivalent to "ξ is not principal"). And hence, from (1.37) we find

$$\alpha=(3/2)\mu \tag{1.38}$$

so that

$$\text{grad}\,\alpha=(3/2)\,\text{grad}\,\mu. \tag{1.39}$$

Since $\phi A\xi\neq 0$, the equation (1.36) shows

$$\beta=0. \tag{1.40}$$

From (1.29), (1.38) and (1.40) we get

$$\mu\,\text{grad}\,\alpha=2\phi A\xi. \tag{1.41}$$

Now, from (1.31) and (1.38) we find

$$(3/2)\mu \operatorname{grad} \mu=(-9/2)\mu^2\phi A\xi. \tag{1.42}$$

On the other hand, from (1.39) and (1.41) we see

$$(3/2)\mu \operatorname{grad} \mu=2\phi A\xi. \tag{1.43}$$

Since $\phi A\xi\neq 0$, the equations (1.42) and (1.43) give us a contradiction. Therefore we can conclude that ξ is a principal curvature vector. Q.E.D.

§2. Proof of Theorem

By virtue of Proposition 7 we find that ξ is a principal curvature vector (with principal curvature α). From Propositions 1 and 5, our discussion is divided into two cases: (i) $\alpha=0$ and (ii) $\alpha\neq 0$.

Case of (i) $\alpha=0$. Statement (i) of Proposition 6 asserts that our real hypersurface M lies on a tube of radius $\pi/4$ over a Kaehler submanifold $\tilde{N}$ in $P_n(C)$. But the converse is not true. Note that, in general a tube of radius $\pi/4$ over an arbitrary Kaehler submanifold $\tilde{N}$ is not a real hypersurface of $P_n(C)$. In fact, for example let $\tilde{N}$ be a complex quadric Q_{n-1}. Then a tube of radius $\pi/4$ over $\tilde{N}$ is $P^n(R)$ (which is the real part of $P_n(C)$) (cf. [4]). Statement (ii) of Proposition 6 shows the following:

Let $\tilde{N}$ be a Kaehler submanifold (with unit normal vector N) in $P_n(C)$. Suppose that the shape operator (with respect to N) A_N does not have the principal curvature 1. Then a tube (in the direction of N) of radius $\pi/4$ over $\tilde{N}$ is a real hypersurface M.

As a matter of course the real hypersurface M admits the vector ξ as a principal curvature vector with principal curvature 0 (cf. [1]).

Finally we remark that a homogeneous real hypersurface M with $A\xi=0$ lies on a tube of radius $\pi/4$ over a totally geodesic $P_k(C)$ $(1\leqq k\leqq n-1)$ (cf. [11]).

Case of (ii) $\alpha\neq 0$. See, Proposition 4. Q.E.D.

REMARK 1. Our Theorem gives the following

COROLLARY. *Let M be a real hypersurface of $P_n(C)$. Suppose that $A\xi\neq 0$. If $\nabla_\xi A=0$, then M lies on a tube of radius r over a totally geodesic $P_k(C)$ $(1\leqq k\leqq n-1)$, where $0<r<\pi/2$ and $r\neq\pi/4$.*

§3. Proof of Proposition A

First of all we shall show that the real hypersurface which satisfies the

hypothesis of Proposition A must be homogeneous.

The equation (1.8), together with Proposition 1 and the hypothesis that $h(=\text{trace } A)$ is constant, shows that

$$\begin{aligned}(\nabla_\xi S)\xi &= \nabla_\xi(S\xi) - S\nabla_\xi\xi \\ &= \nabla_\xi(S\xi) \\ &= \nabla_\xi\{(2n-2+\alpha h-\alpha^2)\xi\} = 0.\end{aligned}$$

And hence "$\nabla_\xi S=0$" implies

$$g((\nabla_\xi S)X, Y)=0 \qquad (\text{for any } X, Y \perp \xi). \tag{3.1}$$

Then from (1.9), (3.1) and the hypothesis that ξ is principal and h is constant, we have to show that the real hypersurface M, which satisfies the following equation (3.2), must be homogeneous.

$$g((\nabla_\xi A)X, (hI-A)Y) - g((\nabla_\xi A)AX, Y)=0 \qquad (\text{for any } X, Y \perp \xi). \tag{3.2}$$

Let X be a principal curvature (unit) vector orthogonal to ξ with principal curvature r. Putting $Y=\phi X$ in (3.2), we see

$$\{h-r-(\alpha r+2)/(2r-\alpha)\}\, g((\nabla_\xi A)X, \phi X)=0.$$

On the other hand, from (1.7) and Proposition 2 we find

$$\begin{aligned}g((\nabla_\xi A)X, \phi X) &= g((\nabla_X A)\xi + \phi X, \phi X) \\ &= g(\nabla_X(\alpha\xi) - A\nabla_X\xi + \phi X, \phi X) \\ &= \alpha r - r(\alpha r+2)/(2r-\alpha) + 1 \\ &= \alpha(r^2-\alpha r-1)/(2r-\alpha).\end{aligned}$$

By hypothesis that $\alpha \neq 0$, the above computation yields

$$h = r + (\alpha r+2)/(2r-\alpha) \quad \text{or} \quad r^2-\alpha r-1=0$$

so that the principal curvatures r and α are constant. Therefore Theorem K asserts that our real hypersurface must be homogeneous. In the following, we shall check the equation (3.2) one by one for six model spaces of type A_1, A_2, B, C, D and E:

Let M be of type A_1 or A_2. Then Proposition 3 tells us that $\nabla_\xi A=0$ so that (3.2) holds.

Let M be of type B (which is a tube of radius r). Let $x=\cot r$. Then M has three distinct constant principal curvatures $r_1=(1+x)/(1-x)$ with multiplicity $n-1$, $r_2=(x-1)/(x+1)$ with multiplicity $n-1$ and $\alpha=x-1/x$ so that $T_pM = V_{r_1}+V_{r_2}+\{\xi\}_R$ at any point p of M. Here note that $\phi V_{r_1}=V_{r_2}$ (cf. Proposition 2). It is sufficient to consider the following equations in order to check

(3.2):

(3.3) $g((\nabla_\xi A)X, (hI-A)X)-g((\nabla_\xi A)AX, X)=0$ (for any $X\in V_{r_1}$),

(3.4) $g((\nabla_\xi A)X, (hI-A)Y)-g((\nabla_\xi A)AX, Y)=0$ (for any $X, Y\in V_{r_1}$, $X\perp Y$),

(3.5) $g((\nabla_\xi A)X, (hI-A)\phi X)-g((\nabla_\xi A)AX, \phi X)=0$ (for any $X\in V_{r_1}$),

(3.6) $g((\nabla_\xi A)X, (hI-A)Y)-g((\nabla_\xi A)AX, Y)=0$

(for any $X\in V_{r_1}$, $Y\in V_{r_2}$ and $\phi X\perp Y$).

We shall calculate the left hand side of (3.3).

$$\begin{aligned}
&g((\nabla_\xi A)X, (hI-A)X)-g((\nabla_\xi A)AX, X)\\
&\quad=(h-2r_1)g((\nabla_\xi A)X, X)\\
&\quad=(h-2r_1)g((\nabla_X A)\xi+\phi X, X) \qquad \text{(from (1.7))}\\
&\quad=(h-2r_1)g(\nabla_X(A\xi)-A\nabla_X\xi, X)\\
&\quad=(h-2r_1)g(\alpha\phi AX-A\phi AX, X) \qquad \text{(from (1.5) and Proposition 1)}\\
&\quad=(h-2r_1)g(\alpha r_1\phi X-r_1r_2\phi X, X)=0.
\end{aligned}$$

This computation shows that the equation (3.3) holds for any $x(>1)$. Moreover a similar computation yields that the equations (3.4) and (3.6) hold for any $x(>1)$. Next we shall calculate the left hand side of (3.5).

$$\begin{aligned}
&g((\nabla_\xi A)X, (hI-A)\phi X)-g((\nabla_\xi A)AX, \phi X)\\
&\quad=(h-r_1-r_2)g((\nabla_\xi A)X, \phi X)\\
&\quad=(h-r_1-r_2)g((\nabla_X A)\xi+\phi X, \phi X) \qquad \text{(from (1.7))}\\
&\quad=(h-r_1-r_2)g(\nabla_X(A\xi)-A\phi AX+\phi X, \phi X) \qquad \text{(from (1.5))}\\
&\quad=(h-r_1-r_2)g(\alpha\phi AX-r_1A\phi X+\phi X, \phi X) \qquad \text{(from Proposition 1 and (1.5))}\\
&\quad=(h-r_1-r_2)(\alpha r_1-r_1r_2+1).
\end{aligned}$$

Note that $\alpha r_1-r_1r_2+1=-(x^2+1)/x\neq 0$. Here we put $h-r_1-r_2=0$. Then we have the following algebraic equation

$$x^4-2(2n-3)x^2+1=0.$$

And hence we find $x^2=2n-3\pm 2\sqrt{(n-1)(n-2)}$ so that

$$x=\sqrt{n-1}+\sqrt{n-2}, \qquad \text{since } x>1.$$

Therefore a homogeneous real hypersurface of type B with $x=\sqrt{n-1}+\sqrt{n-2}$ satisfies the hypothesis of Proposition A.

Now let M be of type C (which is a tube of radius r). Let $x=\cot r$. Then M has five distinct constant principal curvatures $r_1=(1+x)/(1-x)$ with multi-

plicity 2, $r_2=(x-1)/(x+1)$ with multiplicity 2, $r_3=x$ with multiplicity $n-3$, $r_4=-1/x$ with multiplicity $n-3$ and $\alpha=x-1/x$ so that $T_pM=V_{r_1}+V_{r_2}+V_{r_3}+V_{r_4}+\{\xi\}_R$ at any point p of M. Here note that $\phi V_{r_1}=V_{r_2}$, $\phi V_{r_3}=V_{r_3}$ and $\phi V_{r_4}=V_{r_4}$. For simplicity, we set $V_i=\{X \mid AX=r_iX,\ X\perp\xi\}$ for $1\leqq i\leqq 4$. In the following, let $X_i, Y_i\in V_i$ and $\|X_i\|=\|Y_i\|=1$ for $1\leqq i\leqq 4$.

It sufficies to consider the following equations in order to check (3.2):

(3.7) $g((\nabla_\xi A)X_i, (hI-A)X_i)-g((\nabla_\xi A)AX_i, X_i)=0$ (for $1\leqq i\leqq 4$),

(3.8) $g((\nabla_\xi A)X_i, (hI-A)Y_i)-g((\nabla_\xi A)AX_i, Y_i)=0$ (for $X_i\perp Y_i$; $i=1, 2$),

(3.9) $g((\nabla_\xi A)X_i, (hI-A)\phi X_i)-g((\nabla_\xi A)AX_i, \phi X_i)=0$ (for $i=3, 4$),

(3.10) $g((\nabla_\xi A)X_i, (hI-A)Y_i)-g((\nabla_\xi A)AX_i, Y_i)=0$
(for $X_i\perp Y_i$ and $Y_i\perp\phi X_i$; $i=3, 4$),

(3.11) $g((\nabla_\xi A)X_i, (hI-A)\phi X_i)-g((\nabla_\xi A)AX_i, \phi X_i)=0$ (for $i=1, 2$),

(3.12) $g((\nabla_\xi A)X_1, (hI-A)Y_2)-g((\nabla_\xi A)AX_1, Y_2)=0$ (for $\phi X_1\perp Y_2$),

(3.13) $g((\nabla_\xi A)X_1, (hI-A)Y_i)-g((\nabla_\xi A)AX_1, Y_i)=0$ (for $i=3, 4$),

(3.14) $g((\nabla_\xi A)X_2, (hI-A)Y_i)-g((\nabla_\xi A)AX_2, Y_i)=0$ (for $i=3, 4$),

(3.15) $g((\nabla_\xi A)X_3, (hI-A)Y_4)-g((\nabla_\xi A)AX_3, Y_4)=0$.

We here calculate the left hand side of (3.15).

$$\begin{aligned}
&g((\nabla_\xi A)X_3, (hI-A)Y_4)-g((\nabla_\xi A)AX_3, Y_4)\\
&\quad=(h-r_3-r_4)g((\nabla_\xi A)X_3, Y_4)\\
&\quad=(h-r_3-r_4)g((\nabla_{X_3}A)\xi+\phi X_3, Y_4) \qquad \text{(from (1.7))}\\
&\quad=(h-r_3-r_4)g(\nabla_{X_3}(\alpha\xi)-A\nabla_{X_3}\xi, Y_4)\\
&\quad=(h-r_3-r_4)g(\alpha\phi AX_3-A\phi AX_3, Y_4) \qquad \text{(from Proposition 1 and (1.5))}\\
&\quad=(h-r_3-r_4)g(\alpha r_3\phi X_3-(r_3)^2\phi X_3, Y_4)=0.
\end{aligned}$$

This computation shows that the equation (3.15) holds for any $x(>1)$. Moreover a similar computation yields that the equations (3.7)$\sim$(3.10) and (3.12)$\sim$(3.14) hold for any $x(>1)$. Next we shall calculate the left hand side of (3.11) in the case of $i=1$.

$$\begin{aligned}
&g((\nabla_\xi A)X_1, (hI-A)\phi X_1)-g((\nabla_\xi A)AX_1, \phi X_1)\\
&\quad=(h-r_1-r_2)g((\nabla_\xi A)X_1, \phi X_1)\\
&\quad=(h-r_1-r_2)g((\nabla_{X_1}A)\xi+\phi X_1, \phi X_1) \qquad \text{(from (1.7))}\\
&\quad=(h-r_1-r_2)g(\nabla_{X_1}(\alpha\xi)-A\nabla_{X_1}\xi+\phi X_1, \phi X_1)
\end{aligned}$$

$$=(h-r_1-r_2)g(\alpha\phi AX_1-A\phi AX_1+\phi X_1,\ \phi X_1)$$

(from Proposition 1 and (1.5))

$$=(h-r_1-r_2)(\alpha r_1-r_1r_2+1).$$

Note that $\alpha r_1-r_1r_2+1=-(x^2+1)/x\neq 0$. Here we put $h-r_1-r_2=0$. Then we have the following algebraic equation

$$(n-2)x^4-2nx^2+n-2=0.$$

And hence we find $x^2=(n\pm 2\sqrt{n-1})/(n-2)$ so that

$$x=(\sqrt{n-1}+1)/\sqrt{n-2}, \qquad \text{since } x>1.$$

Of course, solving the equation (3.11) in the case of $i=2$, we obtain the same solution x. Therefore a homogeneous real hypersurface of type C with $x=(\sqrt{n-1}+1)/\sqrt{n-2}$ satisfies the hypothesis of Proposition A.

Let M be of type D (which is a tube of radius r). Let $x=\cot r$. Then M has five distinct constant principal curvatures $r_1=(1+x)/(1-x)$ with multiplicity 4, $r_2=(x-1)/(x+1)$ with multiplicity 4, $r_3=x$ with multiplicity 4, $r_4=-1/x$ with multiplicity 4 and $\alpha=x-1/x$. By virtue of the computation in case of type C we have only to solve the equation $h-r_1-r_2=0$. Namely we get the following

$$5x^4-22x^2+5=0 \qquad \text{so that } x=(\sqrt{8}+\sqrt{3})/\sqrt{5}.$$

Therefore a homogeneous real hypersurface of type D with $x=(\sqrt{8}+\sqrt{3})/\sqrt{5}$ satisfies the hypothesis of Proposition A.

Let M of be type E (which is a tube of radius r). Let $x=\cot r$. Then M has five distinct constant principal curvatures $r_1=(1+x)/(1-x)$ with multiplicity 6, $r_2=(x-1)/(x+1)$ with multiplicity 6, $r_3=x$ with multiplicity 8, $r_4=-1/x$ with multiplicity 8 and $\alpha=x-1/x$. Considering the equation $h-r_1-r_2=0$, we have the following

$$9x^4-38x^2+9=0 \qquad \text{so that } x=(\sqrt{5}+\sqrt{14})/3.$$

Therefore a homogeneous real hypersurface of type E with $x=(\sqrt{5}+\sqrt{14})/3$ satisfies the hypothesis of Proposition A. Q. E. D.

§ 4. Proof of Proposition B

First we note that "ξ is principal" is equivalent to "$\nabla_\xi\phi=0$". Since $\nabla_\xi R=0$, from (1.6) we have

$$g((\nabla_\xi A)Y,\ Z)g(AX,\ W)+g(AY,\ Z)g((\nabla_\xi A)X,\ W)-g((\nabla_\xi A)X,\ Z)g(AY,\ W)$$
$$-g(AX,\ Z)g((\nabla_\xi A)Y,\ W)=0,$$

which, together with (1.7), yields

$$g((\nabla_Y A)\xi+\phi Y, Z)g(AX, W)+g(AY, Z)g((\nabla_X A)\xi+\phi X, W)$$
$$-g((\nabla_X A)\xi+\phi X, Z)g(AY, W)-g(AX, Z)g((\nabla_Y A)\xi+\phi Y, W)=0.$$

Therefore, from Proposition 1 and (1.5) we obtain

$$\begin{aligned}(4.1)\quad & g(\alpha\phi AY-A\phi AY+\phi Y, Z)g(AX, W)+g(AY, Z)g(\alpha\phi AX-A\phi AX+\phi X, W)\\ & -g(\alpha\phi AX-A\phi AX+\phi X, Z)g(AY, W)\\ & -g(AX, Z)g(\alpha\phi AY-A\phi AY+\phi Y, W)=0 \quad (\text{for any } X, Y, Z \text{ and } W\in TM).\end{aligned}$$

Let X be a principal curvature (unit) vector orthogonal to ξ with principal curvature r. Putting $Y=Z=\xi$ and $W=\phi X$ in (4.1), from Proposition 2 we find

$$\alpha r-r(\alpha r+2)/(2r-\alpha)+1=0.$$

Since $\alpha\neq 0$, we see $r^2-\alpha r-1=0$, that is, $r=(\alpha r+2)/(2r-\alpha)$. This implies that our real hypersurface M is locally congruent to a homogeneous one of type (A_1) or (A_2) (cf. [8]). Of course a homogeneous real hypersurface of type (A_1) or (A_2) of any radius $r(\neq\pi/4)$ satisfies the hypothesis of Proposition B. Q.E.D.

REMARK 2. The first author classified real hypersurfaces M which satisfy $\phi S=S\phi$ in $P_n(C)$ (cf. [5]). By virtue of his classification we find the following:

Let M be a homogeneous real hypersurface of $P_n(C)$. Then the following two conditions are equivalent.

(i) $\nabla_\xi S=0$.

(ii) $\phi S=S\phi$.

References

[1] T.E. Cecil and P.J. Ryan, Focal sets and real hypersurfaces in complex projective space, Trans. Amer. Math. Soc. **269** (1982), 481-499.

[2] U.H. Ki, Real hypersurfaces with parallel Ricci tensor of a complex space form, Tsukuba J. Math. **13** (1989), 73-81.

[3] U.H. Ki, H. Nakagawa and Y.J. Suh, Real hypersurfaces with harmonic Weyl tensor of a complex space form, Hiroshima Math. J. **20** (1990), 93-102.

[4] M. Kimura, Real hypersurfaces and complex submanifolds in complex projective space, Trans. Amer. Math. Soc. **296** (1986), 137-149.

[5] M. Kimura, Some real hypersurfaces of a complex projective space, Saitama Math. J. **5** (1987), 1-5.

[6] M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space, Math. Z. **202** (1989), 299-311.

[7] S. Maeda and S. Udagawa, Real hypersurfaces of a complex projective space in terms of holomorphic distribution, Tsukuba J. Math. **14** (1990), 39-52.

[8] Y. Maeda, On real hypersurfaces of a complex projective space, J. Math. Soc. Japan **28** (1976), 529-540.

[9] M. Okumura, On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc. **212** (1975), 355-364.
[10] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, Osaka J. Math. **10** (1973), 495-506.
[11] R. Takagi, Real hypersurfaces in a complex projective space with constant principal curvatures I, II, J. Math. Soc. Japan **27** (1975), 43-53, 507-516.
[12] S. Udagawa, Bi-order real hypersurfaces in a complex projective space, Kodai Math. J. **10** (1987), 182-196.

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