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APPROXIMATING THE HOMOTOPY SEQUENCE OF A PAIR OF SPACES

By

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0. Introduction.

In his book [7] H. Toda computed the homotopy groups $\pi_{n+k}(S^n)$ for $k\leqq 19$. Although he permitted himself to use any methods and insights that were available to him at the time, his principal technique was to exploit his "composition method". Very briefly the composition method is an inductive procedure by which composition classes vanishing in a particular "stem" k give rise to secondary and higher order composition classes (Toda brackets) in stem $k+1$ that are "detected by" (i. e. shown to have a non-zero image under) a Hopf invariant homomorphism, after which their orders and relations to other elements are determined. This paper is intended as a contribution toward an analysis of the composition method and its development into a more generally applicable technique for computing homotopy groups.

Let $i: A\rightarrow X$ be an inclusion map and let us suppose that the homotopy groups of X are less well known than those of A. (In Toda's composition method $i: S^n\rightarrow \Omega S^{n+1}$ is the suspension inclusion). Then we have available the relative homotopy sequence

$$(0.1) \qquad \cdots \longrightarrow \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, A) \longrightarrow \pi_{n-1}(A) \longrightarrow \cdots.$$

In general $\pi_n(X, A)$ will not be known but we may be able to approximate it via a map $h: X\rightarrow B$ with $h(A)=*$ (the base point) and where B is a space whose homotopy groups are (better) known. Then we can regard

$$(0.2) \qquad \cdots \longrightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{H} \pi_n(B) \xrightarrow{\Delta} \pi_{n-1}(A) \longrightarrow \cdots$$

(where $H=h_*$) as an approximation to 0.1. Of course if $h_*: \pi_n(X, A)\rightarrow \pi_n(B)$ is not an isomorphism then 0.2 may not be exact. Indeed the operator Δ may not be well-defined. However a partial function (defined on the kernel of i_*) $\Delta^{\leftarrow}: \pi_{n-1}(A)\rightarrow \pi_n(B)$ can be defined (with a degree of indeterminacy) using Toda brackets. If $\mu\in\pi_{n-1}(A)$ is such that $i_*\mu=0$, let

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$$\Delta^{\leftarrow}(\mu) = -\{{}^{0}\{h\}, \{i\}, \mu\} \subseteq \pi_n(B), \tag{0.3}$$

where the little circle decorating the Toda bracket indicates that a "preferred nullhomotopy" of hi (in this case the trivial nullhomotopy) is called for. In consequence, the indeterminacy of $\Delta^{\leftarrow}$ exactly coincides with the image of H. (For conceptual detail see § 2.)

Suppose now that we have classes $\alpha' \in \pi(K, A)$, $\beta' \in \pi_k(K)$, $\gamma' \in \pi_{n-1}(S^k)$ with $i_*(\alpha' \circ \beta') = 0$ and $\beta' \circ \gamma' = 0$. Then the Toda bracket set $\{i_*\alpha', \beta', \gamma'\} \subseteq \pi_n(X)$ is defined. We shall see (Proposition 3.4) that

$$H\{i_*\alpha', \beta', \gamma'\} = \Delta^{\leftarrow}(\alpha' \circ \beta') \circ E\gamma', \tag{0.4}$$

where E denotes the suspension homomorphism. The reader will recognise 0.4 as a primitive form of [7; Proposition 2.6], a cornerstone of Toda's composition method.

The equality 0.4 indicates that H has the power to detect new classes, but this ability will have little practical value unless also it is possible to recognize $\Delta^{\leftarrow}(\alpha' \circ \beta')$ as a specific class (coset) in $\pi_{k+1}(B)$. This can be done if the inclusion $i: A \to X$ factors through a principal cofibration so that we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i'"] & A \cup_\lambda CM \arrow[r, "i''"] \arrow[d, "q"] & X \arrow[d, "h"] \\ & \Sigma M \arrow[r, "j"] & B, \end{tikzcd}$$

where $\lambda \in \pi(M, A)$ and q shrinks A.

0.5. PROPOSITION. *If* $\alpha' \circ \beta' = \lambda \circ \delta$, *where* $\alpha' \in \pi(K, A)$, $\beta' \in \pi_k(K)$, $\delta \in \pi_k(M)$ *then* $\Delta^{\leftarrow}(\alpha' \circ \beta') \ni j_* E\delta$.

Although the sequence 0.2 need not be exact at $\pi_n(X)$ it gives rise to a number of short differential sequences:

$$\mathrm{Im}(i_*) \longrightarrow \pi_n(X) \longrightarrow \mathrm{Im}(H).$$

The following result supplies information concerning the (group) extension associated with an element detected by H.

0.6. PROPOSITION. *If* $i: A \to X$ *factors through* $A \cup_\lambda CM$ *and if* $\gamma \in \pi_n(M)$ *is such that* $\lambda \circ \gamma = 0$, *then there exists an element* $\xi \in \pi_{n+1}(X)$ *such that* $H\xi = j_* E\gamma$. *Moreover if* $m\gamma = 0$ *then* $m\xi \in -i_*\{\{\lambda\}, \{\gamma\}, m\iota_n\}$.

The paradigmatic application of the theory outlined above is to the suspension inclusion $S^n \rightarrow \Omega S^{n+1}$ or, equivalently, to the inclusion $S^n \rightarrow S^n_\infty$ into the James reduced product space, with $B = S^{2n}_\infty$ and $h: S^n_\infty \rightarrow S^{2n}_\infty$ the James map [5]. Here the Whitehead square $[\iota_n, \iota_n]$ plays the role of the class λ. However Proposition 0.6 specialises to a result that appears to be new. The need for such a formula in Toda's book was avoided by relying on the calculus of Toda brackets derived in the course of the computation.

The theory also gives rise to useful formulae in the case of the inclusion $S^n_t \rightarrow S^n_\infty (t>1,\ n$ even). (S^n_t is the tn-skeleton of S^n_∞). In particular, for $t=p-1$ with p an odd prime, a partial technique for computing unstable p-components of the homotopy groups of spheres is obtained. The technique is only partial because of the different behaviour of the odd and even dimensional spheres. It needs to be complemented by a study of the inclusion $S^{n-1} \rightarrow \Omega S^n_{p-1} (n$ even). The present paper considers only a selection of applications of the theory relevant to the homotopy groups of spheres. Other possible applications, e. g. to the study of unstable phenomena associated with the suspension of spaces other than spheres may receive attention in due course.

The convenience of the category of homotopy pairs when dealing with (ternary) Toda brackets has been demonstrated [4]. In §1 this approach is further exploited to derive a number of standard properties of the brackets making the treatment self-contained.

1. Some Toda bracket theory.

In this section we utilise homotopy pair theory to define the secondary homotopy composition operation and to derive certain properties needed in the sequel. Recall that the objects of the *category of homotopy pairs* [3], [4] are (pointed) continuous maps and the morphisms from f to g are equivalence classes of diagrams

(1.1)
$$\begin{tikzcd} X \arrow[r, "\psi_0"] \arrow[d, "f"'] \arrow[dr, phantom, "h_t"] & E \arrow[d, "g"] \\ Y \arrow[r, "\phi_0"'] & B \end{tikzcd}$$

where h_t is a homotopy from $\phi_0 f$ to $g\psi_0$. Specifically the square 1.1 is $\sim$-related to the composite (i. e. outer) square

(1.2)
$$\begin{tikzcd}
X \arrow[r, "\psi_1"] \arrow[d, equal] \arrow[dr, phantom, "\nearrow \psi_t"] & E \arrow[d, equal] \\
X \arrow[r, "\psi_0"] \arrow[d, "f"'] \arrow[dr, phantom, "\nearrow h_{t-1}"] & E \arrow[d, "g"] \\
Y \arrow[r, "\phi_0"] \arrow[d, equal] \arrow[dr, phantom, "\nearrow \phi_{1-t}"] & B \arrow[d, equal] \\
Y \arrow[r, "\phi_1"'] & B
\end{tikzcd}$$

where ψ_t and ϕ_t are homotopies, and also to the square

$$\begin{tikzcd}
X \arrow[r, "\psi_0"] \arrow[d, "f"'] \arrow[dr, phantom, "\nearrow h'_t"] & E \arrow[d, "g"] \\
Y \arrow[r, "\phi_0"] & B
\end{tikzcd}$$

if h_t and h'_t belong to the same *track* (i. e. relative homotopy class of homotopies).

Now let $f: X \to Y$, $h: Y \to E$, $g: E \to B$ be maps such that $hf \simeq *$ and $gh \simeq *$ and let $m_t: X \to E$, $n_t: Y \to B$ be nullhomotopies of hf, gh respectively. Then the composite square

(1.3)
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "X*"'] \arrow[dr, phantom, "\nearrow m_{1-t}"] & Y \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "\nearrow n_t"] & * \arrow[d, "*B"] \\
* \arrow[r] & E \arrow[r, "g"] & B
\end{tikzcd}$$

defines an elements ξ of the homotopy pair set $\pi(X*, *B)$. As discussed in [4], the operator

$$cP: \pi(X*, *B) \longrightarrow \pi(\Sigma X, B),$$

applied to the composite square 1.3 selects the homotopy class of the map induced by 1.3 from the cofibre of $X*$ to the cofibre of $*B$. The *Toda bracket set*

(1.4) $$\{\{g\}, \{h\}, \{f\}\} \subseteq \pi(\Sigma X, B)$$

is defined to be the set $\{cP(\xi) \mid \text{nullhomotopies } m_t, n_t\}$, which turns out to be a double coset of the subgroups $\pi(\Sigma Y, B) \circ \{\Sigma f\}$ and $\{g\} \circ \pi(\Sigma X, E)$ in $\pi(\Sigma X, B)$.

Toda [7] verifies that the coset 1.4 is independent of the choice of f, h, g within their homotopy classes. We can note that these facts are implicit in the definition above. For example, suppose f_t is a homotopy from f to f' then the homotopy pair relation

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "f_t"] & Y \arrow[r] \arrow[d, equal] & * \arrow[d, equal] \\ X \arrow[r] \arrow[d] \arrow[dr, phantom, "m_{1-t}"] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "n_t"] & * \arrow[d] \\ * \arrow[r] & E \arrow[r, "g"'] & B \end{tikzcd} \sim \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "m_{1-t}"] & Y \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "n_t"] & * \arrow[d] \\ * \arrow[r] & E \arrow[r] & B \end{tikzcd}$$

indicates that the relevant brackets have a common element. Other standard properties of the brackets can be derived in a rather simple way. We give two examples. Let $k: W \to X$ be a map. Then the following diagram

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d] & X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "m_{1-t}"] & Y \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "n_t"] & * \arrow[d] \\ * \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & E \arrow[r, "g"'] & B \arrow[d] \\ \Sigma W \arrow[r, "\Sigma k"'] & \Sigma X \arrow[rr] & & B \end{tikzcd}$$

in which the bottom line contains induced maps between respective cofibres yields the inclusion

(1.5) $$\{\{g\}, \{h\}, \{f\}\} \circ \{\Sigma k\} \subseteq \{\{g\}, \{h\}, \{f\} \circ \{k\}\}$$

(cf. [7], Proposition 1.2 (i)). Suppose next that $fk \simeq 0$ and let $p_t: W \to Y$ be a nullhomotopy of fk. Then the following composite squares clearly represent elements of opposite sign in $\pi(W*, *B)$.

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "p_{1-t}"] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "m_t"] & * \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[d, equal] & Y \arrow[r, "h"] \arrow[d] \arrow[dr, phantom, "n_{1-t}"] & E \arrow[r, "g"] \arrow[d, "g"] & B \arrow[d, equal] \\ * \arrow[r] & * \arrow[r] & B \arrow[r, equal] & B \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, equal] \arrow[d, equal] & W \arrow[r] \arrow[d] \arrow[dr, phantom, "p_t"] & * \arrow[r] \arrow[d] & * \arrow[d, equal] \\ W \arrow[r, "k"] \arrow[d] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "m_{1-t}"] & Y \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "n_t"] & * \arrow[d] \\ * \arrow[r] & * \arrow[r] & E \arrow[r] & B \end{tikzcd}$$

If we bear in mind the $\sim$ relation this yields the identity

$$\{\{g\},\{h\},\{f\}\}\circ\{\Sigma k\}=-\{g\}\circ\{\{h\},\{f\},\{k\}\} \tag{1.6}$$

(cf. [7], Proposition 1.4).

We now give two further results needed in the proof of Proposition 0.6. Suppose that the maps f, h and g are only known to satisfy

$$\{\Sigma g\}\circ\{\Sigma h\}=0,\ \{\Sigma h\}\circ\{\Sigma f\}=0. \tag{1.7}$$

and let $i_E: E\to\Omega\Sigma E$ denote the unit of the loop suspension adjunction. Then equivalent to 1.7 we have

$$\{\Omega\Sigma g\}\circ\{i_E h\}=0,\ \{i_E h\}\circ\{f\}=0.$$

Let $\theta: \pi(\Sigma^2 X, \Sigma B)\to\pi(\Sigma X, \Omega\Sigma B)$ denote the adjunction isomorphism.

1.8. Proposition. $\theta\{\{\Sigma g\},\{\Sigma h\},\{\Sigma f\}\}=-\{\{\Omega\Sigma g\},\{i_E h\},\{f\}\}$.

1.9. Remark. Proposition 1.8 is consistent with and can be regarded as a variant of Toda's Proposition 1.3 [7].

Proof of 1.8. It can be checked that the indeterminacies of the brackets are equivalent under θ hence it remains to show that the subsets have an element in common. An element of the left hand side is given by the composite class on the top row of the following diagram

$$\begin{tikzcd}
\Sigma X \arrow[r] \arrow[d, equal] & F_{\Sigma h} \arrow[r] \arrow[d] & \Omega\Sigma B \arrow[d] \\
\Sigma X \arrow[r, "\Sigma f"] \arrow[d] \arrow[dr, phantom, "m_{1-t}"] & \Sigma Y \arrow[r] \arrow[d, "\Sigma h"'] \arrow[dr, phantom, "n_t"] & * \arrow[d] \\
* \arrow[r] & \Sigma E \arrow[r, "\Sigma g"] & \Sigma B
\end{tikzcd}$$

where m_t and n_t denote nullhomotopies and the top row contains the induced maps between the respective homotopy fibres. An element of the right hand bracket is given by the bottom row of the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "i_E h"] \arrow[dl, phantom, "m'_{1-t}"] & * \arrow[d] \arrow[dl, phantom, "n'_t"] \\
* \arrow[r, "\Sigma f"] \arrow[d] & \Omega\Sigma E \arrow[r, "\Omega\Sigma g"] \arrow[d] & \Omega\Sigma B \arrow[d] \\
\Sigma X \arrow[r] & C_{i_E h} \arrow[r] & \Omega\Sigma B,
\end{tikzcd}$$

the bottom row consisting of the induced maps between the respective mapping cones. Comparing the homotopies $(\Omega\Sigma g)m'_{1-t}+n'_t f: X\to\Omega\Sigma B$ and $(\Sigma g)m_{1-t}+n_t(\Sigma f): \Sigma X\to\Sigma B$, note that these coincide (under adjoint correspondence) if we choose $m'_t x(s)=m_t(x, s)$ and $n'_t y(s)=n_t(y, s)(x\in X, y\in Y, s\in I, t\in I)$. The sign in 1.8 arises from the choice of orientation involved in the standard representation of ΣX as the mapping cone of $X\to *$.

1.9.1. REMARK. The equality in Proposition 1.8 remains valid if Σg is replaced by a non-suspension.

If, instead of 1.7, the maps f, h and g satisfy

(1.10) $$\{\Sigma g\}\circ\{\Sigma h\}=0, \qquad \{h\}\circ\{f\}=0$$

then we have:

1.1.1. COROLLARY. $\theta\{\{\Sigma g\}, \{\Sigma h\}, \{\Sigma f\}\}\supseteqq-\{\{i_B g\}, \{h\}, \{f\}\}$

PROOF. By Proposition 1.8,

$$\theta\{\{\Sigma g\}, \{\Sigma h\}, \{\Sigma f\}\}=-\{\{\Omega\Sigma g\}, \{i_E h\}, \{f\}\}\supseteqq-\{\{\Omega\Sigma g\}\circ\{i_E\}, \{h\}, \{f\}\}$$

(by [7; Proposition 1.2 (iii)])$=-\{\{i_B g\}, \{h\}, \{f\}\}$.

1.12. REMARK. The coset $-\theta^{-1}\{\{i_B g\}, \{h\}, \{f\}\}$ coincides with $\{\{\Sigma g\}, \{\Sigma h\}, \{\Sigma f\}\}_1$ in the sense of Toda's notation [7; page 9]. Moreover if f, h, g satisfy $\{g\}\circ\{h\}=0$ and $\{h\}\circ\{f\}=0$ we can recover from 1.11 Toda's inclusion [7; Proposition 1.3]:

(1.13) $$-\Sigma\{\{g\}, \{h\}, \{f\}\}\subseteqq\{\{\Sigma g\}, \{\Sigma h\}, \{\Sigma f\}\}.$$

2. Preferred nullhomotopies.

Let $h, h': Y\to E$ and $g, g': E\to B$ be maps and let n_t(respectively n'_t) be a a nullhomotopy of gh (respectively of $g'h'$). Then the squares

(2.1)
$$
\begin{tikzcd}
Y \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "\nearrow n_t"] & * \arrow[d] \\
E \arrow[r, "g"'] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Y \arrow[r] \arrow[d, "h'"'] \arrow[dr, phantom, "\nearrow n'_t"] & * \arrow[d] \\
E \arrow[r, "g'"'] & B
\end{tikzcd}
$$

are *coherent* there exists a homotopy h_t from h to h' such that

$$
\begin{tikzcd}
Y \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\nearrow h_t"] & Y \arrow[r] \arrow[d, "h'"'] \arrow[dr, phantom, "\nearrow n'_t"] & * \arrow[d] \\
E \arrow[r, equal] & E \arrow[r, "g'"'] & B
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "\nearrow n_t"] & * \arrow[d] \\
E \arrow[r] & B
\end{tikzcd}
$$

in $\pi(h, *B)$. (If the condition is satisfied, note that g and g' are necessarily homotopic.) If the squares are coherent then we also say (with some abuse) that the nullhomotopies n_t and n'_t are coherent. Systematic use of coherent nullhomotopies enables some sharpening of the secondary composition operation. For example, let C denote the coherence class of the nullhomotopy n'_t. Then, given also $f: X \to Y$ with $hf \cong *$, we define

(2.2) $$\{^C\{g\}, \{h\}, \{f\}\} = \{cP(\xi)|m_t : hf \simeq *, n_t \in C\},$$

where ξ is the element referred to in 1.3. Note that the indeterminacy of the bracket 2.2 is the subgroup $\{g\} \circ \pi(\Sigma X, E)$. When the coherence class in question is clearly understood it is convenient to use the nonspecific notation $\{^0\{g\}, \{h\}, \{f\}\}$. For example (as occurred in the introduction) if for particular representative maps g and h it is known that $gh = *$ then the coherence class of the trivial homotopy may be understood. Another situation giving rise to a preferred homotopy is the composition

$$X \xrightarrow{f} Y \xrightarrow{Pf} Y \cup_f CX$$

of a map with the inclusion of codomain into mapping cone. A standard nullhomotopy of $(Pf)f$ is available which defines a coherence class. Note that a bracket $\{\{g\}, \{h\}, \{f\}^0\}$ is defined whenever a preferred nullhomotopy of hf is understood and that it's indeterminacy is the subgroup $\pi(\Sigma Y, B) \circ \{\Sigma f\}$. Versions of the standard inclusion properties of Toda brackets [7; Proposition 1.2] can be derived for brackets involving preferred nullhomotopies. For example, corresponding to 1.6, and respectively to [7; Proposition 1.2 (ii)] the following can be proved.

(2.3) $$\{\{g\},\{h\},\{f\}^0\}\circ\{\Sigma k\}=-\{g\}\circ\{^0\{h\},\{f\},\{k\}\}.$$

(2.4) $$\{^0\{g\},\{h\}\circ\{f\},\{k\}\}=\{^0\{g\},\{h\},\{f\}\circ\{k\}\}.$$

3. The operator $\Delta^{\leftarrow}$.

Recall that an element $\eta\in\pi_n(X, A)$ is a pair-homotopy equivalence class of a commutative diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d] & A \arrow[d, "i"] \\ V^n \arrow[r] & X, \end{tikzcd}$$

where V^n is the n-dimensional ball bounded by S^{n-1}.

Equivalently we may regard η as the homotopy pair class of a homotopy commutative diagram

(3.1)
$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "m_t"] & A \arrow[d, "i"] \\ * \arrow[r] & X. \end{tikzcd}$$

The boundary operator $\partial: \pi_n(X, A)\to\pi_{n-1}(A)$ has the property that $\partial\eta=\pm\{f\}$, the sign depending on orientation conventions. The homomorphism $h_*: \pi_n(X, A)\to\pi_n(B)$ determined by a map $h:(X, A)\to(B, *)$ is that which associates with η the composite square

(3.2)
$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "m_t"] & A \arrow[r] \arrow[d, "i"] \arrow[dr, phantom, "*"] & * \arrow[d] \\ * \arrow[r] & X \arrow[r, "h"] & B \end{tikzcd}$$

regarded as an element of $\pi_n(B)$ via the bijection $cP: \pi(S^{n-1}*, *B)\to\pi_n(B)$. (Note that the right hand square in 3.2 commutes via the trivial homotopy). An immediate consequence of the definition of the operator $\Delta^{\leftarrow}: \pi_{n-1}(A)\to\pi_n(B)$ is the following result.

3.3. PROPOSITION. $\Delta^{\leftarrow}\partial\eta=\pm h_*\eta+H\pi_n(X)\in\pi_n(B)/H\pi_n(X)$.

Proposition 3.3 may be regarded as providing some justification for the notation $\Delta^{\leftarrow}$. We now also state:

3.4. PROPOSITION. *If* $\alpha' \in \pi(K, A)$, $\beta' \in \pi_k(K)$, $\gamma' \in \pi_{n-1}(S^k)$ *are such that* $i_*(\alpha' \circ \beta') = 0$ *and* $\beta' \circ \gamma' = 0$ *then* $H\{i_*\alpha', \beta', \gamma'\} = \Delta^{\leftarrow}(\alpha' \circ \beta') \circ E\gamma'$.

PROOF. By [7; Proposition 1.2 (iii)] we have $\{\{i\} \circ \alpha', \beta', \gamma'\} \subseteqq \{\{i\}, \alpha' \circ \beta', \gamma'\}$ and hence $H\{i_*\alpha', \beta', \gamma'\} = \{h\} \circ \{\{i\}, \alpha' \circ \beta', \gamma'\}$, since the respective indeterminacies coincide. Applying 1.6 we have

$$H\{i_*\alpha', \beta', \gamma'\} = -\{\{h\}, \{i\}, \alpha' \circ \beta'\} \circ E\gamma' \supseteqq -\{{}^0\{h\}, \{i\}, \alpha' \circ \beta'\} \circ E\gamma',$$

but again the indeterminacies coincide, yielding the required result.

PROOF OF PROPOSITION 0.5. Since $-\Delta^{\leftarrow}(\alpha' \circ \beta') = \{{}^0\{h\}, \{i\}, \alpha' \circ \beta'\} = \{{}^0\{h\}, \{i\}, \lambda \circ \delta\} \supseteqq \{{}^0\{h\}, \{i\}, \lambda\} \circ E\delta$, we need to show that $-j_* 1_{\Sigma M} \in \{{}^0\{h\}, \{i\}, \lambda\}$. Now we have

$$\begin{aligned}\{{}^0\{h\}, \{i\}, \lambda\} &= \{{}^0\{h\}, \{i''\} \circ \{i'\}, \lambda\} \supseteqq \{{}^0\{h\} \circ \{i''\}, \{i'\}, \lambda\} \\ &= \{{}^0\{j\} \circ \{q\}, \{i'\}, \lambda\} \supseteqq j_*\{{}^0\{q\}, \{i'\}, \lambda\},\end{aligned}$$

(we note that the preferred nullhomotopies are compatible).

But $\{{}^0\{q\}, \{i'\}, \lambda\} \ni \{{}^0\{q\}, \{i'\}, \lambda^0\}$ and hence the desired result is a consequence of the following lemma.

3.5. LEMMA. *Let* $f: X \to Y$ *be a map, let* $Pf: Y \to Y \cup_f CX$ *be the inclusion of* Y *into the cofibre of* f *and let* $Qf: Y \cup_f CX \to \Sigma X$ *be the projection shrinking* Y. *Then* $(Pf)f \simeq *$ *and* $(Qf)(Pf) \simeq *$ *via standard nulhomotopies and*

$$\{{}^0\{Qf\}, \{Pf\}, \{f\}^0\} = -1_{\Sigma X}.$$

PROOF. The reader may first check that the element of $\pi(X*, *\Sigma X)$, defined by the composite diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\nearrow"] & Y \arrow[r] \arrow[d, "Pf"] \arrow[dr, phantom, "\nearrow"] & * \arrow[d] \\ * \arrow[r] & Y \cup_f CX \arrow[r, "Qf"'] & \Sigma X \end{tikzcd}$$

in which the standard homotopies are used, coincides with the element defined by the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "(-,t)"] & * \arrow[d] \\ * \arrow[r] & \Sigma X . \end{tikzcd}$$

The desired result is now a consequence of the definition of P as a functor [3], essentially going back to [6].

When applicable, a rather efficient detection technique can be based on the following result which can be regarded as a slightly more general version of Proposition 0.6.

3.6. THEOREM. *If $i: A\to X$ factors through $A\cup_\lambda CM$ and if $\gamma\in\pi(W, M)$ is such that $\lambda\circ\gamma=0$ then $j_*E\gamma=H\{{}^0\{i\}, \lambda, \gamma\}$. The coset $\{{}^0\{i\}, \lambda, \gamma\}$ is contained in $i''_*\pi(\Sigma W, A\cup_\lambda CM)$ and its indeterminancy is $i_*\pi(\Sigma W, A)$. Moreover if $\delta\in\pi(V, W)$ is such that $\gamma\circ\delta=0$ and if $\xi\in\{{}^0\{i\}, \lambda, \gamma\}$ then $\xi\circ E\delta\in i_*\{\lambda, \gamma, \delta\}$.*

PROOF. Applying 2.3, we have $H\{{}^0\{i\}, \lambda, \gamma\}=-\{\{h\}, \{i\}, \{\lambda\}^0\}\circ E\gamma=-\{{}^0\{h\}, \{i\}, \{\lambda\}^0\}\circ E\gamma$, since the respective indeterminacies are trivial. But argueing as in the proof of 0.5 we find that $\{{}^0\{h\}, \{i\}, \{\lambda\}^0\}=-j_*1_{\Sigma M}$, so that $j_*E\gamma=H\{{}^0\{i\}, \lambda, \gamma\}$. Moreover, applying 1.6 we have

$$\{{}^0\{i\}, \lambda, \gamma\}\circ E\delta\subseteq -i_*\{\lambda, \gamma, \delta\},$$

hence the result.

4. The $E_t-H_t-\Delta_t$ sequence

For a pointed, locally countable CW-complex A, I. M. James [5] has described a space A_∞ and a homotopy equivalence $A_\infty\to\Omega\Sigma A$ which induces a canonical isomorphism

$$\Omega_1: \pi_{n+1}(\Sigma A)\longrightarrow \pi_n(A_\infty). \tag{4.1}$$

The suspension inclusion $A\to\Omega\Sigma A$ is equivalent via 4.1 to a natural inclusion $i: A\to A_\infty$. In the case $A=S^n$ the homotopy sequence

$$\cdots\to\pi_r(S^n)\longrightarrow\pi_r(S^n_\infty)\longrightarrow\pi_r(S^n_\infty, S^n)\longrightarrow\pi_{r-1}(S^n)\cdots$$

in which the James map $h: (S^n_\infty, S^n)\to(S^{2n}_\infty, *)$ is used to approximate $\pi_r(S^n_\infty, S^n)$ gives rise (via the isomorphism 4.1) to the $E-H-\Delta$ sequence

$$\cdots\to\pi_r(S^n)\xrightarrow{E}\pi_{r+1}(S^{n+1})\xrightarrow{H}\pi_{r+1}(S^{2n+1})\xrightarrow{\Delta}\pi_{r-1}(S^n)\to . \tag{4.2}$$

A little more generally, starting instead with the inclusion

$$i: S^n_{t-1} \longrightarrow S^n_\infty (t \geqq 2)$$

and utilising the James map $h_t: (S^n_\infty, S^n_{t-1}) \to (S^{tn}_\infty, *)$, one obtains the sequence

(4.3) $$\cdots \to \pi_r(S^n_{t-1}) \xrightarrow{E_t} \pi_{r+1}(S^{n+1}) \xrightarrow{H_t} \pi_{r+1}(S^{tn+1}) \xrightarrow{\Delta_t} \pi_{r-1}(S^n_{t-1}) \to .$$

As in the case of the sequence 0.2 the operator Δ_t is imperfectly defined and the sequence fails to be exact (to the extent that h_{t*} fails to be an isomorphism). However

$$\Delta_t^\tau(\mu) = -\Omega_1^{-1}\{{}^0\{h_t\}, \{i\}, \mu\} \subseteqq \pi_{r+1}(S^{tn+1})$$

is defined, with indeterminacy $H_t \pi_{r+1}(S^{n+1})$, whenever $\mu \in \pi_{r-1}(S^n_{t-1})$ belongs to the kernel of E_t. Since the generalized Whitehead product $[\iota_n]^t$ is the attaching class of the tn-cell of S^n_∞ the general theory of § 3 is applicable with $[\iota_n]^t$ playing the role of the class λ. Interpreting Proposition 3.4 in this case and taking into account Remark 1.12, we obtain the following.

4.4. COROLLARY. *If* $\alpha \in \pi(K, S^n_{t-1})$, $\beta \in \pi_k(K)$, $\gamma \in \pi_r(S^k)$ *are such that* $E_t(\alpha \circ \beta) = 0$ *and* $\beta \circ \gamma = 0$ *then* $H_t\{E_t\alpha, E\beta, E\gamma\}_1 = -\Delta_t^\tau(\alpha \circ \beta) \circ E^2\delta$.

In the case $t=2$ we recover [7; Proposition 2.6]. For $t>2$, however, the formula appears to be new. The corresponding version of Proposition 0.5 is as follows.

4.5. COROLLARY. *If* $\alpha \in \pi(K, S^n_{t-1})$, $\beta \in \pi_k(K)$ *are such that* $\alpha \circ \beta = [\iota_n]^t \circ \delta$, *where* $\delta \in \pi_k(S^{tn-1})$, *then* $\Delta_t^\tau(\alpha \circ \beta) \ni E^2\delta$.

The following two examples show that the corollaries can indeed be used to detect elements if $t>2$.

4.6. EXAMPLE. ($t=4$) Toda's generator $\bar{\varepsilon} \in \pi_{18}(S^3; 2)$ is an element of the bracket $\{\varepsilon_3, 2\iota_{11}, \nu_{11}^2\}_6$. To see that $\bar{\varepsilon}_3$ can be detected by H_4 via Corollary 4.4 we state the following lemma.

4.7. LEMMA. $\varepsilon_3 = E_4\hat{\varepsilon}$, *where* $\hat{\varepsilon} \in \pi_{10}(S^2_3)$, $8\hat{\varepsilon} = 0$ *and* $2\hat{\varepsilon} = [\iota_2]^4 \circ \nu_7$.

Applying 4.4, 4.5 and 4.7 we find $H_4\{E_4\hat{\varepsilon}, E2\iota_{10}, \nu_{11}^2\}_1 \ni -\nu_9^3 \neq 0$, as required.

PROOF OF LEMMA 4.7. In the truncated $EH\Delta$ sequence

$$\pi_{10}(S^2) \longrightarrow \pi_{10}(S^2_3) \longrightarrow \pi_{10}(S^4) \longrightarrow \pi_9(S^2)$$

the 2-components of $\pi_{10}(S^2)$ and $\pi_9(S^2)$ are trivial. Hence there is an isomorphism

$\pi_{10}(S_3^2; 2) \approx \pi_{10}(S^4; 2) = \{\nu_4{}^2\} \approx Z_8$. In the $E_4 - H_4 - \Delta_4$ sequence

$$\pi_{12}(S^3) \xrightarrow{H_4} \pi_{12}(S^9) \xrightarrow{\Delta_4} \pi_{10}(S_3^2) \xrightarrow{E_4} \pi_{11}(S^3)$$

it is known that $\mu_3 \in \pi_{12}(S^3)$ is such that $H_4\mu_3 = 4\nu_9[2;\ 4.3]$. It follows that the kernel of $E_4 = \{\Delta_4\nu_9\} = \{[i_2]^4 \circ \nu_7\} \approx Z_4$. Moreover $H_4(\varepsilon_3) = H(H\varepsilon_3) = 0$ so that $\varepsilon_3 \in E_4(\pi_{10}(S_3^2))$ and the existence of $\hat{\varepsilon}$, as claimed, is assured.

4.8. EXAMPLE. ($t = p$, an odd prime) As discovered by J. P. Serre, there is an element $\alpha \in \pi_{2p}(S^3)$ of order p. The facts stated in the following lemma are well known.

4.9. LEMMA. *$\alpha = E_p\bar{\alpha}$, where $\bar{\alpha} \in \pi_{2p-1}(S_{p-1}^2)$ is of infinite order and $p\bar{\alpha} = [\iota_2]^p$.*

Let $\gamma \in \pi_{r-1}(S^{2p-1})$ be an element of order p (for example we may take $\gamma = E^{2p-2}\alpha$) then

$$H_p\{E_p\bar{\alpha},\ p\iota_{2p},\ E\gamma\}_1 = -E^2\gamma\,.$$

Corollaries 4.4 and 4.5 offer a useful detection technique but to extract a maximum of information it is preferable to utilize the sharper construction (with insight into the group extension problem) offered by the following application of Theorem 3.6. We use 1 to denote the identity class $S_{t-1}^n \to S_{t-1}^n$.

4.10. COROLLARY. *If $\gamma \in \pi_k(S^{tn-1})$ is such that $[\iota_n]^t \circ \gamma = 0$ then $H_t\{{}^0E_t(1), E[\iota_n]^t, E\gamma\}_1 = -E^2\gamma$. Moreover if $\delta \in \pi_r(S^k)$ is such that $\gamma \circ \delta = 0$ and if $\xi \in \{{}^0E_t(1), E[\iota_n]^t, E\gamma\}_1$ then $\xi \circ E^2\delta \in E_t\{[\iota_n]^t, \gamma, \delta\}$.*

4.11. REMARK. The indeterminacy of the bracket $\{{}^0E_t(1), E[\iota_n]^t, E\gamma\}_1$ is the subgroup $\Omega_1 i_* \pi_{k+1}(S_{t-1}^n)$ of $\pi_{k+2}(S^{n+1})$. Note that this is precisely the subgroup of James filtration $t-1$[5]. Since $[\iota_n]^t$ is the attaching class of the tn-cell of S_t^n, each element ξ detected in this way has James filtration t. This fact can be used to determine the James filtrations of many of Toda's generators. For example it can be shown in this way that the James filtration of $\bar{\varepsilon}_3$ is 4. The first assertion of Corollary 4.10 has much overlap with [2; Theorem 3.3]. Indeed the bracket can be regarded as a generalization of the Hopf construction, c. f. [1], [2].

4.12. EXAMPLE. To illustrate the use that can be made of Corollary 4.10 to determine group extensions, let us study the element $\nu' \in \pi_6^3$. (We recall that $\pi_r^n = \pi_r(S^n; 2)$ unless $r = n$ or $r = 2n-1$ in which cases $\pi_r^n = \pi_r(S^n)$.) It is known (from an earlier stage in the systematic computation) that $[\iota_2, \iota_2] \circ \eta_3 = 0$ and that

$\pi_6^5 \approx Z/2$, generated by η_5. Applying 4.10 (with $t=2$) we obtain, for $\xi \in \{{}^0E(1), E[\iota_2]^2, E\eta_3\}_1$, $H\xi = -\eta_5$ and $\xi \circ 2\iota_6 = E\{[\iota_2, \iota_2], \eta_3, 2\iota_4\}$, which has trivial indeterminacy. But

$$\{[\iota_2, \iota_2], \eta_3, 2\iota_4\} \supseteqq \eta_2 \circ \{2\iota_2, \eta_3, 2\iota_4\} = \eta_2 \circ \eta_3.$$

It follows that ξ is of order 4 and is the element ν'.

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