TSUKUBA J. MATH.
Vol. 14 No. 2 (1990), 423–433

A GENERALIZATION OF HEREDITY IDEALS

(Dedicated to Professor Manabu Harada on his 60th birthday)

By

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1. Introduction

Throughout this note, A stands for a basic left and right artinian ring, J its Jacobson radical and $\{e_1, \cdots, e_n\}$ the complete set of orthogonal primitive idempotents in A. Let c_{ij} denote the composition length of ${}_{e_iAe_i}e_iAe_j$ for $1 \leqq i,\ j \leqq n$. The matrix $C(A)=(c_{ij})$ is called the left Cartan matrix of A.

Does $\operatorname{gl\,dim} A<\infty$ imply $\det C(A)=1$? This problem has been partially settled by several authors (e. g., Zacharia [7], Wilson [6], Burgess et al. [2], Fuller and Zimmermann-Huisgen [5] and so on), but is still open. There is a way to reduce the size of the matrix $C(A)$. Namely, if $\operatorname{proj\,dim}_A Ae_1/Je_1<\infty$ and $\operatorname{Ext}_A^k(Ae_1/Je_1, Ae_1/Je_1)=0$ for $k>0$, then $\operatorname{gl\,dim}(1-e_1)A(1-e_1) \leqq \operatorname{gl\,dim} A + \operatorname{proj\,dim}_A Je_1$ and $\det C((1-e_1)A(1-e_1))=\det C(A)$. This reduction was effectively used by Zacharia [7] to show that $\operatorname{gl\,dim} A \leqq 2$ implies $\det C(A)=1$ (see also Burgess et al. [2]). Unfortunately, as will be seen, Zacharia's reduction is not necessarily applicable if $\operatorname{gl\,dim} A \geqq 3$.

The aim of this note is to provide another type of reduction. To do this, we will generalize the notion of a heredity ideal which was first introduced by Cline, Parshall and Scott [3]. We are interested in a two-sided ideal I of A such that $\det C(A/I)=\det C(A)$ (of course, we claim $\operatorname{gl\,dim} A/I<\infty$ whenever $\operatorname{gl\,dim} A<\infty$). We will show that the trace ideal of a certain left A-module enjoys this property. We will prove the following

THEOREM. *Let Q be a torsionless left A-module and I its trace ideal. Suppose the following conditions:*

(a) $D=\operatorname{End}_A(Q)$ *is a division ring,*

(b) *the evaluation map* $Q\otimes_D \operatorname{Hom}_A(Q, A)\to A$ *is monic.*

(c) $\operatorname{Tor}_k^A(\operatorname{Tr} Q, Q)=0$ *for* $k\geqq 2$, *where* Tr *is the transpose, and*

(d) $\operatorname{proj\,dim}_A Q<\infty$.

Received May 18, 1989. Revised November 1, 1989.

Then we have

(1) $\operatorname{gl\,dim} A/I \leqq \operatorname{gl\,dim} A + \operatorname{proj\,dim}_A Q$,

(2) $\operatorname{gl\,dim} A \leqq \operatorname{gl\,dim} A/I + \max\{2, \operatorname{proj\,dim}_A Q+1\}$ *and*

(3) $\det C(A/I) = \det C(A)$.

It should be noted that the size of $C(A/I)$ equals that of $C(A)$ unless ${}_AQ$ is projective. Note also that, if $\operatorname{proj\,dim}_A Q \leqq 1$, the condition (c) is automatically satisfied. In case ${}_AQ$ is projective, the ideal I is just a heredity ideal and the statements (1) and (2) have been known (see Dlab and Ringel [4])*.

At present, we do not know whether $\operatorname{gl\,dim} A < \infty$ ensures the existence of a torsionless left A-module which satisfies all the conditions in the above theorem. Note however that, if this is always affirmative, so is the Cartan determinant problem.

In case $\operatorname{gl\,dim} A \leqq 2$, by Dlab and Ringel [4, Theorem 2], there always exists a projective left A-module which satisfies the conditions (a) and (b) in Theorem. Thus, our reduction yields a new proof of Zacharia's result [7]. Another example is the case of A being left serial. In that case, $\operatorname{gl\,dim} A < \infty$ ensures the existence of a simple torsionless left A-module Q with $\operatorname{proj\,dim}_A Q \leqq 1$ (cf. Burgess et al. [2, Lemma 3]).

In what follows, we will denote by $\operatorname{mod} A$ the category of all finitely generated left A-modules, by $(\ )^*$ the A-dual functor, by Tr the transpose and, for any $X \in \operatorname{mod} A$, by $[X]$ its image in the Grothendieck group of $\operatorname{mod} A$. Also, for any module X, we will denote by $|X|$ its composition length. Then, for any $X \in \operatorname{mod} A$, we may identify $[X]$ with the integral column vector

$$^t(|_{e_1Ae_1}e_1X|, \cdots, |_{e_nAe_n}e_nX|).$$

2. Proof of Theorem

Let Q, I and D be as in Theorem. For any $X \in \operatorname{mod} A$, denote by $\varepsilon_X: Q \otimes_D \operatorname{Hom}_A(Q, X) \to X$ the usual evaluation map and define $\alpha_X: Q^* \otimes_A X \to \operatorname{Hom}_A(Q, X)$ by $\alpha_X(f \otimes x)(q) = f(q)x$ for $f \in Q^*$, $x \in X$ and $q \in Q$. Note that $\operatorname{Im} \varepsilon_P = IP$ for all projective $P \in \operatorname{mod} A$.

We divide the proof into several steps. For the benefit of the reader, we do not exclude the case of ${}_AQ$ being projective in the proofs of statements (1) and (2).

We start with recalling a few well-known facts.

* After completing this note, the authors found that, in case ${}_AQ$ is projective, the statement (3) has also been proved by Burgess, W.D. and Fuller, K.R., On quasihereditary rings, Proc. Amer. Math. Soc. 106 (1989), 321-328.

CLAIM 1. $\operatorname{Ker}\varepsilon_X \in \operatorname{mod} A/I$ *for all* $X\in \operatorname{mod} A$.

PROOF. Let $\sum_{i=1}^{r} q_i\otimes f_i \in \operatorname{Ker}\varepsilon_X$ and $q\otimes f\in Q\otimes_D Q^*$. Then

$$\begin{aligned} f(q)\Big(\sum_{i=1}^{r} q_i\otimes f_i\Big) &= \sum_{i=1}^{r} f(q)q_i\otimes f_i \\ &= \sum_{i=1}^{r} q\alpha_Q(f\otimes q_i)\otimes f_i \\ &= q\otimes\Big(\sum_{i=1}^{r}\alpha_Q(f\otimes q_i)f_i\Big) \\ &= q\otimes\alpha_X\Big(f\otimes\Big(\sum_{i=1}^{r} f_i(q_i)\Big)\Big) \\ &= 0\,. \end{aligned}$$

Thus I annihilates $\operatorname{Ker}\varepsilon_X$.

CLAIM 2. $\operatorname{proj\,dim}_{A/I}X \leqq \operatorname{proj\,dim}_A X$ *for all* $X\in \operatorname{mod} A/I$ *with* $\operatorname{Tor}_k^A(A/I, X)=0$ *for* $k>0$.

PROOF. Let $X\in \operatorname{mod} A/I$ with $\operatorname{Tor}_k^A(A/I, X)=0$ for $k>0$. When the functor $A/I\otimes_A -$ is applied, the minimal projective resolution of ${}_AX$ yields a projective resolution of ${}_{A/I}X$.

CLAIM 3. $\operatorname{proj\,dim}_A X \leqq \operatorname{gl\,dim} A/I + \operatorname{proj\,dim}_A Q+1$ *for all* $X\in \operatorname{mod} A/I$.

PROOF. Note first that $\operatorname{proj\,dim}_A A/I \leqq \operatorname{proj\,dim}_A Q+1$. Since $\operatorname{proj\,dim}_A X\leqq \operatorname{proj\,dim}_{A/I}X+\operatorname{proj\,dim}_A A/I$ for all $X\in \operatorname{mod} A/I$, the assertion follows.

CLAIM 4. *Suppose* ${}_AQ$ *is projective. Then* $I^2=I$ *and* $\operatorname{Tor}_k^A(A/I, X)=0$ *for all* $X\in \operatorname{mod} A/I$ *and* $k>0$.

PROOF. Since ${}_AI$ is projective, $I=\operatorname{Im}\varepsilon_I=I^2$. Also, since I_A is projective, $\operatorname{proj\,dim}(A/I)_A\leqq 1$. It only remains to show $\operatorname{Tor}_1^A(A/I, X)=0$ for all $X\in \operatorname{mod} A/I$. Let $0\to Y\to P\to X\to 0$ be an exact sequence in $\operatorname{mod} A$ with P projective. Suppose $IX=0$. Then $IP\subset Y$, thus $IP=I^2P\subset IY$. Hence $\operatorname{Tor}_1^A(A/I, X)=0$, as required.

CLAIM 5. *Suppose* ${}_AQ$ *is not projective. Then* $Q^*\otimes_A Q=0$. *Consequently,* $I^2=0$ *and* $Q\in \operatorname{mod} A/I$.

PROOF. Since $\operatorname{End}_A(Q)$ is a division ring, the non-projectivity of ${}_AQ$ implies that no non-zero $f\in \operatorname{End}_A(Q)$ factors through projective modules. Thus, by Auslander [1, Proposition 7.1], we conclude $Q^*\otimes_A Q\cong \operatorname{Tor}_2^A(\operatorname{Tr} Q, Q)=0$.

CLAIM 6. *Suppose* $_AQ$ *is not projective. Then* $\mathrm{Tor}_k^A(A/I, X)\cong Q\otimes_D\mathrm{Tor}_{k+1}^A(\mathrm{Tr}\,Q, X)$ *for all* $X\in\mathrm{mod}\,A/I$ *and* $k>0$.

PROOF. Let $X\in\mathrm{mod}\,A/I$. Note that $\alpha_X=0$. Thus, by Auslander [1, Proposition 7.1], $Q^*\otimes_A X\cong\mathrm{Tor}_2^A(\mathrm{Tr}\,Q, X)$. Hence

$$\begin{aligned}\mathrm{Tor}_1^A(A/I, X)&\cong Q\otimes_D Q^*\otimes_A X\\ &\cong Q\otimes_D\mathrm{Tor}_2^A(\mathrm{Tr}\,Q, X).\end{aligned}$$

For $k\geqq 2$, since Q^* is a second syzygy of $\mathrm{Tr}\,Q$, we have

$$\begin{aligned}\mathrm{Tor}_k^A(A/I, X)&\cong\mathrm{Tor}_{k-1}^A(Q\otimes_D Q^*, X)\\ &\cong Q\otimes_D\mathrm{Tor}_{k-1}^A(Q^*, X)\\ &\cong Q\otimes_D\mathrm{Tor}_{k+1}^A(\mathrm{Tr}\,Q, X).\end{aligned}$$

CLAIM 7. $\mathrm{proj\,dim}_{A/I}X\leqq\mathrm{proj\,dim}_A X+\mathrm{proj\,dim}_A Q$ *for all* $X\in\mathrm{mod}\,A/I$. *Consequently,* $\mathrm{gl\,dim}\,A/I\leqq\mathrm{gl\,dim}\,A+\mathrm{proj\,dim}_A Q$.

PROOF. In case $_AQ$ is projective, by Claims 2 and 4, the assertion follows. Suppose $_AQ$ is not projective. Then, by Claims 2, 5 and 6, $\mathrm{proj\,dim}_{A/I}Q\leqq\mathrm{proj\,dim}_A Q$. Thus, it suffices to show

$$\mathrm{proj\,dim}_{A/I}X\leqq\mathrm{proj\,dim}_A X+\mathrm{proj\,dim}_{A/I}Q$$

for all $X\in\mathrm{mod}\,A/I$. Let $X\in\mathrm{mod}\,A/I$ with $\mathrm{proj\,dim}_A X=m<\infty$. Note that, if $_AX$ is projective, so is $_{A/I}X$. So we may assume $m>0$. Let

$$0\longrightarrow P_m\xrightarrow{f_m}\cdots\longrightarrow P_1\xrightarrow{f_1}P_0\longrightarrow X\longrightarrow 0$$

be the minimal projective resolution of $_AX$. Put

$$B_k=\mathrm{Im}\,(A/I\otimes f_k)\quad\text{and}\quad Z_{k-1}=\mathrm{Cok}\,(A/I\otimes f_k)$$

for $1\leqq k\leqq m$. Then $Z_0\cong X$ and, by Claim 6, $B_m\cong A/I\otimes_A P_m$. We have exact sequences

$$0\longrightarrow B_k\longrightarrow A/I\otimes_A P_{k-1}\longrightarrow Z_{k-1}\longrightarrow 0$$

for $1\leqq k\leqq m$ and, by Claim 6,

$$0\longrightarrow Q\otimes_D\mathrm{Tor}_{k+1}^A(\mathrm{Tr}\,Q, X)\longrightarrow Z_k\longrightarrow B_k\longrightarrow 0$$

for $1\leqq k\leqq m-1$. Now, one can make an induction on k to prove $\mathrm{proj\,dim}_{A/I}B_{m-k}\leqq k+\mathrm{proj\,dim}_{A/I}Q$ for $1\leqq k\leqq m-1$. Thus

$$\begin{aligned}\mathrm{proj\,dim}_{A/I}X&\leqq 1+\mathrm{proj\,dim}_{A/I}B_1\\ &\leqq m+\mathrm{proj\,dim}_{A/I}Q,\end{aligned}$$

as required.

CLAIM 8. $\operatorname{gl\,dim} A \leqq \operatorname{gl\,dim} A/I + \max\{2, \operatorname{proj\,dim}_A Q+1\}$.

PROOF. Let $X \in \operatorname{mod} A$. Since

$$\operatorname{proj\,dim}_A X \leqq \max\{\operatorname{proj\,dim}_A IX, \operatorname{proj\,dim}_A X/IX\},$$

by Claim 3, we have only to show

$$\operatorname{proj\,dim}_A IX \leqq \operatorname{gl\,dim} A/I + \max\{2, \operatorname{proj\,dim}_A Q+1\}.$$

In case ${}_AQ$ is not projective, by Claims 3 and 5, the assertion follows. Suppose ${}_AQ$ is projective. Then $\operatorname{Ker} \varepsilon_{IX}$ is a first syzygy of IX. Thus, by Claims 1 and 3, we get

$$\begin{aligned} \operatorname{proj\,dim}_A IX &\leqq \operatorname{proj\,dim}_A \operatorname{Ker} \varepsilon_{IX} + 1 \\ &\leqq \operatorname{gl\,dim} A/I + 2, \end{aligned}$$

as required.

CLAIM 9. $\det C(A/I) = \det C(A)$.

PROOF. Put $c_i = |{}_D\operatorname{Hom}_A(Q, Ae_i)|$ for $1 \leqq i \leqq n$. Since

$$Q \otimes_D \operatorname{Hom}_A(Q, Ae_i) \xrightarrow{\sim} Ie_i,$$

we have

$$c_i[Q] = [Ie_i]$$

for $1 \leqq i \leqq n$.

Consider first the case of ${}_AQ$ being projective. We may assume $Q = Ae_1$. Then we have

$$e_1Ae_i \xrightarrow{\sim} e_1Ie_i$$

for $1 \leqq i \leqq n$. Thus, since $c_{11} = 1$, we have

$$\begin{aligned} \det C(A) &= \det([Ae_1], [Ae_2], \cdots, [Ae_n]) \\ &= \det([Ae_1], [Ae_2] - c_2[Ae_1], \cdots, [Ae_n] - c_n[Ae_1]) \\ &= \det([Ae_1], [Ae_2] - [Ie_2], \cdots, [Ae_n] - [Ie_n]) \\ &= \det \left[\begin{array}{c|c} 1 & 0 \\ \hline * & C(A/I) \end{array} \right] \\ &= \det C(A/I). \end{aligned}$$

Suppose next that ${}_AQ$ is not projective. Let

$$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Q \longrightarrow 0$$

be the minimal projective resolution of ${}_AQ$. Put

$$d_i=\sum_{k\geq 0}(-1)^k|\mathrm{Ext}_A^k(Q,\ Ae_i/Je_i)_{e_iAe_i}|$$

for $1\leqq i\leqq n$. Since

$$[P_k]=\sum_{i=1}^{n}|\mathrm{Ext}_A^k(Q,\ Ae_i/Je_i)_{e_iAe_i}|[Ae_i]$$

for all $k\geqq 0$, we have

$$\begin{aligned}[Q]&=\sum_{k\geq 0}(-1)^k[P_k]\\&=\sum_{i=1}^{n}d_i[Ae_i].\end{aligned}$$

Also, since by Claims 5 and 6 the functor $A/I\otimes_A-$ acts exactly on the above projective resolution, we have

$$\begin{aligned}[Q]&=\sum_{k\geq 0}(-1)^k[P_k/IP_k]\\&=\sum_{i=1}^{n}d_i([Ae_i]-[Ie_i]).\end{aligned}$$

After permutation, we may assume $d_1\neq 0$. Then

$$\begin{aligned}d_1\det C(A)&=\det(d_1[Ae_1],\ [Ae_2],\ \cdots,\ [Ae_n])\\&=\det\left(\sum_{i=1}^{n}d_i[Ae_i],\ [Ae_2],\ \cdots,\ [Ae_n]\right)\\&=\det([Q],\ [Ae_2],\ \cdots,\ [Ae_n])\\&=\det([Q],\ [Ae_2]-c_2[Q],\ \cdots,\ [Ae_n]-c_n[Q])\\&=\det\left(\sum_{i=1}^{n}d_i([Ae_i]-[Ie_i]),\ [Ae_2]-[Ie_2],\ \cdots,\ [Ae_n]-[Ie_n]\right)\\&=\det(d_1([Ae_1]-[Ie_1]),\ [Ae_2]-[Ie_2],\ \cdots,\ [Ae_n]-[Ie_n])\\&=d_1\det C(A/I).\end{aligned}$$

Thus $\det C(A/I)=\det C(A)$.

This finishes the proof of Theorem.

3. Concerning the existence

In this section, we will show that $\mathrm{gl\,dim}\,A<\infty$ ensures the existence of a torsionless left A-module which satisfies the conditions (a) and (b) in Theorem. Such a module can be characterized by a certain type of torsion theory on mod A.

LEMMA 1. *Let* $(\mathcal{T}, \mathcal{F})$ *be a torsion theory on* $\operatorname{mod} A$ *and* $Q \in \mathcal{F}$ *a non-zero module. Suppose that no proper factor module of* Q *belongs to* $\mathcal{F}$ *and that* $\operatorname{Cok} f \in \mathcal{F}$ *for all* $f \in \operatorname{Hom}_A(Q, X)$ *with* $X \in \mathcal{F}$. *Then* $D = \operatorname{End}_A(Q)$ *is a division ring and, for every* $X \in \mathcal{F}$, *the evaluation map* $Q \otimes_D \operatorname{Hom}_A(Q, X) \to X$ *is monic.*

PROOF. Let $0 \neq f \in \operatorname{Hom}_A(Q, X)$ with $X \in \mathcal{F}$. We claim f is monic. Since $0 \neq \operatorname{Im} f \in \mathcal{F}$, f induces $Q \xrightarrow{\sim} \operatorname{Im} f$. Thus f is monic. In particular, $D = \operatorname{End}_A(Q)$ is a division ring. Now the last assertion is a consequence of the following

CLAIM. *Let* $X \in \mathcal{F}$ *and* $f_1, \cdots, f_r \in \operatorname{Hom}_A(Q, X)$ *be linearly independent over* D. *Then* $f = (f_1, \cdots, f_r) : \overset{r}{\oplus} Q \to X$ *is monic.*

PROOF. Replacing X by $\operatorname{Im} f$, we may assume f is epic. Note that, if $r=1$, the assertion has been proved. Suppose $r \geqq 2$. Since f_r is monic, we have the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} f & \longrightarrow & \overset{r-1}{\oplus} Q & \xrightarrow{(g_1, \cdots, g_{r-1})} & \operatorname{Cok} f_r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (f_1, \cdots, f_{r-1})} & & \| & & \\
0 & \longrightarrow & Q & \xrightarrow{f_r} & X & \longrightarrow & \operatorname{Cok} f_r & \longrightarrow & 0 .
\end{array}
$$

It is easy to see that $g_1, \cdots, g_{r-1} \in \operatorname{Hom}_A(Q, \operatorname{Cok} f_r)$ are linearly independent over D. Since $\operatorname{Cok} f_r \in \mathcal{F}$, by induction hypothesis, $g = (g_1, \cdots, g_{r-1})$ is monic, so is f.

PROPOSITION 1. *Let* M *be a left* A-*module with* $2 \leqq \operatorname{inj\,dim}_A M = m < \infty$. *Let* Q *be minimal with respect to inclusions in the class of all non-zero torsionless* $X \in \operatorname{mod} A$ *with* $\operatorname{Ext}_A^k(X, M) = 0$ *for* $k \geqq m-1$. *Then* $D = \operatorname{End}_A(Q)$ *is a division ring and the evaluation map* $Q \otimes_D \operatorname{Hom}_A(Q, A) \to A$ *is monic.*

PROOF. Since the functor $\operatorname{Ext}_A^m(-, M)$ is left exact, there is a torsion theory $(\mathcal{T}, \mathcal{F})$ on $\operatorname{mod} A$ such that the torsionfree class $\mathcal{F}$ consists of all $X \in \operatorname{mod} A$ with $\operatorname{Ext}_A^m(X, M) = 0$. Since ${}_A A \in \mathcal{F}$, it suffices to check that Q enjoys the properties in the above lemma.

CLAIM 1. *No proper factor module of* Q *belongs to* $\mathcal{F}$.

PROOF. Let $0 \to Q' \to Q \to Q'' \to 0$ be an exact sequence in $\operatorname{mod} A$ with Q', Q'' non-zero. Applying the functor $\operatorname{Hom}_A(-, M)$, we get $\operatorname{Ext}_A^{m-1}(Q', M) \cong$

$\mathrm{Ext}_A^m(Q'', M)$. Thus, by the minimality of Q, we conclude $\mathrm{Ext}_A^m(Q'', M)\neq 0$.

CLAIM 2. *$\mathrm{Cok}\, f\in\mathcal{F}$ for all $f\in\mathrm{Hom}_A(Q, X)$ with $X\in\mathcal{F}$.*

PROOF. Let $f\in\mathrm{Hom}_A(Q, X)$ with $X\in\mathcal{F}$. We may assume $f\neq 0$. As in the proof of Lemma 1, by Claim 1 we conclude f is monic. Now, the exact sequence $0\to Q\to X\to\mathrm{Cok}\, f\to 0$ yields $\mathrm{Ext}_A^m(\mathrm{Cok}\, f, M)=0$.

This finishes the proof of Proposition 1.

REMARK. Suppose $2\leqq\mathrm{gl\,dim}\, A=m<\infty$ and take ${}_AA$ as an M in the above proposition. Then we have $\mathrm{proj\,dim}_A Q\leqq m-2$. Thus, if $m\leqq 3$, Q satisfies all the conditions in Theorem.

4. In case of algebras

Throughout this section, A is assumed to be a finite dimensional algebra over an algebraically closed field F. Assume further that $\mathrm{gl\,dim}\, A<\infty$.

We intend to replace Q^* by its submodules. As shown in the second section, in case ${}_AQ$ is not projective, the condition (c) in Theorem can be replaced by the condition that $\mathrm{Tor}_k^A(Q^*, Q)=0$ for all $k\geqq 0$. So we are interested in a pair of a left A-module Q and a right A-module R such that there is a bilinear monomorphism $Q\otimes_F R\to A$ and $\mathrm{Tor}_k^A(R, Q)=0$ for all $k\geqq 0$. It should be noted that the existence of a bilinear monomorphism $Q\otimes_F R\to A$ implies R is imbedded into Q^*.

LEMMA 2. *Let $Q\in\mathrm{mod}\, A$ and R a submodule of Q^*. Let $\varepsilon: Q\otimes_F R\to A$ be the induced bilinear map and put $I=\mathrm{Im}\,\varepsilon$. Suppose that ε is monic and that $\mathrm{Tor}_k^A(R, Q)=0$ for all $k\geqq 0$. Then $\mathrm{gl\,dim}\, A/I\leqq 1+\mathrm{gl\,dim}\, A+\min\{\mathrm{proj\,dim}_A Q, \mathrm{proj\,dim}\, R_A\}$.*

PROOF. We may assume $\mathrm{proj\,dim}_A Q\leqq\mathrm{proj\,dim}\, R_A$. One can employ the argument in the second section to conclude that

$$\mathrm{proj\,dim}_{A/I} X\leqq 1+\mathrm{proj\,dim}_A X+\mathrm{proj\,dim}_A Q$$

for all $X\in\mathrm{mod}\, A/I$. The only difference is that $\mathrm{Tor}_m^A(A/I, X)$ may not vanish, where $m=\mathrm{proj\,dim}_A X$.

PROPOSITION 2. *Let $Q\in\mathrm{mod}\, A$ be indecomposable and non-projective and R a submodule of Q^*. Let $\varepsilon: Q\otimes_F R\to A$ be the induced bilinear map and put $I=$*

Im ε. *Suppose that* ε *is monic and that* $\mathrm{Tor}_k^A(R, Q)=0$ *for all* $k\geqq 0$. *Then* $\mathrm{gl\,dim}\, A/I<\infty$ *and* $\det C(A/I)=\det C(A)$.

PROOF. The first assertion follows from the above lemma. For the last assertion, the argument in the second section remains valid in this setting.

Finally, as an example, we prove the following

PROPOSITION 3. *Let* $I=AaA$ *with* $a\in e_iAe_j$. *Suppose that the left multiplication map* $\lambda_a: e_jA\to e_iA$ *is monic and that* $Ja=0$. *Then* $\mathrm{gl\,dim}\, A/I<\infty$ *and* $\det C(A/I)=\det C(A)$.

PROOF. Put $Q=Aa$ and $R=e_jA$. Since ${}_AQ$ is simple, $\mathrm{End}_A(Q)\cong F$ and $|Q_F|=1$. Thus ${}_AQ\otimes_F R_A \simeq {}_AI_A$. In case $i=j$, $Q\cong Ae_i$ and $R\cong Q^*$. Hence, one can apply Theorem to this case. Suppose $i\neq j$. Then $\mathrm{Tor}_k^A(R, Q)=0$ for all $k\geqq 0$. The first assertion follows from Lemma 2. For the last assertion, either ${}_AQ$ is projective or not, the argument in the second section is applicable.

5. Zacharia's reduction

In this final section, we review Zacharia's reduction [7]. His argument remains valid in more general setting.

PROPOSITION 4. *Suppose that* $\mathrm{Ext}_A^k(Ae_1/Je_1, Ae_1/Je_1)=0$ *for* $k>0$ *and that* $\mathrm{proj\,dim}_A Ae_1/Je_1<\infty$. *Then we have*

(1) $\mathrm{gl\,dim}\,(1-e_1)A(1-e_1)\leqq \mathrm{gl\,dim}\, A+\mathrm{proj\,dim}_A Je_1$ *and*

(2) $\det C((1-e_1)A(1-e_1))=\det C(A)$.

PPOOF. Since Ae_1 does not appear as a direct summand of any term in the minimal projective resolution of ${}_AJe_1$, we have

$$\begin{aligned}\mathrm{proj\,dim}_{(1-e_1)A(1-e_1)}(1-e_1)A&=\mathrm{proj\,dim}_{(1-e_1)A(1-e_1)}(1-e_1)Ae_1\\&=\mathrm{proj\,dim}_{(1-e_1)A(1-e_1)}(1-e_1)Je_1\\&\leqq \mathrm{proj\,dim}_A Je_1 .\end{aligned}$$

Hence, the first assertion follows.

Consider now the last assertion. Put

$$d_i=\sum_{k\geqq 0}(-1)^k\,|\mathrm{Ext}_A^k(Ae_1/Je_1, Ae_i/Je_i)_{e_iAe_i}|$$

for $1 \leqq i \leqq n$. Then, as in the second section, we have

$$[Ae_1/Je_1] = \sum_{i=1}^{n} d_i [Ae_i].$$

Thus, since $d_1 = 1$, we get

$$[Ae_1/Je_1] = [Ae_1] + \sum_{i=2}^{n} d_i [Ae_i].$$

Note that $C((1-e_1)A(1-e_1))$ coincides with the (1, 1)th principal minor of $C(A)$. Hence

$$\begin{aligned} \det C(A) &= \det([Ae_1], [Ae_2], \cdots, [Ae_n]) \\ &= \det([Ae_1] + \sum_{i=2}^{n} d_i [Ae_i], [Ae_2], \cdots, [Ae_n]) \\ &= \det \left[\begin{array}{c|c} 1 & * \\ \hline 0 & C((1-e_1)A(1-e_1)) \end{array} \right] \\ &= \det C((1-e_1)A(1-e_1)). \end{aligned}$$

We end with giving an example of an algebra of global dimension three for which Zacharia's reduction is of no use.

EXAMPLE. Let A be a subalgebra of $(F)_8$, the 8×8 matrix algebra over a field F, with the basis elements

$$e_1 = \sum_{i=1}^{5} e_{ii}, \qquad e_2 = \sum_{i=6}^{8} e_{ii}, \qquad a = e_{26},$$

$$e_{36} + e_{47} + e_{58}, \quad e_{41} + e_{52}, \quad e_{71} + e_{82}, \quad e_{56} \text{ and } e_{86},$$

where e_{ij} are matrix units. Then $\operatorname{gl\,dim} A = 3$ and, for both $i=1$ and 2, $\operatorname{Ext}_A^2(Ae_i/Je_i, Ae_i/Je_i) \neq 0$. On the other hand, one can take Ae_1/Je_1 or Ae_2/Aa as a torsionless left A-module which satisfies all the conditions in Theorem. We notice also that A does not have any heredity ideal.

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