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# ON COMPLETE SPACE-LIKE SURFACES WITH CONSTANT MEAN CURVATURE IN A LORENTZIAN 3-SPACE FORM

By

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## Introduction

By a *Lorentzian* $(n+1)$ *-space form* $M_1^{n+1}(c)$ we mean a Minkowski space $R_1^{n+1}$, a de Sitter space $S_1^{n+1}(c)$ or an anti-de Sitter space $H_1^{n+1}(c)$, according as $c>0$, $c=0$ or $c<0$, respectively. That is, a Lorentzian space form $M_1^{n+1}(c)$ is a complete connected $(n+1)$-dimensional Lorentzian manifold with constant curvature $c$.

A hypersurface in a Lorentzian manifold is said to be *space-like* if the induced metric on the hypersurface is positive definite. On a space-like hypersurface, the first fundamental form, the second fundamental form and the mean curvature are defined in the same way as those on a hypersurface in a Riemannian manifold [§1].

It has been proved by Bernstein and others that the only entire minimal hypersurface in a Euclidean space $R^{n+1}$ is a linear hyperplane for $n\leqq 7$, but there are other examples for $n>7$. So, Calabi proposed to study a Lorentzian analoge, called the Bernstein type problem, in Minkowki space $R_1^{n+1}$, and this was solved by Cheng and Yau [4] for every $n$.

More precisely, a space-like hypersurface in a Lorentzian manifold is said to be *maximal*, if the mean curvature is zero. The Bernstein type problem has led to the conclusion that the only entire maximal space-like hypersurface in $R_1^{n+1}$ is a linear hyperplane. In order to prove this, Cheng and Yau [4] established the following result:

(*) *If an entire space-like hypersurface $M$ in $R_1^{n+1}$ has a constant mean curvature $H$, then the induced Lorentzian metric on $M$ is a complete Riemannian metric and the length of second fundamental form of $M$ is bounded from above by $n|H|$.*

It follows from this result that if $M$ is maximal, then it is totally geodesic.

Moreover, Nishikawa [11] studied the Bernstein type problem for complete maximal space-like hypersurfaces in other Lorentzian maniflolds, and Ishihara

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[8] found a similar result for complete maximal space-like submanifolds $M^n$ in a semi-Riemannian space form $M_p^{n+p}(c)$.

On the other hand, in the theory of relativity, certain space-like hypersurfaces with constant mean curvature in arbitrary space-times are also investigated. For instance, Choque-Bruhat, Fischer and Marsden [5] studied the Bernstein type problem in a space-time $M_1^4(c)$ $(c \geqq 0)$ in connection with the positivity of mass, and proved that a compact maximal space-like hypersurface in $M_1^4(c)$ $(c \geqq 0)$ must be totally geodesic.

We shall consider, in this paper, complete space-like hypersurfaces with non-zero constant mean curvature in a Lorentzian space form $M_1^{n+1}(c)$. The well-known standard models of these are the totally umbilical space-like hypersurfaces and the following product manifolds:

$$H^k(c_1)\times M^{n-k}(c_2)$$
$$=\begin{cases} H^k(c_1)\times S^{n-k}(c_2) & \textit{in } S_1^{n+1}(c) \quad \left[\dfrac{1}{c_1}+\dfrac{1}{c_2}=\dfrac{1}{c},\ c_2>0\right], \\ H^k(c_1)\times R^{n-k} & \textit{in } R_1^{n+1} \quad [c=c_2=0], \\ H^k(c_1)\times H^{n-k}(c_2) & \textit{in } H_1^{n+1}(c) \quad \left[\dfrac{1}{c_1}+\dfrac{1}{c_2}=\dfrac{1}{c},\ c_2<0\right]. \end{cases}$$

where $k=1, \cdots, n-1$. $H^1(c_1)\times M^{n-1}(c_2)$ is, in particular, called a *hyperbolic cylinder*.

Goddard [6] conjectured that the only complete space-like hypersurfaces of constant mean curvature in $M_1^4(c)$ $(c \geqq 0)$ are the above standard models. However, it is proved by Treibergs [16] that many other examples of complete space-like surfaces with constant mean curvature exist in $R_1^3$. Thus, conversely, it seems to be interesting to characterize the above standard models among these space-like surfaces.

In this direction, Akutagawa [2], Ramanathan [14] and Cheng and Nakagawa [3] obtained the conditions for a complete space-like hypersurfaces with constant mean curvature in $S_1^{n+1}(c)$ to be totally umbilical.

On the other hand, K. Milnor [10] and Yamada [17] characterized the hyperbolic cylinder $H^1(c_1)\times R^1$ in $R_1^3$ as the only complete "uniformly" non-umbilical space-like surface with non-zero constant mean curvature. In particular, K. Milnor proved this result by making use of the Cheng-Yau result (*).

The purpose of this paper is to prove a certain extension of the Cheng-Yau result (*) as stated in Theorem 1 [§2]. This theorem means that a complete space-like surface with constant mean curvature in $M_1^3(c)$ is totally umbilical, or the Gaussian curvature is non-positive. Furthermore, by applying theorem 1, a characterization of the hyperbolic cylinder $H^1(c_1)\times M^1(c_2)$ in $M_1^3(c)$ is obtained in Theorem 2 [§3].

THEOREM 1. *Let $M$ be a complete space-like surface with constant mean curvature $H$ in a Lorentzian 3-space form $M_1^3(c)$. Let $\alpha$ be the second fundamental form of $M$. Then the following hold:*

(1) *If $c$ is non-positive, then $|\alpha|^2\leqq 4H^2-2c$.*

(2) *If $c$ is positive, then $M$ is totally umbilical or $|\alpha|^2\leqq 4H^2-2c$.*

THEOREM 2. *The hyperbolic cylinder is the only complete space-like surface in $M_1^3(c)$ with non-zero constant mean curvature whose principal curvatures $\lambda$ and $\mu$ satisfy $\inf(\lambda-\mu)^2>0$.*

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## 1. Space-like hypersurfaces in a Lorentzian manifold.

Let $\tilde{M}$ be an $(n+1)$-dimensional Lorentzian manifold and $M$ be a space-like hypersurface in $\tilde{M}$. Throughout this paper, manifolds are always assumed to be connected and geometric objects are assumed to be smooth, unless otherwise stated. We choose a local field of Lorentzian orthonormal frames $\{E_A\}=\{E_0, E_1, \cdots, E_n\}$ defined on a neighborhood of $\tilde{M}$ in such a way that, restricted to $M$, $\{E_1, \cdots, E_n\}$ are space-like and tangent to $M$ and $E_0$ is time-like and normal to $M$. Let $\tilde{\nabla}$ (resp. $\nabla$) denote the Levi-Civita connection of $\tilde{M}$ (resp. $M$).

We use the following convention on the ranges of indices throughout this paper, unless otherwise stated:

$$A, B, \cdots=0, 1, \cdots, n\,; \qquad i, j, \cdots=1, \cdots, n\,.$$

With respect to the frame field $\{E_A\}$, let $\{\omega_A\}=\{\omega_0, \omega_i\}$ denote its dual frame field. Then the Lorentzian metric tensor $\tilde{g}$ of $\tilde{M}$ is given by $\tilde{g}=\sum\varepsilon_A\omega_A\otimes\omega_A$, where $\varepsilon_A$ is defined by $\varepsilon_0=-1$ and $\varepsilon_i=1$. The connection forms on $\tilde{M}$ are denoted by $\omega_{AB}$, that is, $\omega_{AB}$ is defined by $\omega_{AB}(E_C)=\tilde{g}(E_A, \tilde{\nabla}_{E_C}E_B)$. The canonical forms $\omega_A$ and the connection forms $\omega_{AB}$ of the ambient space satisfy the structure equations

$$d\omega_A+\sum\varepsilon_B\omega_{AB}\wedge\omega_B=0\,, \qquad \omega_{AB}+\omega_{BA}=0\,, \tag{1.1}$$

(1.2)
$$d\omega_{AB}+\sum\varepsilon_C\omega_{AC}\wedge\omega_{CB}=\tilde{\Omega}_{AB},$$
$$\tilde{\Omega}_{AB}=-\frac{1}{2}\sum\varepsilon_C\varepsilon_D\tilde{R}_{ABCD}\omega_C\wedge\omega_D.$$

where $\tilde{\Omega}_{AB}$ is called the Riemannian curvature form on $\tilde{M}$, and $\tilde{R}_{ABCD}$ denotes the component of the Riemannian curvature tensor $\tilde{R}$ on $\tilde{M}$. That is, $\tilde{R}_{ABCD}$ is defined by

$$\tilde{R}_{ABCD}=\tilde{g}(\tilde{R}(E_A, E_B)E_C, E_D),$$
$$\tilde{R}(E_A, E_B)E_C=\tilde{\nabla}_{E_A}\tilde{\nabla}_{E_B}E_C-\tilde{\nabla}_{E_B}\tilde{\nabla}_{E_A}E_C-\tilde{\nabla}_{[E_A,E_B]}E_C.$$

Restricting these forms to the hypersurface $M$, we have

(1.3)
$$\omega_0=0,$$

and the Riemannian metric $g$ of $M$ induced from the Lorentzian metric $\tilde{g}$ on $\tilde{M}$ is given by $g=\sum\omega_j\otimes\omega_j$. Then, with respect to this metric, $\{E_j\}$ becomes a local orthonormal frame field and $\{\omega_j\}$ is a local dual frame field of $\{E_j\}$. Further, $\omega_{ij}$ is the connection form on $M$ satisfying $\omega_{ij}(E_k)=g(E_i, \nabla_{E_k}E_j)$. From the structure equations of $\tilde{M}$ it follows that the structure equations for $M$ are given by

(1.4)
$$d\omega_i+\sum\omega_{ij}\wedge\omega_j=0, \qquad \omega_{ij}+\omega_{ji}=0,$$

(1.5)
$$d\omega_{ij}+\sum\omega_{ik}\wedge\omega_{kj}=\Omega_{ij},$$
$$\Omega_{ij}=-\frac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l,$$

where $\Omega_{ij}$ is the Riemannian curvature form on $M$ and $R_{ijkl}$ is the component of the Riemannian curvature tensor $R$ on $M$. That is, $R_{ijkl}$ is definded by

$$R_{ijkl}=g(R(E_i, E_j)E_k, E_l),$$
$$R(E_i, E_j)E_k=\nabla_{E_i}\nabla_{E_j}E_k-\nabla_{E_j}\nabla_{E_i}E_k-\nabla_{[E_i,E_j]}E_k.$$

It follows from (1.3) and Cartan's lemma that the exterior derivative of (1.3) gives rise to

(1.6)
$$\omega_{0i}=\sum h_{ij}\omega_j, \qquad h_{ij}=h_{ji}.$$

On the other hand, the second fundamental form $\alpha$ of $M$ is defined by

$$\tilde{\nabla}_XY=\nabla_XY+\alpha(X, Y),$$

where $X$ and $Y$ are local vector fields on $M$. Then $\alpha$ is the symmetric bilinear form with values in the normal bundle and it can be written as

$$\alpha=\varepsilon_0\sum h_{ij}\omega_i\otimes\omega_jE_0.$$

It follows from (1.2), (1.5) and (1.6) that the Gauss equation is given by

$$R_{ijkl}=\tilde{R}_{ijkl}+\varepsilon_0(h_{il}h_{jk}-h_{ik}h_{jl}). \tag{1.7}$$

The components of a Ricci tensor $S$ are given by

$$S_{ij}=\sum R_{kijk}=\sum \tilde{R}_{kijk}+\varepsilon_0 h h_{ij}-\varepsilon_0 h_{ij}^2, \tag{1.8}$$

where $h=\text{trace}\, h=\sum h_{kk}$ is $n$ times the mean curvature function $H$ of $M$ and $h_{ij}^2=\sum h_{ik}h_{kj}$.

Now, the components $h_{ijk}$ of the covariant derivative $\nabla\alpha$ of the second fundamental form $\alpha$ of $M$ are given by

$$\sum h_{ijk}\omega_k=dh_{ij}-\sum(h_{kj}\omega_{ki}+h_{ik}\omega_{kj}).$$

Then, by substituting $dh_{ij}$ in this definition into the exterior derivative of (1.6), we obtain the Codazzi equation

$$h_{ijk}-h_{ikj}=\tilde{R}_{0ijk}. \tag{1.9}$$

Similarly, the components $h_{ijkl}$ of the second covariant derivative $\nabla^2\alpha$ of $\alpha$ can be defined by

$$\sum h_{ijkl}\omega_l=dh_{ijk}-\sum(h_{ljk}\omega_{li}+h_{ilk}\omega_{lj}+h_{ijl}\omega_{lk}),$$

and the simple calculation gives rise to the Ricci formula

$$h_{ijkl}-h_{ijlk}=\sum(h_{mj}R_{milk}+h_{im}R_{mjlk}). \tag{1.10}$$

In particular, let the ambient space $\tilde{M}$ be a Lorentzian space from $M_1^{n+1}(c)$ of constant curvature $c$. In this case, the Riemannian curvature $\tilde{R}$ of $\tilde{M}$ is given by

$$\tilde{R}_{ABCD}=c\varepsilon_A\varepsilon_B(\delta_{AD}\delta_{BC}-\delta_{AC}\delta_{BD}).$$

Then the Gauss equation and the Codazzi equation are given by

$$R_{ijkl}=c(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})+\varepsilon_0(h_{il}h_{jk}-h_{ik}h_{jl}), \tag{1.11}$$

$$h_{ijk}=h_{ikj}. \tag{1.12}$$

The Ricci curvature is given by

$$S_{ij}=c(n-1)\delta_{ij}+\varepsilon_0 h h_{ij}-\varepsilon_0 h_{ij}^2. \tag{1.13}$$

By means of (1.9) and (1.10), the Laplacian $\Delta h_{ij}=\sum h_{ijkk}$ of the function $h_{ij}$ is given by

$$\Delta h_{ij}=(h)_{ij}+c(nh_{ij}-h\delta_{ij})-\varepsilon_0 h_2 h_{ij}+\varepsilon_0 h h_{ij}^2,$$

where $(h)_{ij}=\nabla_{E_j}\nabla_{E_i}h$ and $h_2$ is a function on $M$ defined by $h_2=|\alpha|^2=\sum h_{ij}^2=\sum h_{ij}h_{ij}$. Then the Laplacian $\Delta h_2$ of the function $h_2$ is given by

$$\Delta h_2=2\sum(h)_{ij}h_{ij}+2c(nh_2-h^2)-2\varepsilon_0(h_2)^2+2\varepsilon_0 hh_3+2|\nabla\alpha|^2,$$

where $h_3=\sum h_{ij}h_{jk}h_{ki}$ and $|\nabla\alpha|^2=\sum h_{ijk}h_{ijk}$.

Now, let the mean curvature $H$ of $M$ be constant. Then, since $(h)_{ij}=0$, the Laplacian of $h_2$ is given by

$$\Delta h_2=2c(nh_2-h^2)-2\varepsilon_0(h_2)^2+2\varepsilon_0 hh_3+2|\nabla\alpha|^2. \tag{1.14}$$

These formulas are obtained by Cheng and Yau [4].

## 2. Proof of Theorem 1.

Let $M$ be a space-like surface with constant mean curvature $H$ in a Lorentzian 3-space form $M_1^3(c)$, and let $\lambda$ and $\mu$ be the principal curvatures of $M$. We can choose a local field of Lorentzian orthonormal frames $\{E_0, E_1, E_2\}$ on $M_1^3(c)$ in such a way that, restricted to $M$, $\{E_1, E_2\}$ are tangent to $M$ and

$$h_{11}=\lambda, \qquad h_{12}=h_{21}=0, \qquad h_{22}=\mu. \tag{2.1}$$

In this case, the Gaussian curvature $G=R_{1221}$ of $M$ is given by

$$G=c-\lambda\mu, \tag{2.2}$$

and the constant mean curvature $H$ is represented as

$$H=\frac{h}{2}, \qquad h=\lambda+\mu. \tag{2.3}$$

The function $h_2=|\alpha|^2$ is given by

$$h_2=\lambda^2+\mu^2=2G+h^2-2c \qquad (\geqq 0). \tag{2.4}$$

It follows from (1.14) that the Laplacian of $h_2$ is calculated as

$$\Delta h_2=2G(\lambda-\mu)^2+2|\nabla\alpha|^2\geqq 2G(\lambda-\mu)^2. \tag{2.5}$$

In this section, we prove Theorem 1 which gives the estimate of the function $h_2$ on a complete space-like surface with constant mean curvature in $M_1^3(c)$. For this purpose, the following generalized maximum principle due to Omori [12] and Yau [18] is needed for the estimate of the Laplacian of the function of class $C^2$.

THEOREM (Omori and Yau). *Let $N$ be a complete Riemannian manifold whose Ricci curvature is bounded from below and let $F$ be a function of class $C^2$ on $N$. If $F$ is bounded from below, then for any $\varepsilon>0$ there exists a point $q$ such that*

$$|\nabla F(q)|<\varepsilon, \qquad \Delta F(q)>-\varepsilon, \qquad F(q)<\inf F+\varepsilon. \tag{2.6}$$

In fact, since $M$ is a complete space-like surface with constant mean curvature $H$, it follows from (1.13) that the Ricci curvature tensor $S_{ij}=\sum_k R_{kijk}$ is given by

$$(2.7)\qquad \begin{aligned} &S_{11}=S_{22}=G=c-\lambda\mu=c-\lambda(h-\lambda)=c-h\lambda+\lambda^2=c+(\lambda-H)^2-H^2\geqq c-H^2\,,\\ &S_{12}=S_{21}=0\,. \end{aligned}$$

implying that the Ricci curvature is bounded from below by constant $c-H^2$. Accordingly, we can apply this theorem to prove Theorem 1.

PROOF OF THEOREM 1. Given any positive number $a$, we define a smooth function $F$ on $M$ by $(h_2+a)^{-1/2}$, which is positive and is also bounded from above by positive constant $a^{-1/2}$. So we can apply the generalized maximum principle due to Omori and Yau to $F$.

First, we compute the gradient and the Laplacian of $F$:

$$\nabla F=-\frac{1}{2}(h_2+a)^{-3/2}\nabla h_2=-\frac{1}{2}F^3\nabla h_2\,,$$

$$\Delta F=-\frac{3}{2}F^2\nabla F\nabla h_2-\frac{1}{2}F^3\Delta h_2=3F^{-1}|\nabla F|^2-\frac{1}{2}F^3\Delta h_2\,.$$

Consequently, the following inequality

$$(2.8)\qquad F^4G(\lambda-\mu)^2\leqq 3|\nabla F|^2-F\Delta F$$

is obtained by (2.5).

For a convergent sequence $\{\varepsilon_m\}$ such that $\varepsilon_m>0$ and $\varepsilon_m\to 0$ $(m\to\infty)$, by the theorem due to Omori and Yau, there is a point sequence $\{q_m\}$ such that $F$ satisfies (2.6) at each $q_m$ for $\varepsilon_m$:

$$(2.6')\qquad |\nabla F(q_m)|<\varepsilon_m,\qquad \Delta F(q_m)>-\varepsilon_m\,,\qquad \inf F\leqq F(q_m)<\inf F+\varepsilon_m\,.$$

Then the sequence $\{F(q_m)\}$ converges to $\inf F$, which implies by the definition of $F$ that $h_2(q_m)\to\sup h_2 (m\to\infty)$. We shall prove that $h_2$ is bounded.

Suppose $\sup h_2=+\infty$. Since $h_2=\lambda^2+\mu^2=2(\lambda-H)^2+2H^2$, the sequence $\{\lambda(q_m)\}$ then diverges to positive infinity, by taking a subsequence if necessary. Moreover, we have

$$\frac{\mu(q_m)}{\lambda(q_m)}\longrightarrow -1\qquad (m\to\infty),$$

for $\mu/\lambda+1=(\mu+\lambda)/\lambda=h/\lambda$. On the other hand, from the inequality (2.8), we get the relation

$$F(q_m)^4G(q_m)\{\lambda(q_m)-\mu(q_m)\}^2<3\varepsilon_m^2+\varepsilon_m F(q_m)\,, \tag{2.9}$$

in which the right hand side converges to 0, because the function $F$ is bounded. Hence the left hand side of (2.9) converges to a non-positive number. But, since the left hand side is

$$\frac{\{c-\lambda(q_m)\mu(q_m)\}\{\lambda(q_m)-\mu(q_m)\}^2}{\{\lambda(q_m)^2+\mu(q_m)^2+a\}^2}=\frac{\left\{\frac{c}{\lambda(q_m)^2}-\frac{\mu(q_m)}{\lambda(q_m)}\right\}\left\{1-\frac{\mu(q_m)}{\lambda(q_m)}\right\}^2}{\left\{1+\frac{\mu(q_m)^2}{\lambda(q_m)^2}+\frac{a^2}{\lambda(q_m)^2}\right\}},$$

it should converge to 1 as $m\to\infty$. This is a contradiction. Accordingly, $h_2$ is bounded.

This implies that the sequence $\{G(q_m)\}$ converges to $sup\,G$ which is bounded. So we have

$$\{\lambda(q_m)-\mu(q_m)\}^2 \longrightarrow sup\,(\lambda-\mu)^2<\infty \qquad (m\to\infty)\,,$$

since $(\lambda-\mu)^2=(\lambda+\mu)^2-4\lambda\mu=h^2+4G-4c$. Then if follows from (2.9) that we have

$$sup\,G\cdot sup\,(\lambda-\mu)^2\leqq 0\,. \tag{2.10}$$

Hence, if $sup\,(\lambda-\mu)^2$ is positive, then $G$ is non-positive. On the other hand, when $sup\,(\lambda-\mu)^2=0$, $\lambda-\mu$ is identically zero. In consequence, under the assumption of Theorem 1, $M$ is either totally umbilical or $G\leqq 0$. Note that, when $c$ is non-positive, if $M$ is totally umbilical then $G$ is non-positive, for $G=c-\lambda\mu=c-\lambda^2$. Finally, it follows from (2.4) that the condition $G\leqq 0$ is equivalent to $h_2=|\alpha|^2\leqq 4H^2-2c$. q.e.d.

REMARK 1. Cheng and Nakagawa [3] extend the Cheng-Yau result and give an estimate of $|\alpha|$ for a complete space-like hypersurface with constant mean curvature in $M_1^{n+1}(c)$, $c\leqq 0$. In the case $c\leqq 0$, Theorem 1 is equivalent to of their result, but the method of proof is different from theirs.

REMARK 2. In the case $c>0$, a totally umbilical surface $S^2(c_2)$ in $S_1^3(c)$ has positive Gaussian curvature $c_2$, and the other surfaces in $S_1^3(c)$ have non-positive curvature. On the other hand, Akutagawa [2] gave the condition for a complete space-like hypersurface $M^n$ in $S_1^{n+1}(c)$ to be totally umbilical. In the case $n=2$, Akuatgawa's theorem can also be proved by Theorem 1:

COROLLARY (Akutagawa). *Let $M$ be a complete space-like surface with constant mean curvature $H$ in $S_1^3(c)$. Suppose $c \geqq H^2$, then $M$ is totally umbilical.*

PROOF. Since the Gaussian curvature $G$ is given by $G=c-\lambda\mu=(\lambda-H)^2+c-H^2$, $G$ is non-negative by the assumption $c \geqq H^2$. Then, it follows from Theorem 1 that $G$ is positive constant or identically zero. Hence, $M$ is a totally umbilical surface in $S_1^3(c)$. q.e.d.

## 3. Proof of Theorem 2.

In this section, we prove Theorem 2 which characterizes a hyperbolic cylinder in a Lorentzian space form $M_1^3(c)$.

First, it is to be remarked that hyperbolic cylinders are the only flat space-like surfaces with non-zero constant mean curvature in $M_1^3(c)$. This fact is proved by the use of a theorem due to Abe, Koike and Yamaguchi [1]. Hence we have only to prove that the Gaussian curvature of a "uniformly" non-umbilical space-like surface with constant mean curvature in $M_1^3(c)$ is identically zero. On the other hand, Theorem 1 asserts that if a space-like surface with constant mean curvature in $M_1^3(c)$ is not totally umbilical, then the Gaussian curvature is non-positive. Accordingly, Theorem 2 will follow immediately from the following lemma.

LEMMA. *Let $M$ be a complete space-like surface with constant mean curvature $H$ in $M_1^3(c)$. If the principal curvatures $\lambda$ and $\mu$ of $M$ satisfy*

$$inf\,(\lambda-\mu)^2>0\,,$$

*then the Gaussian curvature $G$ of $M$ is non-negative.*

In order to prove this lemma, the generalized maximum principle due to Omori [12] and Yau [18] is used here again. So, we are going to compute the Laplacian of the Gaussian curvature $G$ of $M$.

Now, since the mean curvature $H=h/2$ and $c$ are constant, the relation $\Delta h_2=2\Delta G$ is obtained from (2.4). Then it follows from (2.5) that the Laplacian $\Delta G$ is given by

$$\Delta G=G(\lambda-\mu)^2+|\nabla\alpha|^2\,, \tag{3.1}$$

where $|\nabla\alpha|^2=(h_{111})^2+3(h_{221})^2+(h_{222})^2+3(h_{112})^2$. Since the principal curvatures $\lambda$ and $\mu$ are mutually different everywhere by the assumption $inf\,(\lambda-\mu)^2>0$, it is known that they are both smooth functions on $M$ (see Szabó [15], for example).

Recalling the definition of the components $h_{ijk}$, the derivatives of $\lambda$ and $\mu$ are given by

$$d\lambda=dh_{11}=h_{111}\omega_1+h_{112}\omega_2\,,$$

$$d\mu=dh_{22}=h_{221}\omega_1+h_{222}\omega_2\,.$$

Since $h$ is constant, the derivative $dh=d\lambda+d\mu$ is identically zero, and hence the following relations are obtained;

$$h_{111}+h_{221}=0 \quad \text{and} \quad h_{112}+h_{222}=0\,.$$

Also, from (2.2), the derivative of $G$ is given by

$$\nabla G=dG=-(d\lambda)\mu-\lambda(d\mu)=(\lambda-\mu)d\lambda\,.$$

Hence we have

$$|\nabla\alpha|^2=4\{(h_{111})^2+(h_{112})^2\}=4|d\lambda|^2=\frac{4}{(\lambda-\mu)^2}|\nabla G|^2\,,$$

which combined with (3.1), implies that the Laplacian $\Delta G$ is given by

$$\Delta G=G(\lambda-\mu)^2+\frac{4}{(\lambda-\mu)^2}|\nabla G|^2\,. \tag{3.2}$$

PROOF OF LEMMA. It follows from (2.7) that we can apply the generalized maximum principle due to Omori and Yau to a smooth function $F$ bounded from below. Here, we define $F$ to be $exp[aG]$ for any given positive number $a$. Note that $F$ is a smooth function bounded from below by a positive constant $F_0=exp[a(c-H^2)]$, because of (2.7).

The gradient and the Laplacian of $F$ are then given by

$$\nabla F=a\ exp[aG]\nabla G=aF\nabla G\,,$$

$$\Delta F=a\nabla F\nabla G+aF\Delta G=a^2F|\nabla G|^2+aF\Delta G\,.$$

Further, it follows from (3.2) that the Laplacian $\Delta F$ is given by

$$\Delta F=aFG(\lambda-\mu)^2+\left\{2a^2-\left(a^2-\frac{4a}{(\lambda-\mu)^2}\right)\right\}F|\nabla G|^2\,. \tag{3.3}$$

We put $k=inf(\lambda-\mu)^2$, which is positive by the assumption of the lemma. Let $a$ be greater than $4/k$. Then

$$a^2-\frac{4a}{(\lambda-\mu)^2}\geqq a^2-\frac{4a}{k}=a\left(a-\frac{4}{k}\right)>0\,.$$

Accordingly, from (3.3), the Laplacian $\Delta F$ is evaluated by

$$\Delta F\leqq aFG(\lambda-\mu)^2+2a^2F|\nabla G|^2\,,$$

which implies, since $\nabla F=aF\nabla G$ and $F>0$, the following inequality;

$$aF^2G(\lambda-\mu)^2\geqq F\Delta F-2|\nabla F|^2 . \tag{3.4}$$

For a convergent sequence $\{\varepsilon_m\}$ such that $\varepsilon_m>0$ and $\varepsilon_m\to 0$ $(m\to\infty)$, the theorem due to Omori and Yau implies that there is a point sequence $\{q_m\}$ such that $F$ satisfies (2.6′). Then the sequence $\{F(q_m)\}$ converges to $\mathit{inf}\, F$, which satisfies $\mathit{inf}\, F\geqq F_0>0$. So the definition of $F$ implies that $G(q_m)\to \mathit{inf}\, G$ $(m\to\infty)$, where $\mathit{inf}\, G$ is bounded.

Moreover, by taking subsequences if necessary, $\lambda(q_m)$ and $\mu(q_m)$ tends to some numbers;

$$\lambda(q_m)\longrightarrow\lambda_1, \qquad \mu(q_m)\longrightarrow\mu_1=h-\lambda_1 \quad (m\to\infty) .$$

This is proved in the following way. Suppose $\{\lambda(q_m)\}$ is not bounded. Then we can regard $\{\lambda(q_m)\}$ and $\{\mu(q_m)\}$ as sequences which diverge to positive infinity and negative infinity, respectively. It follows from (2.2), that $G(q_m)$ must diverge to positive infinity. This contradicts the fact that $G(q_m)$ converges to its infimum. Thus $\{\lambda(q_m)\}$ is bounded and hence it containes a subsequence converging to some finite number.

On the other hand, from the inequality (3.4), the following relation is obtained;

$$aF(q_m)^2G(q_m)\{\lambda(q_m)-\mu(q_m)\}^2>-\varepsilon_m\{F(q_m)+2\varepsilon_m\},$$

in which the right hand side converges to 0 as $m$ tends to $\infty$, since the function $F$ is bounded. Accordingly, we get

$$a(\mathit{inf}\, F)^2(\mathit{inf}\, G)(\lambda_1-\mu_1)^2\geqq 0 . \tag{3.5}$$

Since $a>0$, $\mathit{inf}\, F>0$ and $(\lambda_1-\mu_1)^2\geqq k>0$, the inequality (3.5) now implies that $\mathit{inf}\, G$ is non-negative. Hence the Gaussian curvature $G$ is non-negative everywhere. q.e.d.

As mentioned above, Theorem 2 is proved by this lemma and Theorem 1 immediately.

REMARK 1. Recently, various kinds of surfaces of revolution with constant mean curvature in Minkowski space $R_1^3$ are constructed by Hano and Nomizu [7] and Ishihara and Hara [9], which shows that the condition $\mathit{inf}\,(\lambda-\mu)^2>0$ in this theorem cannot be omitted in the case $c=0$.

REMARK 2. The examples given by Akutagawa [2], each of which is a space-like rotation surface in $S_1^3(c)$, are complete space-like surfaces with constant mean curvature and negative Gaussian curvature. They are not totally umbilical and satisfy $inf(\lambda-\mu)^2=0$. This shows that there are many surfaces with constant mean curvature in $S_1^3(c)$ such that $G\leqq 0$ which are different from the hyperbolic cylinders.

Finally, it is to be noted that the fact that all the above examples of complete space-like surfaces in $R_1^3$ and $S_1^3(c)$ have negative Gaussian curvature leads us to the following conjecture: Let $M$ be a complete space-like surface with constant mean curvature in $M_1^3(c)$. If there is a point $p$ in $M$ at which the Gaussian curvature is zero, the Gaussian curvature is identically zero on $M$.

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**Added in proof.**

1. There are also many complete space-like surfaces with constant mean curvature in $H_1^3(c)$, which are not totally umbilical and satisty $inf(\lambda-\mu)^2=0$. These examples are constructed by the method similar to Akutagawa's one.

2. Recently, Ki, Kim and Nakagawa [19] gave an estimate of $|\alpha|$ for a complete space-like hypersurface with constant mean curvature in $M_1^{n+1}(c)$ for any $c$ and $n\geqq 2$. If $n=2$, their result are equivalent to Theorem 1.

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