IDEALS ON $\omega$ WHICH ARE OBTAINED FROM HAUSDORFF-GAPS

By

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Let $\mathcal{G}$ be a Hausdorff gap in $^{\omega}\omega$. Hart and Mill [2] defined the ideal $I_{\mathcal{G}}$ which is the family of all subsets of $\omega$ whose restriction of $\mathcal{G}$ is filled. In this paper, we shall show two results (Theorems 1, 6) about these ideals.

Our notions and terminology follow the usual use in set theory. Let $X$ be a subset of $\omega$ and $f, g$ functions from $X$ to $\omega$. $g$ dominates $f$ (denoted by $f < g$), if $\{n \in X; g(n) \leq f(n)\}$ is finite. Let $\kappa$ and $\lambda$ be infinite cardinals. A pair of sequence $\langle\langle f_{\alpha}|\alpha<\kappa\rangle|\langle g_{\beta}|\beta<\lambda\rangle\rangle$ is called a $(\kappa, \lambda)$-gap, if the following (1), (2) are satisfied.

1. $f_{\alpha}, g_{\beta} : \omega \rightarrow \omega$, for any $\alpha<\kappa, \beta<\lambda$.
2. $f_{\alpha}<f_{\gamma}<g_{\delta}<g_{\beta}$, for any $\alpha<\gamma<\kappa, \beta<\delta<\lambda$.

A $(\kappa, \lambda)$-gap $\langle\langle f_{\alpha}|\alpha<\kappa\rangle|\langle g_{\beta}|\beta<\lambda\rangle\rangle$ is unfilled, if there does not exist a function $h : \omega \rightarrow \omega$ such that, for all $\alpha<\kappa, \beta<\lambda$, $f_{\alpha}<h<g_{\beta}$. We call an unfilled $(\omega_{1}, \omega_{1})$-gap a Hausdorff gap (H-gap). The following fact is well-known.

FACT. For any regular cardinals $\kappa$ and $\lambda$ with $(\kappa, \lambda) \neq (\omega_{1}, \omega_{1})$, there exists a generic extension $W$ such that $W$ preserves all cardinals and, in $W$, there are no unfilled $(\kappa, \lambda)$-gap.

In contrast to this fact, the following theorem holds about $H$-gaps.

THEOREM (Hausdorff [1, Theorem 4.3]). There is an $H$-gap.

Let $\mathcal{G}=\langle\langle f_{a}|a<\omega_{1}\rangle|\langle g_{a}|a<\omega_{1}\rangle\rangle$ be a $(\omega_{1}, \omega_{1})$-gap. Following [2], we define the ideal $I_{\mathcal{G}}$ by

$$I_{\mathcal{G}}=\{x \subseteq \omega; \exists h : x \rightarrow \omega \forall \alpha<\omega_{1}(f_{a}|x<h<g_{a}|x)\}.$$ 

It is easy to see that

$\omega \in I_{\mathcal{G}}$ if and only if $\mathcal{G}$ is filled,

$\text{Fin} = \{x \subseteq \omega; x \text{ is finite}\} \subseteq I_{\mathcal{G}}.$
In this paper, we shall show two result about these ideals $I_\mathfrak{g}$.

**Theorem 1.** Assume the Continuum Hypothesis $(CH)$. For any ideal $\mathfrak{i}$ with $\text{Fin} \subset \mathfrak{i}$, there exists an $(\omega_1, \omega_1)$-gap $\mathcal{G}$ such that $\mathfrak{i} = I_\mathcal{G}$.

We need the several lemmas and corollaries to show Theorem 1. Let $\Gamma = \{h : \exists x \subset \omega \langle h : x \rightarrow \omega \rangle\}$. For any $f, g \in \Gamma$, $f \ll g$ means that, for any $k < \omega$, \{n \in \text{dom}(f) \cap \text{dom}(g); g(n) < f(n) + k\} is finite. For any $X, Y \subset \Gamma$, $X \ll Y$ means that, for all $f \in X$ and $g \in Y$, $f \ll g$.

**Lemma 2.** Let $X, Y$ be countable subsets of $^\omega \omega$, $X \neq \emptyset$, and $X \ll Y$. Then there exists an $h : \omega \rightarrow \omega$ such that $X \ll \{h\} \ll Y$.

**Proof.** The case of $Y = \emptyset$ is clear. So, we may assume that $Y \neq \emptyset$. Take an enumeration $<f_j|j < \omega>$ of $X$, and an enumeration $<g_j|j < \omega>$ of $Y$. For any $k < \omega$, since $X \ll Y$, it holds that
\[
\lim_{n \rightarrow \omega} (\min\{g_i(n); i \leq k\} - \max\{f_j(n); j \leq k\}) = \omega.
\]
So, we can take a sequence of natural numbers $n_k$ (for $k < \omega$) such that
\[
n_k < n_{k+1}
\]and
\[
\forall n \in [n_k, n_{k+1}) (\min\{g_i(n); i \leq k\} - \max\{f_j(n); j \leq k\} \geq 2k).
\]
Define $h : \omega \rightarrow \omega$ by
\[
h(n) = \max\{f_j(n); j \leq k\} + k, \text{ if } n \in [n_k, n_{k+1}).
\]
It is easy to see that $X \ll \{h\} \ll Y$. \square

**Corollary 3.** Let $X, Y \subset \Gamma$. Suppose that $|X| \leq \omega$, $|Y| \leq \omega$, $X \ll Y$, and $\exists f \in X \langle f : \omega \rightarrow \omega \rangle$. Then, there exists an $h : \omega \rightarrow \omega$ such that $X \ll \{h\} \ll Y$.

**Proof.** For each $f \in X$, define $f_* : \omega \rightarrow \omega$ by
\[
f_*(n) = \begin{cases} f(n), & \text{if } n \in \text{dom}(f), \\ 0, & \text{otherwise}. \end{cases}
\]
By Lemma 2 there exists $g : \omega \rightarrow \omega$ such that $\{f_* ; f \in X\} \ll \{g\}$. For each $f \in Y$, define $f^* : \omega \rightarrow \omega$ by
\[
f^*(n) = \begin{cases} f(n), & \text{if } n \in \text{dom}(f), \\ g(n), & \text{otherwise}. \end{cases}
\]
Then, since $\{f_* ; f \in X\} \ll \{f_* ; f \in Y\}$, there exists $h : \omega \rightarrow \omega$ such that $\{f_* ; f \in \{f_* ; f \in Y\}$. \square
Ideals on $\omega$ which are obtained

\[ X \ll \{ h \} \ll \{ f^* ; f \in Y \}, \]

by Lemma 2. This $h$ is as required. \qed

**COROLLARY 4.** Let $X, Y, Z$ be countable subsets of $\Gamma$ such that $X \ll Z, Z \ll Y, X \ll Y,$ and \( \exists f \in X(f : \omega \rightarrow \omega) \). Then, there exist $g, h : \omega \rightarrow \omega$ such that $X \ll \{ h \} \ll Z$ and $Z \ll \{ g \} \ll Y$ and $h \ll g$.

**PROOF.** Since $X \ll Z \cup Y$, by Corollary 3, we can take $h : \omega \rightarrow \omega$ such that $X \ll \{ h \} \ll Z \cup Y$. Then $Z \cup \{ h \} \ll Y$ and we can take $g : \omega \rightarrow \omega$ such that $Z \cup \{ h \} \ll \{ g \} \ll Y$. \qed

**LEMMA 5.** Let $b$ be an infinite subset of $\omega$ and $s : b \rightarrow \omega$. Suppose that $X, Y \subset^\omega \omega$ and $Z \subset \Gamma$ satisfy that

(2.1) \( X \neq \emptyset \) \& \( |X| \leq \omega \) \& \( |Y| \leq \omega \) \& \( |Z| \leq \omega \) \& $X \ll Y \& X \ll Z \ll Y$,

(2.2) \( \forall h \in Z(b \cap \text{dom}(h) \text{ is finite}) \).

Then, there are $f, g : \omega \rightarrow \omega$ such that

(2.3) \( X \ll \{ f \} \ll \{ g \} \ll Y \) and $f \ll g$,

(2.4) \( f\nabla b \nleq s \) or $s \nleq g\nabla b$.

**PROOF.** Set $a = \omega \setminus b$. By using Corollary 4, take $f_1, g_1 : a \rightarrow \omega$ such that

\[ X \upharpoonright a \ll \{ f_1 \} \ll \{ g_1 \} \ll Y \upharpoonright a \text{ and } f_1 \ll g_1. \]

Take $f_2, g_2 : b \rightarrow \omega$ such that

\[ X \upharpoonright b \ll \{ f_2 \} \ll \{ g_2 \} \ll Y \upharpoonright b \text{ and } f_2 \nleq s \text{ or } s \nleq g_2 \]

and set

\[ f = f_1 \cup f_2, \quad g = g_1 \cup g_2. \]

Then, $f$ and $g$ are as required. \qed

**PROOF OF THEOREM 1.** Let $l$ be an ideal on $\omega$ such that $\text{Fin} \subseteq l$.

The case of that $\omega \in l$ has no problem. So, we may assume that $\omega \notin l$. Set $\mathcal{X} = \{ s ; \exists x \subset \omega(x \notin l \& s : x \rightarrow \omega) \}$. By CH, take an enumeration $\langle s_\alpha | \alpha < \omega_1 \rangle$ of $\mathcal{X}$ and an enumeration $\langle a_\alpha | \alpha < \omega_1 \rangle$ of $l$. For each $\alpha < \omega_1$, let $b_\alpha = \text{dom}(s_\alpha)$. By induction on $\alpha < \omega_1$, we shall take $f_\alpha, g_\alpha : \omega \rightarrow \omega$ and $h_\alpha : a_\alpha \rightarrow \omega$ which satisfy the following (1) \~(4).

(1) \( f_\xi \ll f_\alpha \ll g_\alpha \ll g_\xi \), for any $\xi < \alpha$.

(2) \( f_\alpha \upharpoonright a_\xi \ll h_\xi \ll g_\alpha \upharpoonright a_\xi \), for any $\xi < \alpha$.

(3) \( f_\alpha \upharpoonright b_\alpha \nsubseteq s_\alpha \text{ or } s_\alpha \nsubseteq g_\alpha \upharpoonright b_\alpha \).
\[ f_{\alpha} \upharpoonright a_{\alpha} \ll h_{\alpha} \ll g_{\alpha} \upharpoonright a_{\alpha}. \]

Assume that we could take such \( f_{\alpha}, g_{\alpha}, h_{\alpha} \) (for \( \alpha < \omega_{1} \)). By (1),
\[ \mathcal{G} = \langle \langle f_{\alpha} \upharpoonright \alpha < \omega_{1} \rangle | \langle g_{\alpha} \upharpoonright \alpha < \omega_{1} \rangle \rangle \]
is a gap. By (2), it holds that
\[ f_{\alpha} \upharpoonright a_{\beta} \ll h_{\beta} \ll g_{\alpha} \upharpoonright a_{\beta}, \quad \text{for any} \quad \alpha, \beta < \omega_{1}. \]
So, it holds that, for all \( \beta < \omega_{1}, a_{\beta} \in I_{\mathcal{G}} \) (i.e., \( l \subset I_{\mathcal{G}} \)). And by (3), we have that \( I_{\mathcal{G}} \subset l \).

It remains to show that we can take such \( f_{\alpha}, g_{\alpha}, h_{\alpha} \) (for \( \alpha < \omega_{1} \)).

Suppose that \( \alpha < \omega_{1} \) and defined \( f_{\xi}, g_{\xi}, h_{\xi} \) (for \( \xi < \alpha \)) satisfying (1)~(4).
Since it holds that
\[ b_{\alpha} \notin l \land \{ a_{\xi} ; \xi < \alpha \} \subset l \land \text{Fin} \subset l, \]
we can take \( b \subset b_{\alpha} \) such that
\[ b \text{ is infinite and } b \cap a_{\xi} \text{ is finite for each } \xi < \alpha. \]
By Lemma 5, take \( f_{\alpha}, g_{\alpha} : \omega \to \omega \) such that
\[ f_{\xi} \ll f_{\alpha} \ll g_{\alpha} \ll g_{\xi} \quad \text{for all} \quad \xi < \alpha, \]
\[ f_{\alpha} \upharpoonright a_{\xi} \gg h_{\xi} \ll g_{\alpha} \upharpoonright a_{\xi} \quad \text{for all} \quad \xi < \alpha, \]
\[ f_{\alpha} \upharpoonright b \ll s_{\alpha} \ll b \quad \text{or} \quad s_{\alpha} \ll b \ll g_{\alpha} \ll b, \]
and take \( h_{\alpha} : a_{\alpha} \to \omega \) such that
\[ f_{\alpha} \upharpoonright a_{\alpha} \ll h_{\alpha} \ll g_{\alpha} \upharpoonright a_{\alpha}. \]
These \( f_{\alpha}, g_{\alpha}, h_{\alpha} \) satisfy (1)~(4). \( \blacksquare \)

Here, we remark that the assumption of CH in Theorem 1 is necessary. To see this, let \( V \) be a ground model which satisfies that \( 2^{\omega} = 2^{\omega_{1}} \). Then, in \( V \), there exists an ideal which is not obtained from any \( (\omega, \omega_{1}) \)-gaps, since the cardinality of the family of ideals on \( \omega \) is greater than the cardinality of the family of \( (\omega, \omega_{1}) \)-gaps. Which ideals are obtained from \( (\omega, \omega_{1}) \)-gaps, under the assumption of \( \neg \text{CH} \)? The following theorem deals a case whose model is obtained by a simple generic extension.

**Theorem 6.** Assume CH. Let \( \kappa \) be a cardinal such that \( \kappa^{\omega} = \kappa \) and \( P \) be the partial ordering \( \{ p : \exists x \subseteq \kappa (|x| < \omega \land p : x \to 2) \} \) which adjoins \( \kappa \)-many Cohen reals. Then, in \( V^{P} \), it holds that the family \( \{ I_{\mathcal{G}} ; \mathcal{G} \text{ is an H-gap} \} \) consists of all ideals \( l \) such that \( \omega \notin l \) and \( \text{Fin} \subset l \) and \( l \) are \( \leq \omega_{1} \)-generated.
We need the following lemma and corollary to show \textbf{Theorem 6}. Let $Q$ be the partial ordering $\{q; \exists x \subset \omega (|x| < \omega \& q : x \rightarrow 2)\}$ which adjoins a Cohen real.

\textbf{Lemma 7}. Let $G = \langle \langle f_{\alpha} | \alpha < \omega_{1} \rangle \mid \langle g_{\alpha} | \alpha < \omega_{1} \rangle \rangle$ be an $H$-gap. Then, it holds that $V^{Q} \models \text{"} I_{G} \text{ is the ideal generated by } (I_{G})^{\mathcal{V}} \text{"}.$

\textbf{Proof}. Set $l = (I_{G})^{\mathcal{V}}$. Since $V^{Q} \models l \subset l_{g}'$, it suffices to show that $\models_{Q} \forall x \in I_{G} \exists y \in (x \subset y)$.

To show this, let $q \in Q \& x : Q$-name $\& q \models x \in I_{G}$.

Take a $Q$-name $h$ such that $q \models h : x \rightarrow \omega \& \forall \alpha < \omega_{1} (f_{\alpha} \prec h < g_{\alpha} \prec x)$.

For each $\alpha < \omega_{1}$, take $q_{\alpha} \leq q$ and $n_{\alpha} < \omega$ such that $q_{\alpha} \models \forall k \in x \& n_{\alpha} (f_{\alpha}(k) < h(k) < g_{\alpha}(k))$.

Since $|Q \times \omega| = \omega$, there exist $r \in Q$ and $m < \omega$ such that $A = \{\alpha < \omega_{1} ; q_{\alpha} = r \& n_{\alpha} = m\}$ is cofinal in $\omega_{1}$.

Set $y = \{k < \omega ; m \leq k \& \exists r' \leq r (r' \models k \in x)\}$. It holds that $r \models x \subset y \cup m$.

\textbf{Claim 1}. For any $\alpha, \beta \in A$ and any $k \in y$, $f_{\alpha}(k) + 1 < g_{\beta}(k)$.

\textbf{Proof of Claim 1}. Let $\alpha, \beta \in A$ and $k \in y$. Take $r' \leq r$ such that $r' \models k \in x$.

Since $k \geq m$, we have that $r' \models f_{\alpha}(k) < h(k) < g_{\beta}(k)$ which implies $f_{\alpha}(k) + 1 < g_{\beta}(k)$.

\textbf{QED of Claim 1}.

By using Claim 1, define $h' : y \rightarrow \omega$ by $h'(k) = \max \{f_{\alpha}(k) ; \alpha \in A\} + 1$.

Then, it holds that $\forall \alpha < \omega_{1} (f_{\alpha} \prec y < h' < g_{\alpha} \prec y)$ and we get $y \in l$. \square

\textbf{Corollary 8}. Let $G = \langle \langle f_{\alpha} | \alpha < \omega_{1} \rangle \mid \langle g_{\alpha} | \alpha < \omega_{1} \rangle \rangle$ be an $H$-gap. Then it holds $V^{P} \models \text{"} I_{G} \text{ is the ideal generated by } (I_{G})^{\mathcal{V}} \text{"}.$
PROOF. This follows from Lemma 7 and the fact that
\[ V^P \cap \mathcal{P}(\omega) \subset \cup \{ V^{P \uparrow a} : a \in V \land a \subseteq \kappa \land |a| \leq \omega \}. \]

PROOF OF THEOREM 6. First we shall show that, in \( V^P \),
\[ \forall \mathcal{G} : H\text{-gap } (I_\mathcal{G} \text{ is } \leq \omega_1\text{-generated}). \]
So, let \( \mathcal{G} \) be a \( P \)-name such that, \( V^P \models \mathcal{G} \) is an \( H \)-gap. Take an \( A \in V \) such that
\[ A \subset \kappa \land |A| \leq \omega_1 \land \mathcal{G} \in V^{P \uparrow A}. \]
Since \( V^{P \uparrow A} \models \text{CH} \), we have
\[ V^{P \uparrow A} \models I_\mathcal{G} \text{ is } \leq \omega_1\text{-generated}. \]
Since \( P \cong (P \upharpoonright A) \times (P \upharpoonright (\kappa \setminus A)) \) and \( P \cong P \upharpoonright (\kappa \setminus A) \), by Corollary 8,
\[ V^P \models I_\mathcal{G} \text{ is } \leq \omega_1\text{-generated}. \]
To show the reverse implication, let \( l \) be a \( P \)-name such that
\[ V^P \models \omega \notin l \text{ and } l \text{ is } \leq \omega_1\text{-generated and } \text{Fin} \subseteq l. \]
Take an \( S \in V^P \) such that
\[ V^P \models |S| \leq \omega_1 \text{ and } l \text{ is generated by } S. \]
Then, there exists an \( A \in V \) such that
\[ A \subset \kappa, \text{ } |A| \leq \omega_1 \text{ and } S \in V^{P \uparrow A}. \]
Since \( V^{P \uparrow A} \models \text{CH} \), there is a \( \mathcal{G} \in V^{P \uparrow A} \) such that
\[ V^{P \uparrow A} \models \mathcal{G} \text{ is an } H\text{-gap and } I_\mathcal{G} \text{ is generated by } S. \]
By Corollary 8, \( V^P \models I_\mathcal{G} = l. \)

References


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