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AN ISOMETRIC EMBEDDING OF THE COMPLEX HYPERBOLIC SPACE IN A PSEUDO-EUCLIDEAN SPACE AND ITS APPLICATION TO THE STUDY OF REAL HYPERSURFACES

By

Oscar J. GARAY and Alfonso ROMERO

0. Introduction.

In the last years, the use of an idea of A. Ros, [11], has meant an interesting progress in the study of several families of submanifolds of the complex projective space CP^n. This idea essentially consists in considering the first standard embedding of CP^n in a certain Euclidean space $\boldsymbol{R}^N$, and contemplating the submanifolds of CP^n in the light of that new embedding. The first standard embedding has parallel second fundamental form and makes CP^n to be a symmetric R-space in $\boldsymbol{R}^N$.

In particular real hypersurfaces of CP^n have been analysed under this point of view, [6], [13], and new characterizations of this important class of hypersurfaces have been obtained.

In 1986, the second author and S. Montiel, [8], made a systematic study of a certain family of real hypersurfaces of the complex hyperbolic space CH^n. In the process of classification of that family they introduced new examples without parallel in CP^n. Therefore if we could get an isometric embedding of CH^n in some Euclidean space $\boldsymbol{R}^N$ provided of as good geometric properties as those of the first standard embedding of CP^n, we could try to profound in the study of real hypersurfaces in CH^n.

On the other hand fully immersed complete submanifolds of a Euclidean space with parallel second fundamental form have been totally classified by D. Ferus, [3], [4]. As a consequence his result implies that a complete irreducible (as a Riemannian manifold) submanifold which is fully immersed in an Euclidean space with parallel second fundamental form is congruent to either an hyperplane or to an irreducible symmetric R-space immersed by means of its standard embedding. Consequentely we see that there exist no an isometric immersion

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of CH^n in an Euclidean space with parallel second fundamental form. Therefore one should rather look for such a good isometric embedding of CH^n in a pseudo-Euclidean space. In this article we propose both to prove that the above mentioned embedding exists and to use that embedding to analize real hypersurfaces of CH^n. In § 1, we firstly construct an isometric embedding φ of CH^n in the pseudo-Euclidean space $\boldsymbol{R}_S^N$, $N=n^2+2n+1$, $S=n^2+1$, with parallel second fundamental form, which makes CH^n to be a space-like submanifold with definite negative normal bundle in the ambient space $\boldsymbol{R}_S^N$. In this way CH^n whitin $\boldsymbol{R}_S^N$ is a pseudo-Riemannian symmetric R-space in the sense of Naitoh, [9], (see also Remark 1.2). Secondly if we take an isometric immersion Φ of a $(2n-1)$-dimensional Riemannian manifold M in CH^n, and consider the induced isometric immersion $\chi=\varphi\circ\Phi$ of M in $\boldsymbol{R}_S^N$ then we ask which real hypersurfaces M of CH^n are minimally immersed by means of χ in some non-flat totally umbilical hypersurface of $\boldsymbol{R}_S^N$. We obtain:

THEOREM 4.2. *There exist no real hypersurtaces M of CH^n, $n\geqq 2$, which are minimal in either a (n^2+2n)-dimensional indefinite sphere or in a (n^2+2n)-dimensional indefinite real hyperbolic space of $\boldsymbol{R}_{n^2+1}^{n^2+2n+1}$; that is to say no real hypersurface M of CH^n is mono-order in $\boldsymbol{R}_{n^2+1}^{n^2+2n+1}$ via the isometric embedding φ.*

This result contrasts with Theorem 3.1 of [6], where certain geodesic hyperspheres were characterized as those whose immersion via the first standard embedding was minimal in some hypersphere of an Euclidean space.

At the same time, we discovered that Theorem 4.2 was accompanied by a seemingly strange fact. Indeed, in computing the Laplacian of the mean curvature vector field $\hat{H}$ of the horosphere M_n^* of CH^n in the pseudo-Euclidean space $\boldsymbol{R}_{n^2+1}^{n^2+2n+1}$ (see § 3) we achieved $\Delta\hat{H}=Q$, where Q was a non-zero constant vector of the ambient space. Fortunately we succed in proving its converse and thus to characterize the horosphere M_n^* as follows:

THEOREM 5.7. *Let M be a real hypersurface of constant mean curvature in CH^n, $n\geqq 2$. Suppose M satisfies the differential equation*

$$\Delta\hat{H}=Q$$

with $\hat{H}$ the mean curvature vector fiel of M in $\boldsymbol{R}_{n^2+1}^{n^2+2n+1}$ and Q is a non-zero constant vector of the ambient space. Then M is locally congruent in CH^n to the horosphere

$$M_n^*=\pi(\{z\in H_1^{2n+1} \mid |z_0-z_1|^2=1\})$$

described in § 3.

Finally one should notice that no real hypersurface of CP^n with constant mean curvature satisfy the above differential equation $\Delta\hat{H}=Q$.

1. *An isometric embedding of the complex hyperbolic space in a pseudo-Euclidean space.* We consider the following Hermitian form in the $(n+1)$-dimensional complex vector space C^{n+1}

$$F(z, w)=-\bar{z}_0 w_0+\sum_{j=1}^{n}\bar{z}_j w_j \tag{1.1}$$

where as usual $(z_0, \cdots z_n)=z^t$ and $(w_0, \cdots, w_n)=w^t$ represent vectors of C^{n+1}, $(\ )^t$ denoting transpose, and $\bar{z}_j$ means the conjugate complex of z_j. The real part of F is an indefinite Riemannian metric on C^{n+1} with index 2, which we call g. The classical definition of the n-dimensional complex hyperbolic space CH^n goes as follows: CH^n is formed by the set consisting of the 1-dimensional complex subspaces $L=\text{span}\,\{z\}$ of C^{n+1} satisfying $F(z, z)<0$, i.e. L is negative definite with respect to g. The real hypersurface $H_1^{2n+1}=\{z\in C^{n+1}\,|\,F(z, z)=-1\}$ of C^{n+1} is, with the induced metric g, a complete Lorentzian manifold of constant sectional curvature -1, which is called the $(2n+1)$-dimensional anti-De Sitter space. In terms of the usual action of the 1-dimensional sphere S^1 on H_1^{2n+1}, we have that H_1^{2n+1} is a S^1-bundle on CH^n with canonical projection $\pi: H_1^{2n+1}\to CH^n$, $z\mapsto[z]=\text{span}\{z\}$. If we additionally take on CH^n the Bergman metric of holomorphic sectional curvature -4, π is a riemannian submersion, [10], with time-like totally geodesic fibers.

Now take $\text{End}(C^{n+1})$ as the space of all the C-linear endomorphisms of C^{n+1}. Then by mapping $L=[z]$, $z\in H_1^{2n+1}$, in the orthogonal projection with respect to L of the metric vector space (C^{n+1}, F), we construct an embedding of CH^n in $\text{End}(C^{n+1})$, whose image is

$$\{p\in\text{End}(C^{n+1})/p\circ p=p,\ F(p(z), w)=F(z, p(w)),\ z,\ w\in C^{n+1},\ \text{trace}(p)=1\} \tag{1.2}$$

By identifying $\text{End}(C^{n+1})$ with the space formed by the complex square matrices of order $n+1$, M_{n+1}, we obtain a differentiable embedding φ of CH^n in M_{n+1}, whose image is

$$\{A\in M_{n+1}\,|\,A^2=A,\ G\bar{A}^t=AG,\ \text{trace}(A)=1\} \tag{1.3}$$

where G is the matrix $\text{diag}(-1, I_n)$, I_n is the matrix identity of order n, and $\bar{A}$ (resp. A^t) is the conjugate (resp. transpose) matrix of A.

Therefore, given any $[z]\in CH^n$ we have

$$\varphi([z])=\left(\begin{array}{c|ccc} |z_0|^2 & -z_0\bar{z}_1 & \cdots & -z_0\bar{z}_n \\ \hline z_1\bar{z}_0 & -|z_1|^2 & \cdots & -z_1\bar{z}_n \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ z_n\bar{z}_0 & -z_n\bar{z}_1 & \cdots & -|z_n|^2 \end{array}\right) \tag{1.4}$$

and so $\varphi([z])$ *is the matrix of the orthogonal projection* p_L *with respect to the line* $L=[z]$ *expressed in the usual basis of* $\boldsymbol{C}^{n+1}$.

Consider now the $(n+1)^2$-dimensional real subspace of M_{n+1} defined by

$$H^1(n+1)=\{A\in M_{n+1}/G\bar{A}^t=AG\}.$$

If we define on $H^1(n+1)$ the symmetric bilinear form

$$\tilde{g}(A, B)=-(1/2)\ \mathrm{trace}(AB) \tag{1.5}$$

A, B in $H^1(n+1)$, then $\tilde{g}$ is a non-degenerate metric of index n^2+1. Consequently $(H^1(n+1), \tilde{g})$ is isometric to the pseudo-Euclidean space $\boldsymbol{R}^{n^2+2n+1}_{n^2+1}$.

If we agree in denoting by $U^1(n+1)$ the Lie group formed by the F-isometric automorphisms of $\boldsymbol{C}^{n+1}$, $U^1(n+1)=\{A\in Gl(n+1, \boldsymbol{C})|\bar{A}^tGA=G\}$, then we see that this group acts transitively on the anti-De Sitter space H^{2n+1}_1 by means of the usual matrix product. $\boldsymbol{C}H^n$ inherits this action in a natural way $(A, [z])\mapsto[Az]$. On the other hand, there is another action of $U^1(n+1)$ on $H^1(n+1)$ represented by $(A, X)\mapsto AXA^{-1}$. This action leaves invariant the metric $\tilde{g}$, (1.5), and φ is a $U^1(n+1)$-equivariant embedding. One can check directely from (1.4) that the differential of φ at $[e_0]$, $e_0^t=(1, 0, \cdots, 0)$, preserves vectors length. Thus φ is an isometric embedding. Moreover, the image of φ given in (1.3) is contained in the hyperplane $H^1_*(n+1)=\{A\in H^1(n+1)|\mathrm{trace}(A)=1\}$.

We end observing that I_{n+1} is normal to $H^1_*(n+1)$ with respect to $\tilde{g}$. Hence $H^1_*(n+1)$ is a non-degenerate hyperplane of index n^2 and there exists no hyperplane of $H^1_*(n+1)$ containing $\varphi(\boldsymbol{C}H^n)$. Finally we summarize these facts in the following:

PROPOSITION 1.1. *The map* φ *written in* (1.4) *defines an* $U^1(n+1)$*-equivariant isometric embedding of the hyperbolic complex space* $\boldsymbol{C}H^n$, *with Bergman metric of holomorphic sectional curvature* -4, *into the pseudo-Euclidean space* $\boldsymbol{R}^{n^2+2n+1}_{n^2+1}$ *(represented as the matrix space* $(H^1(n+1), \tilde{g})$*). In addition* φ *is an* $U^1(n+1)$*-equivariant embedding fully immersed in a non-degenerate hyperplane of index* n^2 *of* $\boldsymbol{R}^{n^2+2n+1}_{n^2+1}$.

At this point we wish to emphasize some useful facts about the embedding we have just defined.

We begin by computing the tangent and normal spaces to CH^n at any point. Identifying CH^n with its image under φ and making as in [6], p. 306, we get

$$(1.6)\qquad \begin{aligned} T_A(CH^n) &= \{X \in H^1(n+1) \mid XA+AX=X\} \\ T_A^{\perp}(CH^n) &= \{X \in H^1(n+1) \mid XA=AX\} \end{aligned}$$

Choosing now any Q of $H^1(n+1)$, its tangential component to CH^n at the point A is

$$(1.7)\qquad QA+AQ+4\tilde{g}(A, Q)A.$$

The complex structure of CH^n is given by

$$(1.8)\qquad JX=\sqrt{-1}(I-2A)X$$

for any X in $T_A(CH^n)$, where I is the identity matrix of order $n+1$.

We denote by $\tilde{\sigma}$, $\tilde{S}$ and $\tilde{H}$ the second fundamental form, the shape operator and the mean curvature vector field associated to φ, respectively. Then if Z is a normal vector to CH^n and A is a point of CH^n, it is not hard to check that

$$(1.9)\qquad \begin{aligned} \tilde{\sigma}(X, Y) &= (XY+YX)(I-2A) \\ \tilde{S}_Z X &= (XZ-ZX)(I-2A) \end{aligned}$$

$$(1.10)\qquad \tilde{\sigma}(JX, JY)=\tilde{\sigma}(X, Y)$$

$$(1.11)\qquad \tilde{H}_A=-(2/n)(I-(n+1)A)$$

for any X, Y tangent to CH^n at the point A.

Hence from (1.10) and Codazzi equation we obtain

$$(1.12)\qquad \nabla\tilde{\sigma}=0,$$

i. e., $\tilde{\sigma}$ is parallel. Now from (1.5) and (1.9) we get

$$(1.13)\qquad \tilde{g}(\tilde{\sigma}(X, Y), I)=0, \qquad \tilde{g}(\tilde{\sigma}(X, Y), A)=-\tilde{g}(X, Y)$$

for any A in CH^n and X, Y in $T_A(CH^n)$. Finally by using that CH^n has holomorphic sectional curvature -4, and Gauss formula, we conclude from (1.8) that

$$(1.14)\qquad \begin{aligned} \tilde{g}(\tilde{\sigma}(X, Y), \tilde{\sigma}(V, W)) = &-2\tilde{g}(X, Y)\tilde{g}(V, W)-\tilde{g}(X, W)\tilde{g}(Y, V) \\ &-\tilde{g}(X, V)\tilde{g}(Y, W)-\tilde{g}(X, JW)\tilde{g}(Y, JV) \\ &-\tilde{g}(X, JV)\tilde{g}(Y, JW), \end{aligned}$$

for any X, Y, V, W of $T_A(\boldsymbol{C}H^n)$.

Perhaps to complete this paragraph we should make you notice some observations which fit in nicely with our discussion. Firstly we consider the Laplacian Δ associated to the Bergman metric of $\boldsymbol{C}H^n$. We take the sign of Δ so as in the standard Euclidean case $\Delta=-\sum_{i=1}^n \partial^2/\partial x_j^2$. Calling χ to the position vector in $H^1(n+1)$ from the point $(1/(n+1))I$, i.e., $\chi([z])=\varphi([z])-(1/(n+1))I$, $[z]\in\boldsymbol{C}H^n$, and using (1.11) we obtain $\Delta\chi=-4(n+1)\chi$ and therefore φ embeds $\boldsymbol{C}H^n$ as a minimal submanifold of the umbilical hypersurface

$$U=\{X\in H^1(n+1)\,|\,\tilde{g}(X-(1/(n+1))I,\ X-(1/(n+1))I)=-n/2(n+1)\}.$$

Identifying $H^1(n+1)$ with the pseudo-Euclidean space $\boldsymbol{R}_{n^2+1}^{n^2+2n+1}$, U is an indefinite hyperbolic space $H_{n^2}^{n^2+2n}$ of sectional curvature $-2(n+1)/n$. Notice that since $\boldsymbol{C}H^n$ is a space-like submanifold of $\boldsymbol{R}_{n^2+1}^{n^2+2n+1}$ (and with negative definite normal bundle) it can not be a minimal submanifold of an indefinite sphere (see [2], Theorem 1). The intersection of U and $H_*^1(n+1)$ (seen as $\boldsymbol{R}_{n^2}^{n^2+2n}$) is the indefinite hyperbolic space $H_{n^2-1}^{n^2+2n-1}$ and $\boldsymbol{C}H^n$ is fully embedded as a minimal submanifold in $H_{n^2-1}^{n^2+2n-1}$. In particular for $n=1$ φ is the usual isometric embedding of the real hyperbolic plane $H_0^2=\boldsymbol{C}H^1$, with Gauss curvature -4, into the Lorentz Minkowski space $\boldsymbol{R}_1^3$.

REMARK 1.2. The isometric embedding φ of $\boldsymbol{C}H^n$ in $H_*^1(n+1)$ is full, and has parallel second fundamental form. In addition the normal vector field ξ defined by $\xi_A=A$, $A\in\boldsymbol{C}H^n$, satisfies $\tilde{S}_\xi X=-X$ for any $X\in T_A(\boldsymbol{C}H^n)$ and any $A\in\boldsymbol{C}H^n$. This means that φ satisfies for any A in $\boldsymbol{C}H^n$ conditions $C_1(A)$ and $C_2(A)$ of [9], p. 739. Then identifying $H_*^1(n+1)$ with the pseudo-Euclidean space $\boldsymbol{R}_{n^2}^{n^2+2n}$, $\boldsymbol{C}H^n$ turns out to be a pseudo-Riemannian symmetric R-space in $\boldsymbol{R}_{n^2}^{n^2+2n}$. In fact setting $\{x, y, z\}=-2(\bar{y}^t\cdot z)x-2(\bar{y}^t\cdot x)\cdot z$ and $\langle x, y\rangle=\mathrm{Real}(\bar{x}^t\cdot y)$ for any $x, y, z\in\boldsymbol{C}^n$ we have that $(\boldsymbol{C}^n, \{\ \}, \langle\ \rangle)$ is a orthogonal Jordan triple system in the sense of [9], p. 736, which satisfies condition (S) of [9], p. 739, and the pseudo-Riemannian symmetric R-space associated to this orthogonal Jordan triple system is precesily $\boldsymbol{C}H^n$ in $\boldsymbol{R}_{n^2}^{n^2+2n}$.

2. Real hypersurfaces of $\boldsymbol{C}H^n$. We would like to study real hypersurfaces of $\boldsymbol{C}H^n$ by making use of the embedding φ. To do that it is worth taking the time to compute several formulae in some detail. They will be needed in the next paragraphs. Whenever we mention $\boldsymbol{C}H^n$ from now on, we shall assume $n\geqq 2$ and it is endowed with the Bergman metric of holomorphic sectional curvature -4. Suppose M is a real hypersurface of $\boldsymbol{C}H^n$ and represent by $\bar{\nabla}$ and ∇ the Levi-Civita connections of $\boldsymbol{C}H^n$ and M respectively. The Gauss and

Weingarten formulae are written as:

$$(2.1)\qquad \bar{\nabla}_X Y=\nabla_X Y+g(SX, Y)N; \qquad \bar{\nabla}_X N=-SX$$

where $\sigma(X, Y)=g(SX, Y)N$ is the second fundamental form of M in $\boldsymbol{C}H^n$, X, Y are tangent vector fields to M, N is a local unit normal vector field to M, and S is the shape operator associated to N.

If J is the complex structure of $\boldsymbol{C}H^n$, then for any tangent vector X to M we put

$$(2.2)\qquad JX=\phi X+f(X)N$$

where ϕX and $f(X)N$ are respectively the tangent and normal components of JX. As it is well known ϕ is a (1, 1) tensor field and f is a 1-form over M. It is not hard to check the following relations:

$$(2.3)\qquad \begin{gathered} f(X)=-g(X, JN);\ f(\phi X)=0;\ \phi^2 X=-X-f(X)JN \\ g(\phi X, Y)+g(X, \phi Y)=0 \\ g(\phi X, \phi Y)=g(X, Y)-f(X)f(Y) \end{gathered}$$

for any X, Y tangent to M.

Now we return to the immersion Φ of M in $\boldsymbol{C}H^n$. If we compose this immersion with the embedding φ given in (1.4) we obtain again an isometric immersion χ of M in $H^1(n+1)$. Calling H and $\hat{H}$ to the mean curvature vector fields of Φ and χ respectively, one gets:

$$(2.4)\qquad H=(1/(2n-1))\textstyle\sum_i \sigma(E_i, E_i)=(1/(2n-1))\sum_i h_{ii}N$$

$$(2.5)\qquad \hat{H}_A=H_A-(1/(2n-1))\{4(I-(n+1)A)+\tilde{\sigma}(N, N)\}$$

at any point A of M, σ being as before the second fundamental form of M in $\boldsymbol{C}H^n$; $h_{ij}=g(SE_i, E_j)$; $\{E_1, \cdots, E_n\}$ is a local orthonormal basis of tangent vector fields to M and $\tilde{\sigma}$ is the second fundamental form of φ.

Moreover using (1.13) we obtain

$$(2.6)\qquad \tilde{g}(\hat{H}, A)=-1$$

for any point A in M, and also

$$(2.7)\qquad \|\hat{H}\|^2=\|H\|^2-4(2n^2-1)/(2n-1)^2.$$

One should notice that $\|\hat{H}\|^2=\tilde{g}(\hat{H}, \hat{H})$ need not be non-negative because the restriction of $\tilde{g}$ to the normal bundle of M in $H^1(n+1)$ is not definite (compare (2.7) with formula (2.21) of [13]).

By a symilar computation to that of [13], pp. 187-188, we are able to obtain:

PROPOSITION 2.1. *Let M be a real hypersurface of CH^n. If D denotes the Levi-Civita connection of the pseudo-Euclidean space $H^1(n+1)$, S_H the shape operator of M associated to H, $\nabla^\perp$ the normal connection of M in CH^n, $\alpha=g(H, N)$, and Δ the Laplacian of M, then the following relations hold*:

$$(2.8)\qquad D_{E_i}\hat{H}=-S_H E_i+\nabla^\perp_{E_i}H+\tilde{\sigma}(E_i, H)+[2(2n+1)/(2n-1)]E_i$$
$$+[2/(2n-1)]\tilde{\sigma}(SE_i, N)-[2/(2n-1)]g(JN, E_i)JN$$

for any $i=1, \cdots, 2n-1$.

$$(2.9)\qquad (\Delta\hat{H})_A=-[4/(2n-1)]JSJN-\{2(3n+1)-\|\sigma\|^2\}H$$
$$+[8(2n+1)/(2n-1)](I-(n+1)A)$$
$$-\{-4(n+1)/(2n-1)+(2n-1)\|H\|^2+(2/(2n-1))\|\sigma\|^2\}\tilde{\sigma}(N, N)$$
$$+2\alpha\textstyle\sum_i\tilde{\sigma}(E_i, SE_i)+(2/(2n-1))\sum_i\tilde{\sigma}(SE_i, SE_i)$$
$$+2S(\mathrm{grad}\,\alpha)+\alpha\textstyle\sum_{i,k}h_{iik}E_k-2\tilde{\sigma}(\mathrm{grad}\,\sigma, N)$$
$$-(2/(2n-1))\textstyle\sum_{i,j}h_{iij}\tilde{\sigma}(E_j, N)+(\Delta\alpha)N$$

at each point A of M. Where $\|\sigma\|^2$ is the square of the length of σ, $\mathrm{grad}\,\alpha$ represents the gradient of α on M and $h_{ijk}=g((\hat{\nabla}_{E_k}\sigma)(E_i, E_j), N)$ and $\hat{\nabla}\sigma$ is the usual Van der Waerden-Bortolotti covariant derivative of the second fundamental form σ of M in CH^n.

In particular, taking $\|H\|$ constant, formula (2.9) can be rewrite as

$$(2.10)\qquad (\Delta\hat{H})_A=(-4/(2n-1))JSJN-\{2(3n+1)-\|\sigma\|^2\}H$$
$$+(8(2n+1)/(2n-1))(I-(n+1)A)+2\alpha\textstyle\sum_i\tilde{\sigma}(E_i, SE_i)$$
$$+(2/(2n-1))\textstyle\sum_i\tilde{\sigma}(SE_i, SE_i)$$
$$-\{-4(n+1)/(2n-1)+(2n-1)\|H\|^2+(2/(2n-1))\|\sigma\|^2\}\tilde{\sigma}(N, N)$$

at each point A of M.

Now we wish to obtain a specially useful formula in our context. By direct computation we obtain from (2.9), (1.13) and (1.14) the following:

PROPOSITION 2.2. *For any real hypersurface M of CH^n the following formula holds*

$$(2.11)\qquad \tilde{g}(\Delta\hat{H}, A)=-(2n-1)\|H\|^2+(4(2n^2-1)/(2n-1)).$$

We should point out that it is possible to obtain formula (2.11) in a different way. Indeed, using (2.6) one sees that the function h defined by $h(A)=g(\hat{H}, A)$ for any A of M is constant. Since $(\Delta h)_A=g(\Delta\hat{H}, A)+(2n-1)g(\hat{H}, \hat{H})$

one gets

$$g(\Delta\hat{H}, A)+(2n-1)g(\hat{H}, \hat{H})=0 \tag{2.12}$$

for A in M. Combining (2.7) and (2.12) one derives (2.11). Observe also that (2.12) remains valid for any submanifold of a pseudo-Euclidean space provided h is constant.

We close this section by analyzing the behavoir of the Laplacian Δ of a real hypersurface M of $\boldsymbol{C}H^n$. We are going to obtain some formulae which will be used in § 3.

Let M' be the Lorentzian hypersurface of the anti-De Sitter space H_1^{2n+1} which is a S^1-bundle on M compatible with the fibration π of H_1^{2n+1} on $\boldsymbol{C}H^n$. Since $\pi: M'\to M$ is a Riemannian submersion with totally geodesic fibers, we have

$$(\Delta f)\circ\pi=\Delta'(f\circ\pi) \tag{2.13}$$

for any differentiable function f on M, where Δ' represents the Laplacian of the Lorentz metric on M'.

If we agree to represent by D° and D the Levi-Civita connections of the pseudo-Euclidean space $\boldsymbol{R}_2^{2n+2}$ and H_1^{2n+1} respectively, then one has

$$D_X^\circ Y=D_X Y+g(X, Y)\chi \tag{2.14}$$

which is nothing but the Gauss formula for H_1^{2n+1} in $\boldsymbol{R}_2^{2n+2}$, for any X, Y tangent to H_1^{2n+1}, where g is the Lorentz metric on H_1^{2n+1} (see § 1), and χ being the position vector at each point.

On the other hand, from the Gauss formula of M' in H_1^{2n+1} one has

$$D_X Y=\nabla'_X Y+\sigma'(X, Y) \tag{2.15}$$

for any X, Y tangent to M', where ∇' and σ' are the Levi-Civita connection and the second fundamental form of M' in H_1^{2n+1} respectively. Now we use formulae (2.14) and (2.15) to compute the Hessian of a differentiable function f on M', represented by Hess (f),

$$\begin{aligned}\mathrm{Hess}(f)(X, Y)=&X(Y(f))-(D_X^\circ Y)(f)\\&+g(X, Y)\chi(f)+\sigma'(X, Y)(f)\end{aligned} \tag{2.16}$$

for any tangent vector fields X, Y to M'.

Remenbering the known formula $\Delta' f=-\mathrm{trace}_g \mathrm{Hess}(f)$ and using (2.16), it turns out that for any function f on M' which is restriction of a linear function of $\boldsymbol{R}_2^{2n+2}$ in $\boldsymbol{R}$ we get

$$\Delta' f=-2nf-2nH'(f) \tag{2.17}$$

H' being the mean curvature vector field of M' in H_1^{2n+1}.

Finally, f being as above the restriction to M' of a linear function $\tilde{f}$ on $\boldsymbol{R}_2^{2n+2}$ we get

$$\|\nabla' f\|^2 = \|D^\circ \tilde{f}\|^2 + f^2 - (\xi'(f))^2 \tag{2.18}$$

$\nabla' f$ and $D^\circ \tilde{f}$ being respectively the gradient of f in M' and the gradient of $\tilde{f}$ in $\boldsymbol{R}_2^{2n+2}$, and where ξ' is a unit normal vector field to M' in H_1^{2n+1}.

3. Examples. In this paragraph we present two families of examples, constructed in [8], which will play a fundamental and quite different role in the final sections. Example 3.2 give us a family $M_{2p+1,2q+1}(r)$, $0<r<1$, $p+q=n-1$, of real hypersurfaces of $\boldsymbol{C}H^n$ whose immersion in $H^1(n+1)$ via φ (stated in § 1) is never mono-order. We recall that an isometric immersion χ into a pseudo-Euclidean space is said to be mono-order if $\chi=\chi_0+y$ where χ_0 is a constant vector and y satisfies $\Delta y=\lambda y$, $\lambda\in\boldsymbol{R}$, with Δ the Laplacian of the submanifold. This property of $M_{2p+1,2q+1}(r)$ is essential in our proof of Theorem 4.2, and it contrasts with the complex projective case, where the analogue family of real hypersurfaces of $\boldsymbol{C}P^n$ has some members which are mono-order via the first standard embedding of $\boldsymbol{C}P^n$. On the other hand, the horosphere M_n^* of the Example 3.1 shows what at first sight is a rather strange behaviour: the second Laplacian of its immersion in $H^1(n+1)$, via φ, is a non-zero constant matrix (that will be proved below in this § 3). Moreover M_n^* has constant mean curvature in $\boldsymbol{C}H^n$. Therefore it seems interesting to study the family of real hypersurfaces of $\boldsymbol{C}H^n$ verifying these conditions. This problem will be treated in last paragraph. Meanwhile we analize these examples in some detail.

EXAMPLE 3.1. Let M_n' the Lorentz hypersurface of the anti-De Sitter space H_1^{2n+1} defined by

$$M_n' = \{z\in H_1^{2n+1} \mid |z_0 - z_1|^2 = 1\}$$

where $z=(z_0, z_1, \cdots, z_n)\in\boldsymbol{C}^{n+1}$ as in § 1. If $z\in M_n$ then

$$\xi_z = (-z_1, z_0 - 2z_1, -z_2, \cdots, -z_n)$$

is a unit vector normal to M_n' at the point z. The corresponding shape operator S' is given by

$$S'(a_0, a_1, \cdots, a_n) = (a_1, 2a_1 - a_0, a_2, \cdots, a_n) \tag{3.1}$$

for any $(a_0, a_1, \cdots, a_n)\in\boldsymbol{C}^{n+1}$ representing a tangent vector to M_n' at the point z.

From (3.1) it is easy to see that trace$(S')=2n$, and so formula (2.17) gives in this case

(3.2) $$\Delta' f = -2nf - 2n(\xi(f))^2$$

for any function $f: M_n' \to \boldsymbol{R}$ which is the restriction of a linear one from $\boldsymbol{R}_2^{2n+2}$ to $\boldsymbol{R}$.

M_n' is S^1-invariant and $M_n^* = \pi(M_n')$ is a real hypersurface of $\boldsymbol{CH}^n$, and $\pi: M_n' \to M_n^*$ is a Riemannian submersion with totally geodesic fibers. Using (3.2) we obtain

(3.3) $$\Delta' z = -2n(z_0 - z_1, z_0 - z_1, 0, \cdots, 0)$$

where we have written $z = (z_0, z_1, \cdots, z_n)$ for the position vector of M_n'. From this last equation we can compute the mean curvature vector field H of M_n' in $\boldsymbol{R}_2^{2n+2}$

(3.4) $$H = (z_0 - z_1, z_0 - z_1, 0, \cdots, 0) = \xi_z + z$$

and then H is an isotropic vector at any point. From (3.2) again together with (3.4) one sees that

(3.5) $$\Delta' H = 0.$$

Now let us write Δ and $\hat{H}$ for the Laplacian of M_n^* and the mean curvature vector field of M_n^* in $H^1(n+1)$ via the embedding φ stated in §1. Thus making use of (1.4), (2.13), (2.18) and (3.2), and the usual properties of Δ acting on the product of functions, we have

(3.6) $$\Delta \hat{H} = Q$$

where we have put

$$Q := -8(n^2-1)/(2n-1) \left(\begin{array}{cc|c} 1 & -1 & \\ 1 & -1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array} \right)$$

EXAMPLE 3.2. Given integers p, q with $p+q = n-1$ and $r \in \boldsymbol{R}$ with $0 < r < 1$, we denote by $M_{2n+1,2q+1}'(r)$ the Lorentz hypersurface of the anti-De Sitter space H_1^{2n+1} defined by

$$M_{2p+1,2q+1}'(r) = \left\{ z \in H_1^{2n+1} \,|\, r\left(-|z_0|^2 + \sum_{j=1}^{p} |z_j|^2\right) = -\sum_{k=p+1}^{n} |z_k|^2 \right\}$$

where $z = (z_0, z_1, \cdots, z_n)$. It is easy to see that $M_{2p+1,2q+1}'(r)$ is isometric to the riemannian product

(3.7) $$H_1^{2p+1}(-1/(1-r)) x S^{2q+1}(r/(1-r))$$

where $-1/(1-r)$ and $r/(1-r)$ are the respective equares of the radii and each factor is embedded in H_1^{2n+1} in a totally umbilical way.

If $z \in M_{2p+1,2q+1}(r)$ we can see that

$$\xi_z = (-1/\sqrt{r})(rz_0, rz_1, \cdots, rz_p, z_{p+1}, \cdots, z_n)$$

is a unit vector normal to $M'_{2p+1,2q+1}(r)$ at the point z.

$M'_{2p+1,2q+1}(r)$ is S^1-invariant, $M_{2p+1,2q+1}(r) = \pi(M'_{2p+1,2q+1}(r))$ is a real hypersurface of CH^n, and π from $M'_{2p+1,2q+1}(r)$ onto $M_{2p+1,2q+1}(r)$ is a Riemannian submersion with totally geodesic fibers.

Representing by (z, w) a generic point of $M'_{2p+1,2q+1}(r)$ and from (3.7) we can rewrite formula (1.4) as

$$\chi = \varphi([z, w]) = \left(\begin{array}{c|c|c} |z_0|^2 & \cdot & \cdot \\ \cdot & -z_i\bar{z}_j & -z_i\bar{w}_k \\ \cdot & \cdot & \cdot \\ z_p\bar{z}_0 & \cdot & \cdot \\ \hline w_0\bar{z}_0 & \cdot & \cdot \\ \cdot & -w_h\bar{z}_j & -w_h\bar{w}_k \\ \cdot & \cdot & \cdot \\ w_q\bar{z}_0 & \cdot & \cdot \end{array}\right)$$

where $0 \leqq i \leqq p$, $0 \leqq h$, $k \leqq q$, $0 \leqq j \leqq p$. In order to simplify computations we put this formula in the following way

$$\chi = \varphi([z, w]) = \left(\begin{array}{c|c} a_{\alpha\beta} & b_{\alpha\gamma} \\ \hline c_{\delta\beta} & d_{\delta\gamma} \end{array}\right) \tag{3.8}$$

with $0 \leqq \alpha$, $\beta \leqq p$ and $0 \leqq \gamma$, $\delta \leqq q$.

We use now the properties of the Laplacian of a Riemannian product, along with formulas (2.17), (2.18) and (3.7) to calculate the Laplacian of χ (χ being the position vector of $M_{2p+1,2q+1}(r)$ in $H^1(n+1)$ via φ). If $p, q > 0$ one has

$$\Delta\chi = \left(\begin{array}{c|c} 2(p+1)(r-1)a_{\alpha\beta} + 4I_{p+1} & (r-1)(p-(q/r))b_{\alpha\gamma} \\ \hline (r-1)(p-(q/r))c_{\delta\beta} & 2(q+1)(1-r)(1/r)d_{\delta\gamma} + 4I_{q+1} \end{array}\right) \tag{3.9}$$

If $p=0$ and $q>0$ we have

$$\Delta\chi = \left(\begin{array}{c|c} 0 & (r-1)(1-(2n-1)/r)b_{0\gamma} \\ \hline (r-1)(1-(2n-1)/r)c_{\delta 0} & 4n((1-r)/r)d_{\delta\gamma} + 4I_n \end{array}\right) \tag{3.10}$$

Finally, if $p>0$ and $q=0$ we have

$$\Delta\chi = \left(\begin{array}{c|c} 4n(r-1)a_{\alpha\beta} + 4I_n & (r-1)(1-(2n-1)/r)b_{\alpha 0} \\ \hline (r-1)(1-(2n-1)/r)c_{0\beta} & 0 \end{array}\right) \tag{3.11}$$

Compare (3.9), (3.10) and (3.12) with [13] formulae (3.4) and (3.5).

Since $0<r<1$ it is easy to see from (3.9), (3.10) and (3.11) that no hypersurface of the family $M_{2p+1,2q+1}(r)$ is mono-order in $H^1(n+1)$. Note that for $p=0$, $M_{1,2n-1}(r)$ is a geodesic hypersphere of CH^n, [8], p. 255. On the other hand certain geodesic hyperspheres of the complex projective space $\boldsymbol{C}P^n$ are mono-order via the first standard embedding. Furthermore this property characterize them completely, [6], Theorem 3.1.

REMARK 3.3. Given an isometric immersion χ of an indefinite Riemannian manifold M in a pseudo-Euclidean space, χ is said of finite type, [2], if the position vection vector χ admits a decomposition $\chi=\chi_0+\chi_{p_1}+\cdots+\chi_{p_k}$ where χ_0 is a constant vector and $\Delta\chi_{p_i}=\lambda_i\chi_{p_i}$, $\lambda_i\in\boldsymbol{R}$ and Δ being the Laplacian of M. In this situation if we additionally have $\Delta H=0$, with H the mean curvature vector field of χ, then we conclude from the linear independence of the eigenfunctions χ_{p_i} that $H=0$. Hence, from this and (3.5) we deduce that M_n' of Example 3.1 is not of finite type in $\boldsymbol{R}_2^{2n+2}$. Similarly from (3.6) we see that the horosphere M_n^* is not of finite type in $H^1(n+1)$. Finally, using (3.8), (3.9) and (3.10), and because p, q are integers and $0<r<1$, we obtain that certain hypersurfaces of the family $M_{2p+1,2q+1}(r)$ are of 2-type and some of 3-type.

4. Mono-order real hypersurfaces.

As we have already said, we will show in this section the non-existence of real hypersurfaces of $\boldsymbol{C}H^n$ whose immersion in the pseudo-Euclidean space $H^1(n+1)$ via the embedding φ stated in § 1, is mono-order.

In trying to find a satisfying proof of this fact we followed two different ways. Firstly we took in consideration the proof of Martinez and Ros for real hypersurfaces of the complex projective space $\boldsymbol{C}P^n$, [6], Theorem 3.1. Keeping this in mind we found a proof similar to that of [6]. To do that, we needed a classification result for the matrices of $H^1(n+1)$ up to the action of the group $U^1(n+1)$. This is shown in Lemma 5.1. The first case of that Lemma is treated in the same form as that of the proof in [6], leading to the fact that our hypersurface should be locally one of the family $M_{2p+1,2q+1}(r)$. But this is impossible as Example 3.2 shows. As for the rest of posibilities of Lemma 5.1, we can get rid of them on the base of algebraic considerations.

What we shall discuss now in detail is our second proof. As we shall notice soon, we are able of adapting this new proof to the case of real hypersurfaces in $\boldsymbol{C}P^n$, giving an alternative proof of Theorem 3.1 in [6]. We begin by rewritting a particular case of Theorem 7.4 of [7] in a more suitable way. Let M be a real hypersurface of $\boldsymbol{C}H^n$, $n\geqq 2$, we say it is η-umbilical if there exist real numbers a, b such that $SX=aX+bg(JN, X)JN$, where S is the shape operator of M in CH^n associated to N and X is any tangent vector field to M. Then we have:

PROPOSITION 4.1 (see [7], Theorem 7.4). *Let M be a connected real hypersurface of CH^n, $n\geqq 2$. Suppose M is η-umbilical. Then M is locally congruent to one of the following spaces:*

(a) *A geodesic hypersphere $M_{0,2n-1}(r)$.*

(b) *A real hypersurface $M_{2n-1,0}(r)$.*

(c) *A horosphere M_n^*.*

PROOF. Notice firstly that a η-umbilical hypersurface has at most two principal curvatures at each point. Go now to Theorem 7.4 of [7]. Case "*d*" of that Theorem is not possible now because it is not η-umbilical. On the other hand, hypothesis $n\geqq 3$ of that Theorem is used only to assure that JN is a principal vector with constant principal curvature. We can deduce both facts from the η-umbilical condition. Finally note that although Theorem 7.4 of [7] is formulated for complete hypersurfaces, it is local in nature because it is proved on the base of Theorem 4.1 of [7].

The above Proposition leads in turn to

THEOREM 4.2. *There exist no real hypersurfaces M of CH^n, $n\geqq 2$, which are minimal in either a (n^2+2n)-dimensional indefinite sphere or in a (n^2+2n)-dimensional indefinite real hyperbolic space of $H^1(n+1)$, that is to say, no real hypersurface M of CH^n is mono-order in $H^1(n+1)$ via the isometric embedding φ stated in* § 1.

PROOF. Suppose M is mono-order in $H^1(n+1)$, then from [2], Theorem 1, the mean curvature vector field $\hat{H}$ of M in $H^1(n+1)$ satisfies

$$\hat{H}=(-1/(2n-1))\Delta\chi=(-1/(2n-1))\lambda(\chi-A_0) \tag{4.1}$$

with $\lambda\neq 0$, where $\chi=\varphi\circ f$, f being the isometric immersion of M in CH^n and φ the embedding of CH^n in $H^1(n+1)$ given in § 1.

Using (2.8) we have from (4.1)

$$\begin{aligned} D_X\hat{H}= & -\alpha SX+\nabla^{\perp}{}_X H+\tilde{\sigma}(X, H)+(2(2n+1)/(2n-1))X \\ & +(2/(2n-1))\tilde{\sigma}(SX, N)-(2/(2n-1))g(JN, X)JN \end{aligned} \tag{4.2}$$

with the same notations as those of Proposition 2.1.

From (4.1) we have $\Delta\hat{H}=\lambda\hat{H}$ and putting this in (2.11) we get

(4.3) $$\lambda=(2n-1)\|H\|^2-4(2n^2-1)/(2n-1)$$

having used (2.6) also. But (4.3) tell us that $\|H\|^2=\alpha^2$ is constant and therefore

(4.4) $$\nabla^{\perp}{}_X H=0$$

for any X tangent to M. Combining (4.1) and (4.4) we can rewrite (4.2) in the following way

(4.5) $$-(\lambda/(2n-1))X=-\alpha SX+\tilde{\sigma}(X, H)+(2(2n+1)/(2n+1))X$$
$$+(2/2n-1))\tilde{\sigma}(SX, N)-(2/(2n-1))g(JN, X)JN$$

so that equaling tangential components

(4.6) $$-(\lambda/(2n-1))X=-\alpha SX+(2(2n+1)/(2n-1))X$$
$$-(2/(2n-1))g(JN, X)JN$$

for any X tangent to M.

We can assume $\alpha\neq 0$. Otherwise from (4.6), one gets

(4.7) $$(2/(2n-1)g(JN, X)JN=((2(2n+1)+\lambda)/(2n-1))X$$

for any X tangent to M. Choosing $X\in(\mathrm{Span}\{JN\})^{\perp}$ we have

(4.8) $$2(2n+1)+\lambda=0$$

Choosing $X=JN$ we obtain from (4.7) again

(4.9) $$2=2(2n+1)+\lambda$$

which are clearly incompatible. Hence $\alpha\neq 0$ and from (4.6) we can write

(4.10) $$SX=aX+bg(JN, X)JN$$

where a, b are real numbers given by

$$a=((2(2n+1)+\lambda)/(2n-1)\alpha \quad \text{and} \quad b=-2/(2n-1)\alpha .$$

Showing that M is η-umbilical. Using Proposition 4.1 we should have that M is an open subset either a geodesic hypersphere $M_{0,2n-1}(r)$ or the horosphere M_n^*. But Examples 3.1 and 3.2 shows that this is not possible, and this concludes the proof of Theorem 4.2.

By using the same method as that of the our Theorem 4.2, and taking into account formula (2.14) of [13], one can prove that a real hypersurface of the complex projective space $\boldsymbol{C}P^n$ which is minimal in a hypersphere of $\boldsymbol{R}^{n^2+2n+1}$ via the first standard embedding, is necessarily η-umbilical. On the other hand, as a consequence of Theorem 3 of [1], a η-umbilical real hypersurface of $\boldsymbol{C}P^n$, $n\geqq 2$, is an open subset of a geodesic hypersphere of $\boldsymbol{C}P^n$. But the only such hypersurfaces which is mono-order via the first standard embedding is the projection by the Hopf fibration of $S^1(\sqrt{(1/(2n+2))}) x S^{2n-1}(\sqrt{(2n+1)/(2n+2)})$, where $\sqrt{(1/(2n+2)}$ and $\sqrt{(2n+1)/(2n+2)}$ are the respective radii, [6], p. 309. This gives another proof of Theorem 3.1 of [6].

5. Real hypersurfaces of CH^n satisfying a certain differential equation. This section is mainly concerned with a characterization of the horosphere M_n^* of the complex hyperbolic space $\boldsymbol{C}H^n$ by means of a certain differential equation. The touchstone for its characterization is the behaviour of the Laplacian of its mean curvature vector in $H^1(n+1)$. As we proved in § 3 the Laplacian of the mean curvature vector field $\hat{H}$ of M_n^* in $H^1(n+1)$ via the embedding φ (stated in § 1) satisfies

$$\Delta\hat{H}=Q \tag{5.1}$$

where Δ is the Laplacian of M_n^* and $Q \in H^1(n+1)$ is a non-zero constant matrix normal to M_n^* at every point. As we mentioned earlier § 3 this situation is not possible for real hypersurfaces of the complex projective space $\boldsymbol{C}P^n$ ($\hat{H}$ denoting the mean curvature vector field via the first standard embedding). Even although one considers a submanifold M of an Euclidean space $\boldsymbol{R}^m$ satisfying $\Delta H=Q$, with Q a constant vector of $\boldsymbol{R}^m$ which is normal to M at any point, we shall prove soon Lemma 5.2 that Q has to be zero. Note that in this case M can not be compact, because if M in addition was compact then H would be zero necessarily and this is not possible. The case $\Delta\hat{H}=0$ is equaly impossible for real hypersurfaces of $\boldsymbol{C}P^n$ (for this consider formula (2.22) of [13]).

In contrast there may be submanifolds in a pseudo-Euclidean space $\boldsymbol{R}_s^m$ satisfying $\Delta H=0$ and $H \neq 0$. It is enough to consider the submanifold M_n' of $\boldsymbol{R}_2^{2n+2}$ described in Example 3.1. We shall treat here the case when $Q \neq 0$. This leaves open the possibility of studying the real hypersurfaces of $\boldsymbol{C}H^n$ verifying $\Delta\hat{H}=0$. (Compare our (2.11) with the above mentioned (2.22) of [13]). We start with a technical Lemma which is an adaptation of Theorem 2, p. 229, of [5], for our purposes, and hence it will not be proved.

LEMMA 5.1. *Let V be an n-dimensional complex vector space, and F a non-degenerate Hermitian form of index two on V. Let f be an F-selfadjoint (complex) endomorphism of V. Then there exists a basis B of V in such way that the matrix of f relative to B, $M(f, B)$, and the matrix of F with respect to B, $M_B(F)$, fall in one of the following cases:*

(i) *$M(f, B)$ real diagonal;* $M_B(F)=\left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & I_{n-1} \end{array}\right)$

(ii) $M(f, B)=\left(\begin{array}{cc|c} a & 1 & 0 \\ 0 & a & \\ \hline \multicolumn{2}{c|}{0} & D \end{array}\right)$; $M_B(F)=\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)$

where $a\in\boldsymbol{R}$ and D a square real diagonal matrix of order $n-2$.

(iii) $M(f, B)=\left(\begin{array}{ccc|c} a & 0 & 1 & \\ 0 & a & 0 & 0 \\ 0 & 1 & a & \\ \hline \multicolumn{3}{c|}{0} & D' \end{array}\right)$; $M_B(F)=\left(\begin{array}{ccc|c} 0 & 1 & 0 & \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \\ \hline \multicolumn{3}{c|}{0} & I_{n-3} \end{array}\right)$

with $a\in\boldsymbol{R}$ and D' a square real diagonal matrix of order $n-3$.

(iv) $M(f, B)=\left(\begin{array}{cc|c} a & b & 0 \\ -b & a & \\ \hline \multicolumn{2}{c|}{0} & D'' \end{array}\right)$; $M_B(F)=\left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1 & \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)$

with $a, b\in\boldsymbol{R}$, $b\neq 0$ and D'' a square real diagonal matrix of order $n-2$.

Note that if f satisfies a polynomial of degree 2, the case (iii) is impossible. On the other hand if $f^2=0$, the case (iv) is not possible too, and either $f=0$ or case (ii) occurs with $a=0$. In this latter case $M(f, B)=\mathrm{diag}\left(\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix};\ 0\right)$ and $M_{\tilde{B}}(F)=\mathrm{diag}(-1\,;\ I_{n-1})$ for a suitable basis $\tilde{B}$ obtained from B of (ii).

Now we prove:

PROPOSITION 5.2. *Let M be a submanifold of a pseudo-Euclidean space $\boldsymbol{R}_s^m$. Denote by H and Δ the mean curvature vector field of M in $\boldsymbol{R}_s^m$ and the Laplacian of M respectively. Suppose that $\Delta H=C$ for some constant vector C of $\boldsymbol{R}_s^m$, which is normal to M at every point. Then either $C=0$ or C is an isotropic vector.*

PROOF. Let g be the usual flat metric of $\boldsymbol{R}_s^m$, and suppose $g(C, C)\neq 0$. Note that $g(H, C)$ can not be a constant function on M, otherwise $0=\Delta g(H, C)$ $=g(\Delta H, C)=g(C, C)$, which contradicts our assumption. Then choose a unitary vector η of $\boldsymbol{R}_s^m$ so that $C=ag(C, \eta)\eta$, $a=+1$ or -1. We can write H as

$$H=(1/n)\sum_{i=1}^{m-n} g(\xi_i, \xi_i)(\mathrm{trace}(S_i))\xi_i$$

where $\{\xi_1=\eta, \xi_2, \cdots, \xi_{m-n}\}$ is a local orthonornal basis formed by normal vector fields to M, and S_i is the shape operator associated to ξ_i. Since η is constant, we have $S_1=0$ and therefore $g(H, C)=0$ which gives a contradiction and proves the Proposition.

In particular if $s=0$ in Proposition 5.2, i.e. if the ambient space is the Euclidean space $\boldsymbol{R}^m$, then we obtain $C=0$.

Now take a real hypersurface M of $\boldsymbol{C}H^n$ satisfying (5.1). Then formula (2.11) turns to be

$$\tilde{g}(Q, A)=-(2n-1)\|H\|^2+(4/(2n-1))(2n^2-1)$$

at any point A of M. So that Q is normal to M at every point if and only if $\|H\|$ is constant. This shows that Q in Example 3.1 is normal to the horosphere M_n^* of $\boldsymbol{C}H^n$.

REMARK 5.3. Let M be a real hypersurface of $\boldsymbol{C}P^n$ with constant mean curvature. Suppose that the immersion of M in $\boldsymbol{R}^{n^2+2n+1}$ via the first standard embedding of the complex projective space $\boldsymbol{C}P^n$ in $\boldsymbol{R}^{n^2+2n+1}$ satisfies the differential equation $\Delta\hat{H}=Q$, where $\hat{H}$ is the mean curvature vector field of M in $\boldsymbol{R}^{n^2+2n+1}$ and Δ its Laplacian. As before, constant mean curvature is equivalent to say that Q is normal to M(see formula (2.22) of [13]). But then using Proposition 5.2 we see that $Q=0$, which is not possible from formula (2.22) of [13]. Therefore there exist no real hypersurface of $\boldsymbol{C}P^n$ with constant mean curvature whose mean curvature vector field in $\boldsymbol{R}^{n^2+2n+1}$ satisfies differential equation (5.1).

Now we reach our goal by using some intermediate steps.

LEMMA 5.4. *Consider a real hypersurface M of $\boldsymbol{C}H^n$ with constant mean curvature. Then*

$$\tilde{g}(\Delta\hat{H}, \tilde{\sigma}(X, N))=-4\alpha\tilde{g}(SJN, \phi X)-(4/(2n-1))\tilde{g}(S^2JN, \phi X) \tag{5.2}$$

(see §2 for notations). If, in addition, there exists a function μ on M such that $SJN=\mu JN$, we have

$$\tilde{g}(\Delta\hat{H}, \tilde{\sigma}(X, N))=0 \tag{5.3}$$

where X is any tangent vector field to M.

PROOF. Use (2.10), (1.13) and (1.14) along with (2.2), (2.3) to get (5.2), (5.3) follows early from (5.2) by using condition $SJN=\mu JN$.

LEMMA 5.5. *Let M be a real hypersurface of $\boldsymbol{C}H^n$ with constant mean curvature and satisfying the differential equation $\Delta\hat{H}=Q$. Then there exists a function μ on M so that $SJN=\mu JN$.*

PROOF. From (2.11) we have that $\tilde{g}(Q, A)=r=$cte. for any A of M. But this means that Q is normal to M everywhere and therefore the tangential component of $\Delta\hat{H}$ is zero. Using now (2.10) we see that $\phi SJN=0$. Hence (2.2) says that $JSJN=\gamma N$ where γ is obtained from (2.3) as $\gamma=-g(SJN, JN)$. Therefore $SJN=\mu JN$ with $\mu=-\gamma$.

LEMMA 5.6. *Given a real hypersurface M of $\boldsymbol{C}H^n$ with constant mean curvature and verifying the differential equation $\Delta\hat{H}=Q$, then the function $\lambda: M\rightarrow \boldsymbol{R}$ defined by $\lambda(A)=\tilde{g}(Q, N_A)$ is constant.*

PROOF. From (2.11) Q is normal to M everywhere. Therefore $X(\lambda)=\tilde{g}(Q, \tilde{\sigma}(X, N))$ for X tangent to M. Hence from Lemma 5.5 and the second part of Lemma 5.4, we obtain $X(\lambda)=0$. Therefore λ is constant on M.

We finally give:

THEOREM 5.7. *Let M a real hypersurface of constant mean curvature in $\boldsymbol{C}H^n$. Suppose M satisfies the differential equation $\Delta\hat{H}=Q$, with $\hat{H}$ the mean curvature vector field of M in $H^1(n+1)$ and Q a non-zero constant matrix of $H^1(n+1)$. Then M is locally congruent in $\boldsymbol{C}H^n$ to the horosphere $M_n^*=\pi(\{z\in H_1^{2n+1} \mid |z_0-z_1|^2=1\})$ described in* § 3.

PROOF. From (1.7) and Lemma 5.6, one sees that Q^2 satisfies

$$\tilde{g}(A, Q^2)=(\lambda^2-4\beta^2)/2$$

for any point A of M, where $\lambda=g(Q, N)$ and $\beta=g(A, Q)$ are real numbers. On the other hand $g(A, I)=-1/2$ for any point A of M. Hence because M is a real hypersurface of $\boldsymbol{C}H^n$ and φ an isometric embedding which is full in $H_*^1(n+1)$ (see § 1), then there exist real numbers r_1, r_2 and r_3 so that

$$r_1Q^2+r_2Q+r_3I=0. \tag{5.4}$$

Moreover by Proposition 5.2 we have

$$\operatorname{trace} Q^2=-2\tilde{g}(Q, Q)=0, \tag{5.5}$$

and from (2.10) we obtain

$$\operatorname{trace} Q=-2\tilde{g}(\Delta\hat{H}, I)=0. \tag{5.6}$$

Now (5.5) and (5.6) means that $r_3=0$ in (5.4). Let us see that r_2 is also zero. Indeed, if r_2 were non-zero then since $Q\neq 0$, r_1 would be non-zero. But in this case Q would be a diagonalizable matrix which only admits as eigenvalues to 0 and $-r_2/r_1$. This contradicts (5.5). Therefore $r_2=0$. Then (5.4) means that

Q is a non-diagonalizable matrix satisfying

(5.7) $$Q^2=0.$$

Call β to $\tilde{g}(Q, A)=-(1/2)\mathrm{trace}(QA)$ and assume β is positive (otherwise we could find $P_1\in U^1(n+1)$ so that $P_1QP_1^{-1}=-Q_1=-\mathrm{diag}\left(\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}; 0\right)$. Because $Q\in H^1(n+1)$ and verifies (5.7) we know from Lemma 5.1 that there exists $P\in U^1(n+1)$ such that

(5.8) $$PQP^{-1}=\left(\begin{array}{cc|c} -1 & 1 & \\ -1 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right)=Q_1$$

(see comentaries following Lemma 5.1). As the metric $\tilde{g}$ is $U^1(n+1)$-invariant, we have from (5.8)

$$\mathrm{trace}(Q_1A)=-|z_0-z_1|^2=-2\beta$$

for every A in M. Thus, M is locally congruent to the real hypersurface $\pi(\{z\in H_1^{2n+1}|\,|z_0-z_1|^2=2\beta\})$ of CH^n, and this one is in turn congruent to $M_n^*=\pi(\{z\in H_1^{2n+1}|\,|z_0-z_1|^2=1\})$. (See [8], Theorem 5.1). This concludes the proof of Theorem 5.7.

COROLLARY 5.8. *$M_n^*=\pi(\{z\in H_1^{2n+1}|\,|z_0-z_1|^2=1\})$ is, up to rigid motions of $\boldsymbol{CH}^n$, the unique complete real hypersurface of $\boldsymbol{CH}^n$ with constant mean curvture which satisfies the differential eqhation $\Delta\hat{H}=Q$, for a non-zero constant matrix Q.*

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Departamento de Geometria y Topologia
Facultad de Ciencias
Universidad de Granada
18071-Granada. Spain.