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COMPLETE SPACE-LIKE HYPERSURFACES OF A DE SPITTER SPACE WITH CONSTANT MEAN CURVATURE

By

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Introduction

Let $M_s^m(c)$ be an m-dimensional connected semi-Riemannian manifold of index s and of constant curvature c, which is called an *indefinite space form of index* s or simply a *space form* according as $s>0$ or $s=0$. An m-dimensional space form of constant curvature c is only denoted by $M^m(c)$. The study of hypersurfaces with constant mean curvature of $M^{n+1}(c)$ was initiated by Nomizu and Smyth [13], who proved some excellent results.

It is seen that a complete space-like hypersurface of a Minkowski space R_1^{n+1} possesses a remarkable Bernstein property in the maximal case by Calabi [3] and Cheng and Yau [5]. As a generalization of the Bernstein type problem a complete space-like maximal submanifold M of $M_p^{n+p}(c)$ was recetly characterized by Ishihara [9] under a certain condition. In particular, it is proved that if c is non-negative, then M it totally geodesic.

On the other hand, it is pointed out by Marsden and Tipler [10] that space-like hypersurfaces with constant mean curvature of arbitrary spacetimes have interest in relativity theory. An entire space-like hypersurface with constant mean curvature of a Minkowski space is investigated by Goddard [8] and Treibergs [19]. It is well known as standard models of space-like hypersurfaces with constant mean curvature of a Minkowski space R_1^{n+1} (resp. a de Sitter space $S_1^{n+1}(c)$) that we have hyperboloids $H^k(c)\times R^{n-k}$ (resp. $H^k(c_1)\times S^{n-k}(c_2)$ and R^n), where $k=0, 1, \cdots, n$. After some perturbations conserving constant mean curvatures, Goddard [8] conjectured the following two results: the only space-like hypersurfaces with constant mean curvature of R_1^4 are the hyperboloids and three classes of space-like hypersurfaces $S^3(c_2)$, R^3 and $H^3(c_1)$ are the only complete space-like hypersurfaces with constant mean curvature which exist in $S_1^4(c)$. Stumbles [18] and Treibergs [19] however constructed many entire such hypersurfaces of R_1^{n+1} different from the hyperboloids.

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The Bernstein-type property was also generalized by Nishikawa [12] from the different point of view, and K. Milnor [11] and Yamada [20] gave a characterization of the hyperbolic cylinder of R_1^3. Complete space-like hypersurfaces with constant mean curvature of a de Sitter space S_1^{n+1} are also studied by many authors [1], [2], [4], [17] and so on. Under this situation it seems to be interesting to investigate whether or not there exist examples of such hypersurfaces of a de Sitter space different from hyperboloids.

In this paper, a class of complete space-like hypersurfaces with constant mean curvature of a de Sitter space is considered. The purpose of this paper has different two directions. One is to generalize the Bernstein-type property in this version, that is, to classify such hypersurfaces of non-negative curvature in the case where a multiplicity of each principal curvature is greater than one. The other is to show that there exist infinitely many space-like hypersurfaces with constant mean curvatures of a de Sitter space.

In the first section, we will simply recall the theory of space-like hypersurfaces of an indefinite Riemannian manifold and in § 2, some standard models of complete space-like hypersurfaces of a de Sitter space whose mean curvatures are constant are introduced. The following main theorem is proved in § 3.

MAIN THEOREM. *Let M be an $n(\geqq 3)$-dimensional complete space-like hypersurface with constant mean curvature of a de Sitter space $S_1^{n+1}(c)$. If the sectional curvature is of non-negative and if a multiplicity of each principal curvature is greater than one, then M is isometric to a Euclidean space R^n or a sphere $S^n(c_1)$, $0<c_1<c$.*

In the last section Goddard's second conjecture for a de Sitter space will be treated and infinitely many complete space-like hypersurfaces with constant mean curvature of a Sitter space are given.

1.Preliminaries

Let (M', g') be an m-dimensional indefinite Riemannian manifold of index $s(>0)$. Throughout this paper, manifolds are always assumed to be connected and geometric objects are assumed to be of class C^∞. We choose a local field of orthonormal frames $e_1, \cdots, e_m$ adapted to the indefinite-Riemannian metric in M' and let $\omega_1, \cdots, \omega_m$ denote the dual coframe. Suppose that we have $g'(e_A, e_B)=\varepsilon_A\delta_{AB}$, $\varepsilon_A=\pm 1$ for $A, B, \cdots=1, \cdots, m$. The connection forms $\{\omega_{AB}\}$ of M' are characterized by the equations

$$(1.1)\qquad \begin{aligned} &d\omega_A+\sum\varepsilon_B\omega_{AB}\wedge\omega_B=0, \qquad \omega_{AB}+\omega_{BA}=0, \\ &d\omega_{AB}+\sum\varepsilon_C\omega_{AC}\wedge\omega_{CB}=\Omega_{AB}, \\ &\Omega_{AB}=(-1/2)\sum\varepsilon_C\varepsilon_D R'_{ABCD}\omega_C\wedge\omega_D, \end{aligned}$$

where Ω_{AB} (resp. R'_{ABCD}) denotes the indefinite Riemannian curvature form (resp. components of the indefinite Riemannian curvature tensor R') of M'. The components of the Ricci curvature tensor Ric$'$ and the scalar curvature r' are defined by respectively by

$$(1.2)\qquad \begin{aligned} &R'_{AB}=R'_{BA}=\sum\varepsilon_C R'_{CABC}, \\ &r'=\sum\varepsilon_A R'_{AA}=\sum\varepsilon_A\varepsilon_C R'_{CAAC}. \end{aligned}$$

An indefinite Riemannian manifold M' of constant sectional curvature is called an *indefinite space form of index* s if M' is of index s. By $M_s^m(c)$ an m-dimensional indefinite space form of index s and of constant curvature c is denoted. Then the components R'_{ABCD} of the indefinite Riemannian curvature tensor R' for an indefinite space form $M_s^m(c)$ are given by

$$(1.3)\qquad R'_{ABCD}=c\varepsilon_A\varepsilon_B(\delta_{AB}\delta_{BC}-\delta_{AC}\delta_{BD}).$$

Therefore the Ricci curvature tensor Ric$'$ and the scalar curvature r' are also given by

$$(1.4)\qquad R'_{AB}=(m-1)c\varepsilon_A\delta_{AB}, \qquad r'=m(m-1)c.$$

In particular, $M_1^m(c)$ is called a *Lorentz space form* and it is called a *Minkowski space* provided that $c=0$.

Standard models of complete Lorentz space forms are given as follows. In an $(n+p)$-dimensional Euclidean space R^{n+p} with a standard basis, a scalar product $\langle\ ,\ \rangle$ is defined by

$$\langle x, y\rangle=-\sum_{i=1}^{s}x_iy_i+\sum_{s+1}^{n+p}x_jy_j,$$

where $x=(x_1, \cdots, x_{n+p})$ and $y=(y_1, \cdots, y_{n+p})$ are in R^{n+p}. This is a scalar product of index p and the space $(R^{n+p}, \langle\ ,\ \rangle)$ is an indefinite Euclidean space, which is simply denoted by R_s^{n+p}. Let $S_1^{n+1}(c)$ be a hypersurface of R_1^{n+2} defined by

$$\langle x, x\rangle=r^2=1/c.$$

Then $S_1^{n+1}(c)$ inherits a Lorentz metric from the ambient space R_1^{n+2} with constant curvature c, which is called a *de Sitter space*. On the other hand, let $H_1^{n+1}(c)$ be a hypersurface of R_2^{n+2} defined by

$$\langle x, x\rangle=-r^2=1/c.$$

Then $H_1^{n+1}(c)$ induces a Lorentz metric from the ambient space R_2^{n+2} with negative constant curvature c, which is called an *anti-de Sitter space*. For indefinite Riemannian manifolds, refer to O'Neill [15].

From now on, let $M'=M_1^{n+1}(c)$ be an $(n+1)$-dimensional Lorontz space form of index 1 and of constant curvature c and let M be a positive definite hypersurface of $M_1^{n+1}(c)$, which is said to be *space-like*. In the sequel, the following convention on the range of indices are used, unless otherwise stated: $A, B, \cdots=0, 1, \cdots, n$; $i, j, \cdots=1, \cdots, n$. By restricting the canonical forms ω_A and the connection forms ω_{AB} to the hypersurface M, they are denoted by the same symbol, respectively. Then we have

$$\omega_0=0, \tag{1.5}$$

and the metric on M induced from the indefinite-Riemannian metric g' on the ambient space M' under the immersion is given by $g=\sum\omega_i\otimes\omega_i$. Then $\{e_1, \cdots, e_n\}$ becomes a field of orthonormal frames on M with respect to this metric and $\{\omega_1, \cdots, \omega_n\}$ is a field of dual frames on M. From (1.1), (1.2) and Cartan's lemma it follows that we have

$$\omega_{0i}=\sum h_{ij}\omega_j, \qquad h_{ij}=h_{ji}. \tag{1.6}$$

The quadratic form $\alpha=\sum\varepsilon h_{ij}\omega_i\omega_j e_0$ with valued in the normal bundle is called the *second fundamental form* on M, where we put $\varepsilon=\varepsilon_0$. That is,

$$\alpha(e_i, e_j)=\varepsilon h_{ij}e_0, \tag{1.7}$$

and the scalar $H=\sum h_{jj}/n$ is called the *mean curvature* of the hypersurface M. The connection forms $\{\omega_{ij}\}$ of M are characterized by the structure equations

$$\begin{aligned} &d\omega_i+\sum\omega_{ij}\wedge\omega_j=0, \qquad \omega_{ij}+\omega_{ji}=0, \\ &d\omega_{ij}+\sum\omega_{ik}\wedge\omega_{ki}=\Omega_{ij}, \\ &\Omega_{ij}=(-1/2)\sum R_{ijkl}\omega_k\wedge\omega_l, \end{aligned} \tag{1.8}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor R) of M. For the semi-Riemannian curvature tensors R' and R of M' and M respectively, it follows from (1.1) and (1.8) that we have the Gauss equation

$$R_{ijkl}=c(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})+\varepsilon(h_{il}h_{jk}-h_{ik}h_{jl}). \tag{1.9}$$

The components of the Ricci curvature tensor Ric and the scalar curvature r are given by

$$R_{jk}=c(n-1)\delta_{jk}+\varepsilon hh_{jk}-(h_{jk})^2, \tag{1.10}$$

(1.11) $$r=n(n-1)c+\varepsilon h^2-h_2,$$

where $h=\sum h_{jj}$, $(h_{jk})^2=\sum\varepsilon h_{ir}h_{rk}$ and $h_2=\sum(h_{jj})^2$.

Now, the covariant derivative $\nabla\alpha$ of the second fundamental form α of M with components h_{ijk} is given by

(1.12) $$\sum h_{ijk}\omega_k=dh_{ij}-\sum h_{kj}\omega_{ki}-\sum h_{ik}\omega_{kj}.$$

Then, differentiating (1.6) exteriorly and making use of the structure equations we have

$$\sum h_{ijk}\omega_j\wedge\omega_k=0.$$

Thus we have the Codazzi equation

(1.13) $$h_{ijk}=h_{ikj}.$$

Similarly the covariant derivative $\nabla^2\alpha$ of $\nabla\alpha$ with components h_{ijkl} is given by

$$\sum h_{ijkl}\omega_l=dh_{ijk}-\sum h_{ljk}\omega_{li}-\sum h_{ilk}\omega_{kj}-h_{ijl}\omega_{lk}.$$

Next, differentiating (1.12) exteriorly and using again the structure equations and the above relationship, we obtain

$$\sum(2h_{ijkl}+h_{rj}R_{rikl}+h_{ir}R_{rjkl})\omega_k\wedge\omega_l=0.$$

Thus we get

(1.14) $$h_{ijkl}-h_{ijlk}=-\sum h_{rj}R_{rikl}-\sum h_{ir}R_{rjkl},$$

which is called the *Ricci formula* for the second fundamental form.

Making use of this relationship repeatly and taking account of the Codazzi equation (1.13), the Gauss equation (1.7) and the first Bianchi identity, one can compute h_{ijkl} as follows:

$$\begin{aligned}h_{ijkl}=h_{klij}&+c(h_{ij}\delta_{kl}-h_{ij}\delta_{jk}-h_{kl}\delta_{ij}+h_{jl}\delta_{ik}-h_{il}\delta_{jk}+h_{ik}\delta_{jl})\\&+(h_{ij})^2h_{kl}-(h_{il})^2h_{jk}-(h_{kl})^2h_{ij}+(h_{jl})^2h_{ik}-(h_{il})^2h_{jk}+(h_{ik})^2h_{jl}.\end{aligned}$$

Accordingly a Laplacian of the second fundamental form is given by

(1.15) $$\Delta h_{ij}=\sum h_{ijkk}=\sum h_{kkij}+c(nh_{ij}-h\delta_{ij})+h(h_{ij})^2-h_2h_{ij}.$$

The Laplacian of the function h_2 may be computed by using (1.9), (1.10) and (1.15):

(1.16) $$(1/2)\Delta h_2=\varepsilon\sum h_{ijk}h_{ijk}+\varepsilon\sum h_{ij}h_{kkij}+c(nh_2-\varepsilon h^2)+\varepsilon hh_3-h_2^2,$$

where $h_3=\sum h_{ij}(h_{ij})^2$.

2. Standard models

In this section, some standard models of complete space-like hypersurfaces with constant mean curvature of R_1^{n+1} and $S_1^{n+1}(c)$ are given. In particular, we take examples of hypersurfaces whose sectional curvature are non-negative. An $(n+1)$-dimensional Minkowski space R_1^{n+1} can first be regarded as a product manifold of R_1^{k+1} and R^{n-k}. With respect to the standard orthonormal basis of R_1^{n+1} a class of space-like hypersurfaces $H^k(c_1)\times R^{n-k}$ of R_1^{n+1} is defined by

$$H^k(c_1)\times R^{n-k}=\{(x, y)\in R_1^{n+1}=R_1^{k+1}\times R^{n-k} : |x|^2=-1/c_1>0\},$$

where $k=0, 1, \cdots, n$ and $|\ |$ denotes the norm determined by the scalar product on R_1^{k+1} which is given by $\langle x, x\rangle=-x_0^2+\sum_{j=1}^k x_j^2$. When $k=0$, this is a family of totally geodesic Euclidean spaces. In particular, if $k=1$, then $H^1(c_1)$ is a part γ of the hyperbolic curve in R_1^2 and $\gamma\times R^{n-1}$ is a class of space-like hypersurfaces of R_1^{n+1}. A number of distinct principal curvatures of each hypersurface in this family are exactly two, say $\pm(c-c_1)^{1/2}$ and 0, with muliplicities 1 and $n-1$, respectively.

We next define a family of space-like hypersurfaces $H^k(c_1)\times S^{n-k}(c_2)$ of $S_1^{n+1}(c)$ by

$$\begin{aligned}&H^k(c_1)\times S^{n-k}(c_2)\\ &=\{(x, y)\in S_1^{n+1}(c)\subset R_1^{n+2}=R_1^{k+1}\times R^{n-k+1} : |x|^2=-1/c_1,\ |y|^2=1/c_2\},\end{aligned}$$

where $c_1<0$, $c_2>0$, $1/c_1+1/c_2=1/c$ and $k=0, 1, \cdots, n$. When $k=0$, it is a family of spheres of constant curvature c_2, which are totally umbilical. In particular, if $k=1$, then $H^1(c_1)$ is a part γ of th hyperbolic curve in R_1^2 and $\gamma\times S^{n-1}(c_2)$ is a class of space-like hypersurfaces of $S_1^{n+1}(c)$ and the number of distinct principal curvatures of each hypersurface in this family are exactly two and they are constant. A principal curvature is equal to $\pm(c-c_1)^{1/2}$ with multiplicity 1 and the other is equal to $\pm(c-c_2)^{1/2}$ with multiplicity $n-1$.

Another family of space-like hypersurfaces $M(s)$ of $S_1^{n+1}(c)$ given. For any positive number s the hypersurface $M(s)$ is defined by

$$M(s)=\{x\in S_1^{n+1}(c)\subset R_1^{n+2} : x_0=x_{n+1}+s\}.$$

Then it is totally umbilical and moreover it is flat. That is, it is a Euclidean space.

3. Hypersurfaces of non-negative curvature

In this section the main theorem mentioned in the introduction is proved.

Let M' be an $(n+1)$-dimensional de Sitter space of constant curvature c and let M be a complete space-like hypersurface with constant mean curvature of M'. Assume that the sectional curvature of M is of non-negative. For any point p in M we can choose a frame field $e_1, \cdots, e_n$ on M so that the matrix (h_{jk}) is diagonalized at that point, say

$$h_{jk}=\lambda_j\delta_{jk}\,. \tag{3.1}$$

Under such frame field at p we have

$$h=\sum\lambda_j,\quad h_2=-\sum\lambda_j^2,\quad h_3=-\sum\lambda_j^3,$$

because M is space-like and hence the normal vector is time-like. Accordingly, we get

$$\begin{aligned} c(nh_2-\varepsilon h^2)+\varepsilon hh_3-h_2^2&=c\{n(-\sum\lambda_j^2)+(\sum\lambda_j)^2\}+\sum\lambda_i\sum\lambda_j^3-(\sum\lambda_j^2)^2\\ &=(1/2)\sum(-c+\lambda_i\lambda_j)(\lambda_i-\lambda_j)^2, \end{aligned}$$

from which together with (1.16) it turns out that

$$\Delta h_2=-\sum(c-\lambda_i\lambda_j)(\lambda_i-\lambda_j)^2-2|\nabla\alpha|^2, \tag{3.2}$$

where $|\nabla\alpha|$ denotes the norm of the covariant derivative $\nabla\alpha$ of the second fundamental form α.

In order to prove the main theorem, we need a fundamental property for the following generalized maximal principle due to Omori [14] and Yau [21].

THEOREM. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. Let f be a C^2-function which is bounded from below on M. Then for any positive number ε there exists a point q at which it satisfies*

$$|\operatorname{grad} f|(q)<\varepsilon,\quad \Delta f(q)>\varepsilon,\quad f(q)<\inf f+\varepsilon\,. \tag{3.3}$$

Since M in space-like, the Ricci curvature tensor R_{ij} is given by

$$R_{ij}=(n-1)c\delta_{ij}-hh_{ij}-(h_{ij})^2$$

by (1.10). Let $\lambda_1, \cdots, \lambda_n$ be principal curvatures of M. Then the Ricci curvature tensor becomes

$$R_{ij}=\{(n-1)c-h\lambda_i+\lambda_i^2\}\delta_{ij},$$

which yields that the Ricci curvature of M is bounded from below by a constant $(n-1)c-h^2/4$. Given any positive number a, a function f is defined by $1/(a-h_2)^{1/2}$. Then it is smooth and positive on M because h_2 is non-positive. Moreover, since the function f is bounded from below, we can apply the theorem due to Omori and Yau to the function f. So, given any point p in M

and any positive number ε there exists a point q at which f satisfies the property (3.3) in the theorem. Consequently the following relationship

$$(1/2)f(q)^4\Delta h_2(q)>f(q)\varepsilon-3\varepsilon^2 \tag{3.4}$$

can be derived by the simple and direct calculation. For a convergent sequence $\{\varepsilon_m\}$ such that $\varepsilon_m>0$ and $\varepsilon_m\to 0$ $(m\to\infty)$ there exists a point sequence $\{q_m\}$ so that the sequence $\{f(q_m)\}$ converges to f_0 by taking a subsequence, if necessary. From the definition of the infimun we have $f_0=\inf f$ and hence the definition of f gives rise to

$$h_2(q_m) \longrightarrow \inf h_2 \qquad (m\to\infty). \tag{3.5}$$

On the other hand, it follows from (3.4) that we have

$$(1/2)f(q_m)^4\Delta h_2(q_m)>f(q_m)\varepsilon_m-3\varepsilon_m^2, \tag{3.6}$$

and the right hand side converges to 0 because the function f is bounded. Since the sectional curvature $K(e_j, e_k)$ of the plane section spanned by e_i and e_k is given by $K(e_j, e_k)=c-\lambda_j\lambda_k$ and since it is assumed to be non-negative, we have $c-\lambda_j\lambda_k\geqq 0$ for any distinct indices j and k. Accordingly (3.2) means that

$$\Delta h_2(q_m)\leqq-\sum(c-\lambda_j\lambda_k)(\lambda_j-\lambda_k)^2\leqq 0, \tag{3.7}$$

where principal curvatures are continuous. Accordingly, for any positive number ε (<2) there is a sufficiently large integer m_0 for which we have

$$f(q_m)^4\Delta h_2(q_m)>-\varepsilon \qquad \text{for} \quad m>m_0. \tag{3.8}$$

On the other hand, it is seen that the sequence $\{f(q_m)\}$ is bounded from below by a positive constant. In fact, we have

$$-h_2h_4-(-h_3)^2=\sum\lambda_j^2\lambda_k^2(\lambda_j-\lambda_k)^2\geqq 0,$$

$$h_4-(-h_2)^2=-\sum_{j\neq k}(\lambda_j\lambda_k)\leqq 0,$$

and hence we get $-(-h_2)^{3/2}\leqq h_3\leqq(-h_2)^{3/2}$, from which together with (1.6) it follows that

$$(1/2)\Delta h_2\leqq c(h_2+h^2)+|h|(-h_2)^{3/2}-(-h_2)^2.$$

This relationship and (3.8) yield the inequality

$$(2-\varepsilon)h_2(q_m)^2-2|h|(-h_2)^{3/2}(q_m)+2(a\varepsilon-nc)h_2(q_m)-)\varepsilon\alpha^2+2ch^2)<0,$$

which implies that $\{h_2(q_m)\}$ is bounded from below. This means that the sequence $\{f(q_m)\}$ is bounded from below by a positive constant. By means of (3.6), (3.7) and the above fact, we get

(3.9) $$\Delta h_2(q_m) \longrightarrow 0 \qquad (m\to\infty).$$

Thus (3.7) and (3.9) give rise to

(3.10) $$(c-\lambda_j\lambda_k)(\lambda_j-\lambda_k)^2(q_m) \longrightarrow 0 \qquad (m\to\infty)$$

for any distinct indices j and k.

Now, the following facts will be proved by (3.10):

(a) Any sequence $\{\lambda_j(q_m)\}$ is bounded for any j;

(b) For any distinct indices j and k there is a subsequence $\{q_{mi}\}$ of the sequence $\{q_m\}$ such that

(3.11) $$(c-\lambda_j\lambda_k)(q_{mi}) \longrightarrow 0 \qquad (i\to\infty) \quad \text{or}$$
$$(\lambda_j-\lambda_k)(q_{mi}) \longrightarrow 0 \qquad (i\to\infty).$$

(c) For any distinct indices j and k there is a subsequence $\{q_{mi}\}$ of the sequence $\{q_m\}$ such that

$$(\lambda_j-\lambda_k)(q_{mi}) \longrightarrow 0 \qquad (i\to\infty).$$

The assertion (a) is first proved. Suppose that there is an index j such that $\{\lambda_j(q_m)\}$ is not bounded. Without loss of generality, we may suppose that $\lambda_j(q_m)\to\infty$ $(m\to\infty)$. Since the mean curvature $H=h/n$ is constant, there is another sequence $\{\lambda_k(q_m)\}$ such that $\lambda_k(q_m)\to-\infty$ $(m\to\infty)$ and hence

$$\lambda_j\lambda_k(q_m) \longrightarrow -\infty \quad \text{and} \quad (\lambda_j-\lambda_k)(q_m) \longrightarrow \infty \quad (m\to\infty).$$

This is a contradiction to (3.10).

Next, in order to prove the assertion (b), we put $a_m=(c-\lambda_j\lambda_k)(q_m)$ and $b_m=(\lambda_j-\lambda_k)^2(q_m)$. For two sequences $\{a_m\}$ and $\{b_m\}$ these are both bounded by the assertion (a) and (3.10) means that a sequence $\{a_m b_m\}$ converges to 0 as m tends to infinity. Suppose that there is a subsequence $\{a_{mj}\}$ such that $a_{mj}\to a\neq 0$ and $|b_m|\leqq B$. Because of $a_{mj}b_{mj}=b_{mj}(a_{mj}-a)+ab_{mj}$, we have

$$|a_{mj}b_{mj}|+|b_{mj}(a_{mj}-a)|\geqq|ab_{mj}|.$$

Since $\{a_{mj}b_{mj}\}$ is also converges to 0, there is a positive integer N for any positive number ε such that $|a_{mj}b_{mj}|<\varepsilon$ and $|a_{mj}-a|<\varepsilon$ for $j>N$, which yields that $|ab_{mj}|\leqq\varepsilon(1+B)$ and hence $b_{mj}\to 0$ $(j\to\infty)$. This leads to $a_{mj}\to 0$ $(j\to\infty)$ or $b_{mj}\to 0$ $(j\to\infty)$, that is, the proof of (b) is complete.

Now, for principal curvatures λ_1 and λ_2 we can regard the subsequence $\{q_{mi}\}$ in the assertion (b) as a sequence $\{q_m\}$. Suppose anew $\lambda_1\lambda_2(q_m)\to c$ $(m\to\infty)$. Since $\{\lambda_1(q_m)\}$ is bounded, it converges to λ_{10} by taking the subsequence $\{q_{mj}\}$ if necessary. Suppose that $\lambda_{10}=0$. Then we have $|\lambda_1\lambda_2(q_{mj})|\leqq A_2|\lambda_1(q_{mj})|$, where Λ_2 denotes the upper bound of $\{\lambda_2(q_m)\}$, which implies that $\lambda_1\lambda_2(q_{mj})\to 0\neq c(j\to\infty)$, a contradiction. Thus $\lambda_{10}\neq 0$ and hence we have

$$|\lambda_{10}|\,|\lambda_2(q_{mj})-c/\lambda_{10}|\leqq|\lambda_2(q_{mj})|\,|\lambda_1(q_{mj})-\lambda_{10}|+|\lambda_1\lambda_2(q_{mj})-c|,$$

from which it follows that $\lambda_2(q_{mj})\to c/\lambda_{10}=\lambda_{20}$. Consequently two limits have the same sign, which shows that, without loss of generality we may suppose that they are positive. The assumption that a multiplicity of all principal curvatures is greater than one and the condition of the sectional curvatures give us the fact $c-\lambda_1(q_{mj})^2\geqq 0$ and $c-\lambda_2(q_{mj})^2\geqq 0$ and therefore $0<\lambda_{10}$, $\lambda_{20}\leqq\sqrt{c}$. This coupled with (3.11) yields that $\lambda_1(q_{mj})\to\sqrt{c}$ and $\lambda_2(q_{mj})\to\sqrt{c}$ $(j\to\infty)$, which means that $\lambda_{10}=\lambda_{20}=\sqrt{c}$, that is, $(\lambda_1-\lambda_2)(q_{mj})\to 0$ $(j\to\infty)$, by taking the subsequence if necessary, Similarly, for principal curvatures λ_1 and λ_3 there is a subsequence $\{q_{mk}\}$ of $\{q_{mj}\}$ such that $(\lambda_1-\lambda_3)(q_{mk})\to 0$ $(k\to\infty)$, by taking the subsequenc if necessary. Since the sequence $\{(\lambda_1-\lambda_2)(q_{mj})\}$ converges to zero, it is see that the subsequence $\{(\lambda_1-\lambda_2)(q_{mk})\}$ converges also to zero as k tends to infinity, which implies that there is a point subsequence $\{q_{mk}\}$ of $\{q_{mj}\}$ so that two sequences $\{(\lambda_1-\lambda_2)(q_{mk})\}$ and $\{(\lambda_1-\lambda_3)(q_{mk})\}$ converges both to zero as k tends to infinity. This means that the assertion (c) is inductively proved.

Thus, for any distinct indices j and k we have $\lambda_{j0}=\lambda_{k0}$ or $\lambda_{j0}\lambda_{k0}=c$ and hence we get $\lambda_{j0}=\lambda_{k0}=\sqrt{c}$ because of $0<\lambda_{j0}\leqq\sqrt{c}$. Furthermore, because of $-nh_2-h^2=\sum(\lambda_j-\lambda_k)^2\geqq 0$, the function h^2 is bounded from above by the constant $-h^2/n$ and on the other hand the sequence $\{h_2(q_m)\}$ converges to $-h^2/n$. It means that we obtain $h_2(q_m)\to -h^2/n=\sup h_2$, from which together (3.5) it follows that $\sup h_2=\inf h_2$, namely, the function h_2 is constant on M. Accordingly (3.2) means that the second fundamental form is parallel and therefore the simple algebraic calculation implies that all principal curvatures are constant on M and the number of distinct principal curvatures are at most 2. By means of the congruence theorem of Abe, Koike an Yamaguchi [1] it completes the proof of Main theorem stated in the introduction.

REMAR 3.1. By the hyperbolic cylinder $\gamma\times S^{n-1}(c_2)$, where γ denotes the space-like curve in R_1^2, it is seen that the assumtion of the multiplicities of principal curvatures in Main theorem can not be omitted.

The case where the ambient space is an anti-de Sitter space shall be considered. Let M be a space-like hypersurface of an anti-de Sitter space $H_1^{n+1}(c)$, whose mean curvature is constant. If M is of non-negative curvature, then the process of the proof of the main theorem in this section holds to the fact that all sequences $\{\lambda_j(q_m)\}$ are convergent as m tends to infinity. Suppose that M

is of non-negative curvature. Then it is easily seen by the simple algebraic consideration that the number of distinct limits of these sequences $\{\lambda_j(q_m)\}$ is at most two, say λ_{10} and λ_{20}, and $c-\lambda_{10}\lambda_{20}\geqq 0$. It turns out that one is positive and the other is negative. Then we may suppose, without loss of generality, that the multiplicity of λ_{10} is greater than one, because of $n\geqq 3$. Then, for a sufficiently large m_2 there are distinct indices j and k such that the sequences $\{\lambda_j(q_m)\}$ and $\{\lambda_k(q_m)\}$ convergent to λ_{10}, and $\lambda_j(q_m)$ and $\lambda_k(q_m)$ are positive for any integer $m>m_2$. Therefore the sectional curvature of the plane at q_m spanned by principal vectors corresponding to $\lambda_j(q_m)$ and $\lambda_k(q_m)$ is given by $c-\lambda_j(q_m)\lambda_k(q_m)$, which is negative. Thus one finds

PROPOSITION 3.1. *Let M be an $n(\geqq 3)$-dimensional space-like hypersurface of an anti-de Sitter space $H_1^{n+1}(c)$ whose mean curvature is constant. Then M is not of non-negative curvature.*

REMARK 3.2. In the case where the ambient space is a Minkowski one, it is trivial that a space-like hypersurface M of non-negative curvature is totally geodesic, if M is of constant mean curvature and if a mnltiplicity of each principal curvature is greater than one. The last assumption can not be omitted. In fact, a hyperbolic cylinder $\gamma\times R^{n-1}$ in R_1^{n+1} defined by $\{x\in R_1^{n+1}: -x_1^2+x_2^2=-b^2\}$, where b is a positive constant and hence γ is a space-like curve in R_1^2, is a complete flat space-like hypersurface of R_1^{n+1} one of whose distinct principal curvatures is 0 with its multiplicity $n-1$ the other is a non-zero constant.

REMARK 3.3. A hypersurface of a Minkowski space R_1^{n+1} is said to be *entire* if it is the graph of a function over the whole R^n. It is seen by Cheng and Yau [5] that in the entire space-like hyersurface M with constant mean curvature of R_1^{n+1} the Ricci curvature of M is non-positive. According to Stumbles and Treibergs' theorem [18] and [19] there are many entire space-like hypersurfaces with constant mean curvature of R_1^{n+1} which are different from the hyperboloids. It means that entire space-like surfaces with non-positive Gauss curvature and constant mean curvature exist in R_1^3 and hence we cannot expect to classify space-like hypersurfaces with constant mean curvature and of non-positive sectional curvature.

By the similar method to the proof of the main theorem, we can verify following fact, which is a complete version of a part of Nomizu and Smyth's theorem [13]. A complete simply connected Riemannian manifold of negative constant curvature is said to be *hyperbolic.*

THEOREM 3.2. *Let M be an $n(\geqq 3)$-dimensional complete hypersurface with constant mean curvature of a hyperbolic space. If the sectional curvature is non-negative and if a muliplicity of each principal curvature is greater than one, then M is isometric to a Euclidean space or a sphere.*

4. Goddard's second conjecture

This section is devoted to the investigation about examples of complete space-like hypersurfaces with constant mean curvature of a de Sitter space $S_1^{n+1}(c)$, which are different from standard models. By taking account of the main theorem, it is seen that at least one principal curvatures ought to be simple. In fact, as we remarked in the last of §3, it suffices to consider hypersurfaces with distinct two principal curvatures one of which is simple of $S_1^{n+1}(c)$. Such hypersurfaces are conformally flat.

Let M be a space-like hypersurface of $S_1^{n+1}(c)$, $n\geqq 3$, and assume that the principal curvatures λ_j's on M satisfy

$$\lambda_1=\cdots=\lambda_{n-1}=\lambda\neq 0,\qquad\qquad (4.1)$$
$$\lambda_n=\mu,$$

such that $\lambda\neq\mu$. Without loss of generality, we may suppose that $\lambda>0$. As is well known, the distribution D of the space of principal vectors at any point corresponding to the principal curvature λ is completely integrable, because the multiplicity of each principal curvature is constant. Now, since λ and μ are smooth functions on M and since the second fundamental form is given by $h_{jk}=\lambda_j\delta_{jk}$, i.e., $h_{aa}=\lambda$, $h_{nn}=\mu$ and $h_{jk}=0$ $(j\neq k)$, we have, by the definition of the covariant derivative $\nabla\alpha$ with components h_{ijk},

$$d\lambda=h_{aaa}\omega_a+\textstyle\sum_{b\neq a}h_{aab}\omega_b+h_{aan}\omega_n, \qquad (4.2)$$

where indices $a, b, \cdots$ run over the range $\{1, \cdots, n-1\}$. Because of $\omega_{0a}=\lambda\omega_a$, we have

$$\begin{aligned} d\omega_{0a}&=d\lambda\wedge\omega_a+\lambda d\omega_a\\ &=d\lambda\wedge\omega_a+\lambda(-\textstyle\sum_b\omega_{ab}\wedge\omega_b-\omega_{an}\wedge\omega_n), \end{aligned}$$

while the restriction of the structure equation for the ambient space to the hypersurface M yields

$$d\omega_{0a}=-\textstyle\sum_k\omega_{0k}\wedge\omega_{ka}+\Omega_{0a}$$

$$=-\lambda\sum_b\omega_b\wedge\omega_{ba}-\mu\omega_n\wedge\omega_{na},$$

because the ambient space is a constant curvature and $\omega_{0n}=\mu\omega_n$. Combining together with above two equations, we have

$$\sum_b\lambda,_b\omega_b\wedge\omega_a+\{(\mu-\lambda)\omega_{an}-\lambda,_n\omega_a\}\wedge\omega_n=0 \tag{4.3}$$

for any index a, where $d\lambda=\sum_b\lambda,_b\omega_b+\lambda,_n\omega_n$. Since 2-forms $\omega_j\wedge\omega_k$ $(j<k)$ are linearly independent, this implies

$$\begin{gathered}\lambda,_a=0,\\ (\mu-\lambda)\omega_{an}-\lambda,_n\omega_a=f_a\omega_n\end{gathered} \tag{4.4}$$

for any index a, where f_a is a function on M. By the second equation of (4.4) we have $d\omega_n=\{f_a/(\mu-\lambda)\}\omega_n\wedge\omega_a$, which inplies that $d\omega_n\equiv 0$ (mod. ω_n). This yields that the distribution D is completely integrable because D is defined by $\omega_n=0$. From (4.2) and the first equation of (4.4) it follows that we have

$$h_{aaa}\omega_a+\sum_{b\neq a}h_{aab}\omega_b+h_{aan}\omega_n=\lambda,_n\omega_n,$$

and hence

$$h_{aaa}=0,\quad h_{aab}=0\ (b\neq a),\quad h_{aan}=\lambda,_n. \tag{4.5}$$

Similarly, for the other principal curvature μ one has

$$d\mu=\sum_b h_{nnb}\omega_b+h_{nnn}\omega_n.$$

Because of $\omega_{0n}=\mu\omega_n$, by the same argument as that of λ we have

$$d\omega_{0n}=-\lambda\sum_b\omega_{nb}\wedge\omega_b=d\mu\wedge\omega_n-\mu\sum_b\omega_{nb}\wedge\omega_b,$$

and hence

$$d\mu\wedge\omega_n+(\lambda-\mu)\sum_b\omega_{nb}\wedge\omega_b=0.$$

We set $d\mu=\sum_b\mu,_b\omega_b+\mu,_n\omega_n$. This together with (4.4) implies

$$\mu,_a=f_a \quad \text{for any index } a. \tag{4.6}$$

By definition, we have

$$h_{nna}=\mu,_a,\qquad h_{nnn}=\mu,_n. \tag{4.7}$$

On the other hand, for distinct indices a and b, one has

$$h_{abk}=0 \quad \text{for any index } k, \tag{4.8}$$

In particular, let M be a space-like hypersurface of $S_1^{n+1}(c)$ whose mean curvature is non-zero constant. Then principal curvatures λ μ and satisfy $h=(n-1)\lambda+\mu$ and hence $(n-1)\lambda,_k+\mu,_k=0$ for any index k. Moreover, by (4.4), (4.6) and (4.7) we have

(4.9) $$\lambda,_a=\mu,_a=h_{ann}=0, \qquad f_a=0$$

for any index a. Thus, the principal curvature λ is constant along each integral submanifold of the corresponding distribution. By (4.4) we have

(4.10) $$\omega_{na}=\lambda,_n(n\lambda-h)^{-1}\omega_a, \qquad -(n-1)\lambda,_n=\mu,_n=h_{nnn}.$$

Consequently, in order for M to satisfy that the mean curvature is constant, their principal curvatures λ and μ must satisfy (4.9) and (4.10). Moreover, by the second equation of (4.4) and (4.9) we have $d\omega_n=0$, which shows that we may put

(4.11) $$\omega_n=dv.$$

Thus we have

(4.12) $$\omega_{na}=\lambda'(n\lambda-h)^{-1}\omega_a,$$

where the prime denotes the derivative with respect to the parameter v. (4.12) shows that the integral submanifold $M^{n-1}(v)$ corresponding to λ and v is umbilical in M and in $S_1^{n+1}(c)$. Substituting the above equation into the structure equation

$$d\omega_{na}+\sum_b\omega_{nb}\wedge\omega_{ba}=(c-\lambda\mu)\omega_n\wedge\omega_a,$$

we have

$$d(\lambda'(n\lambda-h)^{-1}\omega_a)=-\lambda'(n\lambda-h)^{-1}\sum_b\omega_b\wedge\omega_{ba}+(c-\lambda\mu)\omega_n\wedge\omega_a.$$

Since the left hand side is reduced to

$$\{(n\lambda-h)^{-1}\lambda'\}'\omega_n\wedge\omega_a+(n\lambda-h)^{-1}\lambda'(+\sum_b\omega_{ab}\wedge\omega_b+\omega_{an}\wedge\omega_n),$$

we get

(4.13) $$\{(n\lambda-h)^{-1}\lambda'\}'-\{(n\lambda-h)^{-1}\lambda'\}^2-\{c-h\lambda+(n-1)\lambda^2\}=0,$$

which is reformed to

(4.14) $$(n\lambda-h)\lambda''-(n+1)\lambda'^2-\{c--h\lambda+(n-1)\lambda^2\}(n\lambda-h)^2=0,$$

Putting $\sigma=(\lambda-h/n)^{-1/n}$, (4.14) can be replaced by

$$\sigma''+(n-1)\sigma^{-2n+1}+\{(n-2)h/n\}\sigma^{-n+1}+(c-h^2/n^2)\sigma=0.$$

Integrating the differential equation for σ of degree 2, we obtain

(4.15) $$\sigma'^2-\sigma^{-2n+1}-2h/n\sigma^{-n+2}+(c-h^2/n^2)\sigma_z=c_1,$$

where c_1 is the integral constant. Hence, we have the different situation compared with the case of no simple roots. Hence we have many hypersurfaces of $S_1^{n+1}(c)$ whose mean curvature is constant corresponding to the values of the constant c_1.

In the sequel, we suppose that $c=1$ and $S_1^{n+1}(1)$ is an $(n+1)$-dimensional de Sitter space S_1^{n+1} of constant curvature 1 in R_1^{n+2}. We may consider the frame $(e_0, e_1, \cdots, e_n, x)$ in R_1^{n+2} such that $x=e_{n+1}$. Since the second fundamental form of S_1^{n+1} as the hypersurface R_1^{n+2} is given by $h_{AB}=-\sum_B \varepsilon_B \delta_{AB}$, we have

$$\omega_{n+10}=0, \qquad \omega_{n+1k}=-\omega_k .$$

Accordingly, we have the following relations by (4.12),

$$\begin{aligned} de_a &= -\omega_{0a}e_0+\textstyle\sum_{b=a}\omega_{ba}e_b+\omega_{na}e_n+\omega_{n+1a}e_{n+1} \\ &= \textstyle\sum_{b\neq a}\omega_{ba}e_b-\{\lambda e_0+(\log\sigma)'e_n+e_{n+1}\}\omega_a , \\ d\{\lambda e_0 &+(\log\sigma)'e_n+e_{n+1}\} \\ &= \lambda'\omega_n e_0-\lambda(\lambda\textstyle\sum_a\omega_a e_a+\mu\omega_n e_n)+(\log\sigma)''\omega_n e_n \\ &= \{\lambda'-\mu(\log\sigma)'\}\omega_n e_0+\{-\lambda\mu+(\log\sigma)''+1\}\omega_n e_n \\ &\quad -(\log\sigma)'\omega_n e_{n+1} \qquad (\text{mod. } e_1, \cdots, e_{n-1}\} \\ &= -(\log\sigma)'\{\lambda e_0+(\log\sigma)'e_n+e_{n+1}\}dv , \end{aligned}$$

by means of (4.13). Hence, putting

$$W=e_1\wedge \cdots \wedge e_{n-1}\{\lambda e_0+(\log\sigma)'e_n+e_{n+1}\}, \tag{4.16}$$

we get

$$dW=-(\log\sigma)'Wdv, \tag{4.17}$$

which shows that the n-vector W in R_1^{n+2} is constant along $M^{n-1}(v)$. Hence there exists an n-dimensional linear subspace $E_n(v)$ in R_1^{n+2} containing $M^{n-1}(v)$. Since the scalar product $\langle u, u\rangle$ of the vector u defined $\lambda e_0+(\log\sigma)'e_n e_{n+1}$ is given by

$$-\lambda^2+\{(\log\sigma)'\}^2+1=c_1(\lambda-h/n)^{2/n}$$

by means of (4.15), the vector u is space-like, null or time-like, according as the integral constant c_1 is positive, zero or negative. So the linear space $E^n(v)$ is space-like if c_1 is positive. By (4.17) the n-vector field W depends only on v and by integrating in the equation

$$W(v)=\{n\lambda(v)-h\}^{1/n}\{n\lambda(v_0)-h\}^{-1/n}W(v_0)$$

holds. Hence it is seen that $E_n(v)$ is parallel to $E^n(v_0)$ in R_1^{n+2}.

The sectional curvature of $M^{n-1}(v)$ is here considered. Since $M^{n-1}(v)$ is regarded as the submanifold of S_1^{n+1}, its sectional curvature is give by $[-\lambda^2+\{(\log\sigma)'\}^2+1](v)$ because of

$$d\omega_{ab}+\sum_c\omega_{ac}\wedge\omega_{cb}=\omega_{a0}\wedge\omega_{0b}-\omega_{an}\wedge\omega_{nb}+\omega_a\wedge\omega_b$$
$$=[-\lambda^2+\{(\log\sigma)'\}^2+1]\omega_a\wedge\omega_b.$$

Since the vector $u=\lambda e_0+(\log\sigma)'e_n+e_{n+1}$ is space-like, $M^{n-1}(v)$ is the $(n-1)$-dimensional sphere $M^{n-1}(v)=E^n(v)\cap S_1^{n+1}$ of constant curvature $|u(v)|^2=[-\lambda^2+\{(\log\sigma)'\}^2+1](v)$ and the the center $q(v)$ is given by

$$q(v)=x-u(v)/|u(v)|^2.$$

Moreover the center lies in a fixed plane E_1^2 through the origin of R_1^{n+2} and orthogonal to $E^n(v_0)$ because the position vector q in R_1^{n+2} is orthogonal to the vectors e_a for any a and u. Hence the point $q(v)$ makes a plane curve in E_1^2. Thus one find the following

THEOREM 4.1. *Let M be an $n(\geqq 3)$-dimensional space-like hypersurface with constant mean curvature of $S_1^{n+1}(c)$. If it has exactly two distinct principal curvatures one of which is simple and the other λ has no zero points, the following assertions are true:*

(1) *M is a locus of moving $(n-1)$-dimensional submanifold $M^{n-1}(v)$ along which the principal curvature λ is constant and which is umbilic in M and of constant curvature $\{d/dv(\log\lambda)^2+\lambda^2+c)$, where v is the arc length of an orthogonal trajectory of the family $M^{n-1}(v)$, and $\lambda=\lambda(v)$ satisfies the ordinary differential equation* (4.14) *of order* 2.

(2) *If $M=S_1^{n+1}$ in R_1^{n+2}, then $M^{n-1}(v)$ is contained in an $(n-1)$-dimensional sphere $S^{n-1}(v)=E^n(v)\cap S_1^{n+1}$ of the intersection of S^{n+1} and an n-dimensional linear subspace $E^n(v)$ in R_1^{n+2} which is parallel to a fixed E^n. The center q moves on a plane curve in a plane R^2 through the origin of R_1^{n+2} and orthogonal to E^n.*

COROLLARY 4.2. *There exist infinitely many space-like hypersurfaces of $S_1^{n+1}(c)$ whose mean curvature is constant, which is not congruent to each other on it.*

This section is essentially related to Otsuki's theory in [16].

REMARK 4.1. We have no information about the sign of the sectional curvature of the above examples.

REMARK 4.2. According to the Goddard second conjecture the class of totally umbilic hypersurfaces is the only complete space-like hypersurface with constant mean curvature of S_1^4. This is denied by the existence of hyperbolic cylinders, whose principal curvatures are constant. However principal curvatures of the above hypersurfaces are not necessarily constant, which that they are different from the hyperboloids. So they are also conuter-examples of Goddard's second conjecture.

REMARK 4.3. In a Minkowski space the existence is to give the entire solution of the partial differential equation

$$(4.18)\qquad (1-\sum f_j^2)\sum f_{kk}+\sum f_j f_k f_{jk}=h(1-\sum f_j^2)^{3/2},\qquad 1-\sum f_j^2>0$$

on R^n, where $f_j=\partial f/\partial x_j$ and $f_{jk}=\partial^2 f/\partial x_j\partial x_k$. Treibergs [19] considered approximations of entire space-like hypersurfaces with constant mean curvature in R_1^{n+1} by solving a Direchlet problem in an increasing sequence of domains and obtained many essentially different solutions.

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