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GENERALIZED COHOMOLOGICAL DIMENSION OF COMPACT METRIC SPACES

Dedicated to professor Yukihiro Kodama on his 60th birthday

By

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§ 0. Introduction.

The notion of homological dimension was introduced by P. S. Alexandroff in the later 1920's. The further contribution to the development of homological dimension theory on compact metric spaces was made by L. S. Pontrjagin [2], K. Borsuk [3], M. F. Bokshtein [4], V. G. Boltynanski [5], E. Dyer [6], Y. Kodama [7, 8], V. I. Kuz'minov [9] and others. New achievement of the theory are surveyed in [10]. In this paper we do not consider homological dimension theory out of compact metric spaces. Moreover everywhere in this paper cohomological language is used instead of homological one, and therefore a dual notion of cohomological dimension is considered.

A compact metric space X has *cohomological dimension with respect to an abelian group* G *equal or less than* n (written, $c\text{-dim}_G X \leqq n$) iff for an arbitrary closed subset $A \subset X$ and an arbitrary integer $k \geqq n$, the inclusion $A \to X$ induces an epimorphism of k-dimensional Čech cohomology groups with coefficients.

ALEXANDROFF THEOREM [11]. *For every finite-dimensional compact metric space, we hold the equation* $c\text{-dim}_Z X = \dim X$.

As justifiably L. Rubin remarks in [13], we do not have enough information about infinite-dimensional spaces to determine a theory. In this paper a notion of generalized cohomological dimension $c\text{-dim}_{\boldsymbol{E}}$ for some class of spectra $\delta = \{\boldsymbol{E}\}$ is introduced. For all $\boldsymbol{E} \in \mathcal{E}$, those dimensions $c\text{-dim}_{\boldsymbol{E}}$ coincide with the covering dimension dim for finite dimensional compacta. Namely, they are distinguished in the class of infinite-dimensional compacta.

In § 1, the inequalities

$$c\text{-dim}_Z X \leqq c\text{-dim}_{\boldsymbol{E}} X \leqq \dim X$$

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are considered. By Z we will denote the Eilenberg-MacLane spectrum $\{K(Z, n)\}$ as well as the additive group of integers. In the case of sphere spectrum $\boldsymbol{S}$ one gets the notion of stable cohomotopical dimension. In § 2, an example which distinguishes $c\text{-}\dim_Z$ and $c\text{-}\dim_S$ is constructed. Also, in § 2, a strongly infinite-dimensional compactum X with $c\text{-}\dim_Z X=3$ is constructed.

The following theorem is well-known.

THEOREM [11]. *For a compact metric space X the following conditions are equivalent*:

1) $\dim X \leqq n$,

2) *for an arbitrary closed subset $A \subset X$ and any map $\varphi: A \to S^n$, there exists a continuous extension $\bar{\varphi}: X \to S^n$ of φ.*

Such a space S^n is called a *test space* for dimension n. A kind of test spaces was considered in [14]. In § 3, the following statement is proved:

The space $S^n \cup_\alpha B^{n+k+1}$ is a test space for dimension n for any $\alpha \in \pi_{n+k}(S^n)$ and $k>0$.

Here $S^n \cup_\alpha B^{n+k+1}$ is the space obtained from S^n by attaching a cell B^{n+k+1} by a representation $\partial B^{n+k+1}=S^{n+k} \to S^n$ of α.

§ 1. Definition of cohomological dimension by spectra.

The *suspension* of a space X is denoted by ΣX. A *spectrum* $\boldsymbol{E}$ [15] is a sequence of CW-complexes $\{E_n : n \in Z\}$(with base points) and embeddings ε_n: $\Sigma E_n \to E_{n+1}$. Given a spectrum $\boldsymbol{E}$ defines a *generalized cohomology theory* by the formula:

$$E^n(X)=\lim\,[\Sigma^k X, E_{n+k}].$$

Here $[X, Y]$ is the set of all homotopy classes of maps (preserving the base points) from X to Y. There is a group structure on the set $[X, Y]$ in the case of $X=\Sigma Z$ for some space Z. Let ΩX be the *loop space* on X. We define the space

$$\Omega^\infty E_{n+\infty}=\lim\{\Omega^k E_{n+k}\,;\ \Omega^k \varepsilon_n\},$$

where $\Omega^k X=\Omega(\Omega\cdots(\Omega X)\cdots)$ is the k-iterated loop space and $\tilde{\varepsilon}_n$: $E_n \to \Omega E_{n+1}$ is the *adjoint map* of ε_n. Then we have

$$E^n(X)=[X, \Omega^\infty E_{n+\infty}].$$

It can be shown that any generalized cohomology theory which satisfies some axioms can be obtained by the above way from a suitable spectrum $\boldsymbol{E}$ [15].

DEFINITION 1. A compact metric space X has cohomological dimension with respect to a spectrum E equal or less than n (written, $c\text{-}\dim_{\boldsymbol{E}} X \leqq n$) if for an arbitrary closed subset $A \subset X$, the inclusion $A \to X$ induces an epimorphism of cohomology groups E^m for $m \geqq n$.

In other words, $c\text{-}\dim_{\boldsymbol{E}} X \leqq n$ iff for an arbitrary closed subset $A \subset X$ and any map $\varphi: A \to \Omega^\infty E_{m+\infty}$, there exists a continuous extension $\bar{\varphi}: X \to \Omega^\infty E_{m+\infty}$ for $m \geqq n$.

If we choose $\boldsymbol{G} = \{K(G, n)$ for $n \geqq 0$ and a singleton for $n < 0\}$ as a spectrum $\boldsymbol{E}$, we get the notion of the ordinal cohomological dimension with respect to G as the coefficient group. Not every dimension with respect to G as the coefficient group. Every generalized cohomology theory can not define a proper dimension theory because it is periodic. This is a reason why we use some restriction on a class of spectra.

A spectrum $\boldsymbol{E}$ is called *connected* [15] if for every n,

$$\lim_{k \to \infty} \pi_{i+k}(E_{n+k}) = 0 \qquad \text{for every} \quad i < n.$$

DEFINITION 2. A connected spectrum $\boldsymbol{E}$ is called perfectly connected for every $n \geqq 0$, the $(n+1)$-skeleton $E_n^{(n+1)}$ of the CW-complex E_n is a n-dimensional sphere and for every $n < 0$, E_n is the singleton.

EXAMPLES. 1) The sphere spectrum $\boldsymbol{S} = \{S^n$ for $n \geqq 0$ and the singleton for $n < 0\}$ is perfectly connected.

2) The Eilenberg-MacLane spectrum $\boldsymbol{Z} = \{K(Z, n)$ for $n \geqq 0$ and the singleton for $n < 0\}$ is perfectly connected.

3) The Eilenberg-MacLane spectrum $\boldsymbol{Z}_2 = \{K(Z_2, n)$ for $n \geqq 0$ and the singleton $n < 0\}$ is connected but not perfectly connected.

Following K. Kuratowski, by $X\tau Y$, we denote the condition: *every map f: $A \to Y$ from a closed subset A of X to Y admits a continuous extension $\bar{f}: X \to Y$.* Then we note the following equivalences.

1. $X\tau S^n \Longleftrightarrow \dim X \leqq n$.
2. $X\tau K(G, n) \Longleftrightarrow c\text{-}\dim_G X \leqq n$.
3. $\bigwedge_{m \geqq n} X\tau\Omega^\infty E_{m+\infty} \Longleftrightarrow c\text{-}\dim_{\boldsymbol{E}} X \leqq n$.

Moreover, by $X\tau^k Y$, we denote the following condition: *for every closed subset $A \subset X$ and every map $f: \Sigma^k A \to Y$, there is a continuous extension $\bar{f}: \Sigma^k X \to Y$ of f.*

Let $\tilde{\varepsilon}_n: E_n \to \Omega E_{n+1}$ be the adjoint map of ε_n in a spectrum $\boldsymbol{E}$. A spectrum $\boldsymbol{E}$ is called *Ω-spectrum* if for every n, the map $\tilde{\varepsilon}_n$ is a weak homotopy equivalence. Let $\boldsymbol{E}$ be an Ω-spectrum. Then it is easy to see that the condition $X\tau E_m$ implies the condition $X\tau^k E_{m+k}$. Moreover the implication $X\tau E_m \Rightarrow X\tau E_{m+1}$ can be proved by S. Ferry's method [17, Appendix]. On the other hand, there is a weak homotopy equivalence $w_n: E_n \to \Omega^\infty E_{n+\infty}$. Since both E_n and $\Omega^\infty E_{n+\infty}$ are absolute neighborhood extensors for the class of compact metric spaces, the map w_n induces an equivalence $[X, \Omega^\infty E_{n+\infty}] \to [X, E_n]$ for every compactum X. Thereby the equivalence 3 can be simplified for an Ω-spectrum $\boldsymbol{E}$: $c\text{-}\dim_{\boldsymbol{E}} X \leqq n \Leftrightarrow X\tau E_n$.

We remark that every spectrum $\boldsymbol{E}$ is weak homotopy equivalent to an Ω-spectrum. Consequently, every generalized cohomology theory can be defined by some Ω-spectrum.

LEMMA 1. *Suppose that a spectrum E be perfectly connected. Then* $\dim X \geqq c\text{-}\dim_{\boldsymbol{E}} X$.

PROOF. The case of $\dim X = \infty$ is trivial. Hence, suppose that $\dim X = m < <\infty$. By the definition, it suffices to verify the condition $X\tau^k E_{n+k}$ for all k and $n \geqq m$. Let A be an arbitrary closed subset of X and let $f: \Sigma^k A \to E_{n+k}$ be a map. Since $\dim \Sigma^k A \leqq m+k$, there exists a map $g: \Sigma^k A \to E_{n+k}$ such that $g(\Sigma^k A) \subset E_{n+k}^{(m+k)}$ and g is homotopic to f (written, $g \simeq f$). By the perfect connectedness of $\boldsymbol{E}$, we have

$$E_{n+k}^{(m+k)} = (E_{n+k}^{(n+k+1)})^{(m+k)} = \begin{cases} \text{singleton} & \text{if } n > m \\ S^{m+k} & \text{if } n = m. \end{cases}$$

In both cases there exists a continuous extension $\bar{g}: \Sigma^k X \to E_{n+k}^{(m+k)}$. Hence there exists a continuous extension $\bar{f}: \Sigma^k X \to E_{n+k}^{(m+k)}$ of the initial map f.

Let $C(X, Y)$ denote the space of all maps from X to Y. There exists the natural embedding j of Y into $C(X, Y)$ given by

$$j(y)(x) = (x, y).$$

Then the embedding j induces an embedding $j': Y \to C(X, X \wedge Y)$. In the case of $X = S^k$ the embedding j' induces the natural embedding $i: Y \to \Omega^k \Sigma^k Y$.

LEMMA 2. *Let $i: S^n \to \Omega^k \Sigma^k S^n$ be the natural embedding. Then for every map $f: X \to \Omega^k \Sigma^k S^n$ of an $(n+1)$-dimensional compactum (=compact metric space) X, there exists a homotopy $H: X \times I \to \Omega^k \Sigma^k S^n$ such that*

(i) $H_0 = f$,

(ii) $\operatorname{Im} H_1 \subset i(S^n)$,

(iii) $f|f^{-1}(i(S^n))=H_1|f^{-1}(i(S^n))$.

PROOF. It is a consequence of the fact that the embedding i induces an isomorphism of the l-dimensional homotopy groups for $l \leqq n$.

THEOREM 1. *Let* $\boldsymbol{E}$ *be a perfectly connected spectrum and let* X *be a finite-dimensional compactum. Then* $\dim X = c\text{-}\dim_{\boldsymbol{E}} X$.

PROOF. In view of Lemma 1, it is sufficient to prove the inequality $\dim X \leqq c\text{-}\dim_{\boldsymbol{E}} X$. Assume the contrary: $\dim X = m+1$ and $c\text{-}\dim_{\boldsymbol{E}} X \leqq m$. Let A be a closed subset and let $f: A \to E_m^{(m+1)} = S^m$ be a map. By the inequality $c\text{-}\dim_{\boldsymbol{E}} X \leqq m$, there exists a map $\bar{f}: X \to \Omega^\infty E_{m+\infty}$ such that $\bar{f}|A=f$. Then there exists an integer $k \geqq 0$ such that $\bar{f}(X) \subset \Omega^k E_{m+k}$. Then the map $\bar{f}$ associates a map $\varphi: X \times S^k \to E_{m+k}$. Since $\dim X = m+1$, there exists a map $\psi: X \times S^k \to E_{m+k}^{(m+k+1)} = S^{m+k} = \Sigma^k E_m^{(m+1)} = \Sigma^k S^m$ such that

$$\varphi \simeq \psi \text{ maps from } (X \times S^k, A \times S^k) \text{ to } (\Sigma^k S^m, S^m).$$

The map ψ associates a map $\xi: X \to \Omega^k \Sigma^k S^m$ such that

$$\xi \simeq \bar{f} \text{ in } \Omega^k E_{m+k} \quad \text{and} \quad \xi|A \simeq \bar{f}|A = f \text{ in } S^m.$$

Moreover, by Lemma 2, we have a map $\eta: X \to S^m$ such that

$$\eta \simeq \xi \text{ rel. } A.$$

Hence the map f has a continuous extension $\tilde{f}: X \to S^m$. Therefore $\dim X \leqq m$. But it is a contradiction.

THEOREM 2. *Let* $\boldsymbol{E}$ *be a perfectly connected* Ω*-spectrum. Then* $c\text{-}\dim_Z X \leqq c\text{-}\dim_{\boldsymbol{E}} X$ *for any compactum* X.

The proof of Theorem 2 needs the following notion.

DEFINITION 3. *Let* L *be simplicial complex. A map* $f: X \to |L|$ *is called an* E_n*-modification of* L *if*

1) *for each simplex* $\sigma \in L$, *the space* $f^{-1}(|\sigma|)$ *is homotopy equivalent to a finite product of* E_n *and* $f^{-1}(|\sigma|) \in \mathrm{ANR(Comp)}$,

2) *for each simplex* $\sigma \in L$, *the inclusion* $f^{-1}(|\partial\sigma|) \hookrightarrow f^{-1}(|\sigma|)$ *induces an epimorphism*

$$H^n(f^{-1}(|\sigma|; \boldsymbol{Z}) \longrightarrow H^n(f^{-1}(|\partial\sigma|; Z).$$

Here we denote by $|L|$ a *geometric relization* of a simplicial complex L, and by $\partial\sigma$ the *boundary* of a simplex σ. The one-point space is regarded as a

0-*multiple product* of E_n.

LEMMA 3. *Let* $\boldsymbol{E}$ *be a perfectly connected* Ω*-spectrum and let* n *be an integer* >1. *Then every simplicial complex* L *has an* E_n*-modification.*

PROOF OF THEOREM 2. Suppose that $c\text{-}\dim_{\boldsymbol{E}} X=1$. It is easy to see that there exists a retraction $r: E_1 \to E_1^{(2)}=S^1$. This implies the condition $X\tau S^1$ which is equivalent to $c\text{-}\dim_Z X \leqq 1$.

Assume that $c\text{-}\dim_{\boldsymbol{E}} X=n>1$. Clearly, the condition $X\tau E_n$ implies the condition $X\tau(E_n\times\cdots\times E_n)$. Let $A\subset X$ be an arbitrary closed subset of X and let $\varphi: A\to K(Z, n)$ be an arbitrary map. Then there exists a compact polyhedron L and maps $q: X\to L$, $\psi: q(A)\to K(Z, n)$ such that $\varphi\simeq\psi\circ q|A$. Without loss of generality one can regard $q(A)$ as a subpolyhedron of L. Choose a sufficiently small triangulation τ of L. By Lemma 2, we have an E_n-modification $f: Y\to |\tau|=L$. Then by the condition 2) of E_n-modifications, there is a continuous extension $g: Y\to K(Z, n)$ of $\psi\circ f$. By the condition 1) of E_n-modifications, one can construct a map $\eta: X\to Y$ with the following properties: if $q(x)\in|\sigma|$, then $f\circ\eta(x)\in|\sigma|$ for each simplex $\sigma\in\tau$. Then

$$q|A\simeq f\circ\eta|A \quad \text{in} \quad q(A).$$

Hence we have that

$$g\circ\eta|A=\psi\circ f\circ\eta|A\simeq\psi\circ q|A\simeq\varphi.$$

Therefore there is a continuous extension $\bar{\varphi}: X\to K(Z, n)$ of φ. It follows that $c\text{-}\dim_Z X\leqq n$.

LEMMA 4. *For every map* $\varphi: S^n\to\Omega X$ *the following diagram is commutative up to homotopy*:

$$\begin{array}{ccc} S^n & \xrightarrow{-1} & S^n \\ \varphi\downarrow & & \downarrow\phi \\ \Omega X & \xrightarrow[-1]{} & \Omega X \end{array}$$

PROOF. Let us consider the standard n-dimensional sphere $S^n=\{X\in E^{n+1}: |x|=1\}$ in the $(n+1)$-dimensional Euclidean space. Let $s: S^n\to S^n$ be the symmetry in E^{n+1} with respect to the hyperplane $E_0^n\ni\{0\}$. Then the map s has degree $=-1$. A map $\varphi: S^n\to\Omega X$ associates with the adjoint map $\psi: \Sigma S^n\to X$. We regard that the suspension ΣS^n is naturally realized in $E^{n+1}\times E^1=E^{n+2}$ and ψ carries the base-point meridian of ΣS^n into the base point of X.

The map $(-1)\circ\varphi: S^n\to\Omega X$ induces a map $\psi\circ\tau$, where $\tau=id_{E^{n+1}}\times(-1)$:

$E^{n+1}\times E^1 \to E^{n+1}\times E^1$ is the symmetry. The map $\varphi\circ(-1)$ induces a map $\phi\circ(s\times id_{E^1})$. The maps τ and $s\times id_{E^1}$ are homotopic because the hyperplane $E^{n+1}\times\{0\}$ can be continuously removed in E^{n+2} to the hyperplane $E^n_0\times E^1$. Thus the maps $(-1)\circ\varphi$ and $\varphi\circ(-1)$ are carried into the same homotopy class by the adjoint correspondence $[S^n, \Omega X]\to[\Sigma S^n, X]$. Thereby $(-1)\circ\varphi\simeq\varphi\circ(-1)$.

We need the following results from the infinite-dimensional manifold theory.

THEOREM A [19, 20]. *Let A be a compactum and let M be an l_2-manifold. Suppose that F: $A\times I\to M$ is a homotopy such that each of F_0 and F_1 is an embedding of A into M as a deficient set. Then there exists a homeomorphism h: $M\to M$ such that $h|F_0(A)=F_1\circ(F_0|A)^{-1}$.*

Recall that A is a *deficient set* in an l_2-manifold M if there is a homeomorphism w: $M\to M\times l_2$ such that $w(A)\subset M\times\{0\}$.

THEOREM B. *Let f: $M\to M$ be a homotopy equivalence of an l_2-manifold M. Then the map f is homotopic to a homeomorphism.*

A proof of Theorem B can be derived from a proof of [21], Theorem 7.3.

DEFINITION 3. *An embedding $X\hookrightarrow Y$ is called a symmetric embedding if every homeomorphism h: $X\to X$ can be extended to a homeomorphism $\bar{h}: Y\to Y$.*

LEMMA 5. *Let E_n be a countable locally finite CW-complex and let i: $S^n\to E_n$ be an embedding, where $n>1$. Suppose that g: $E_n\to\Omega X$ is a weak homotopy equivalence. Then the embedding $i'=i\times id_{\{0\}}$: $S^n\times\{0\}\to E_n\times l_2$ is symmetric.*

PROOF. Let ϕ: $S^n\to S^n$ be a homeomorphism. Suppose that deg $\phi=1$. In this case there exists a homotopy F: $S^n\times I\to E_n\times l_2=M$ between i' and $i'\circ\phi$. Since M is a l_2-manifold, by Theorem A, there exists a homeomorphism h: $M\to M$ such that $h|i'(S^n)=i'\circ\phi\circ i'^{-1}$.

Next, suppose that deg $\phi=-1$. Let consider the diagram:

$$\begin{array}{ccc}
\Omega X & \xrightarrow{\ -1\ } & \Omega X \\
g\uparrow & & \uparrow g \\
E_n & \overset{f}{\dashrightarrow} & E_n \\
i\uparrow & & \uparrow i \\
S^n & \xrightarrow{\ -1\ } & S^n
\end{array}$$

Since g is a weak homotopy equivalence and E_n is a CW-complex, there exists a map f: $E_n\to E_n$ such that $(-1)\circ g\simeq g\circ f$. It is easy to check that f is a weak homotopy equivalence. Hence f is a homotopy equivalence. Hence, by Theorem B, there exists a homotopy $H: M\times I\to M$ such that $H_0=f\times id_{I_2}$ and H_1 is a homeomorphism. On the other hand, by Lemma 4, $(-1)\circ g\circ i\simeq g\circ i\circ\phi$. Since g is a weak homotopy equivalence and S^n is a polyhedron, $f\circ i\simeq i\circ\phi$. Hence $H_1\circ i'\simeq i'\circ\phi$. Therefore, by Theorem B, there exists a homeomorphism $h: M\to M$ such that $h|i'\circ\phi(S^n)=(H_1\circ i')\circ(i'\circ\phi)^{-1}$. Then the homeomorphism $\Psi=h^{-1}\circ H_1$: $M\to M$ is an extension of ϕ.

PROPOSITION. *For every symmetric embedding $j: S^n\to E$, there exists a functor $\boldsymbol{\Phi}$: **Smp**→**Hom**$_{\mathbf{Top}}$ from the category **Smp** of simplicial complexes to the category **Hom**$_{\mathbf{Top}}$ of maps between topological spaces which carries monomorphisms into monomorphisms and satisfying the following conditions for each simplicial complex K:*

1) *the image* Im $\boldsymbol{\Phi}(K)$ *is a geometric realization* $|K|$ *of* K,

2) *for each simplex* $\sigma\in K$, *the space* $\boldsymbol{\Phi}(K)^{-1}(|\sigma|)$ *is homotopy equivalent to a finite product* $E\times\cdots\times E$.

PROOF. The full subcategory of **Smp** consisting of simplexes of dimension $\leqq m$ is denoted by **Smp**$_m$. We define $\boldsymbol{\Phi}$ by the induction on $m\geqq n$. For $m=n$, we define

$$\boldsymbol{\Phi}(K)=id_{|K|}.$$

Let σ_{n+1} be an $(n+1)$-dimensional simplex. Fix a homeomorphism $h: |\sigma_{n+1}^{(n)}|\to S^n$. The diagram

$$\begin{array}{ccc} |\sigma_{n+1}^{(n)}| & \xrightarrow{j\circ h} & E \\ & \searrow^{c} & \downarrow \phi \\ & & pt \end{array}$$

means the morphism $j\circ h\xrightarrow{(id,\,\phi)} c$ in the category **Hom**$_{\mathbf{Top}}$. Here a morphism of maps

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ f\downarrow & & \downarrow g \\ Y & \xrightarrow[\beta]{} & Y' \end{array}$$

in **Hom**$_{\mathbf{Top}}$ is denoted by

$$(\alpha,\beta): f\longrightarrow g \quad \text{or} \quad f\xrightarrow{(\alpha,\,\beta)} g.$$

Let M: $\mathbf{Hom_{Top}} \to \mathbf{Top}$ be the *mapping cylinder functor*. It is easy to see that $M(c)$ is naturally homeomorphic to $|\sigma_{n+1}|$. We define

$$\Phi(\sigma_{n+1})=M(id,\ \psi).$$

Let α: $\sigma_{n+1}\to\sigma_{n+1}$ be an isomorphism in **Smp**. Since j is a symmetric embedding, there exists a commutative diagram:

$$\begin{array}{ccccc}
E & & \xrightarrow{\quad\gamma\quad} & & E \\
\Big\downarrow{\scriptstyle j\circ h} & \overset{\psi}{\searrow} & pt \longrightarrow pt & \overset{\psi}{\swarrow} & \Big\downarrow{\scriptstyle j\circ h} \\
 & \underset{c}{\nearrow} & & \underset{c}{\nwarrow} & \\
|\sigma^{(n)}_{n+1}| & & \xrightarrow[\quad|\alpha|\quad]{} & & |\sigma^{(n)}_{n+1}|
\end{array}$$

In the language of the category $\mathbf{Hom_{Top}}$ it means the following commutative diagram:

$$\begin{array}{ccc}
j\circ h & \xrightarrow{(|\alpha|,\ \gamma)} & j\circ h \\
{\scriptstyle (id,\ \psi)}\Big\downarrow & & \Big\downarrow{\scriptstyle (id,\ \psi)} \\
c & \xrightarrow[(|\alpha|,\ id)]{} & c
\end{array}$$

Hence the functor M gives the following commutative diagram:

$$\begin{array}{ccc}
M(j\circ h) & \xrightarrow{M(|\alpha|,\ \gamma)} & M(j\circ h) \\
{\scriptstyle \Phi(\sigma_{n+1})}\Big\downarrow & & \Big\downarrow{\scriptstyle \Phi(\sigma_{n+1})} \\
M(c) & \xrightarrow[|\alpha|]{} & M(c)\ .
\end{array}$$

Therefore we define

$$\Phi(\alpha)=(M(|\alpha|,\ \gamma),\ |\alpha|).$$

For an arbitrary $(n+1)$-dimensional simpiicial complex K, the map $\Phi(K)$ is constructed by pasting maps $\Phi(\sigma)$ for all $\sigma\in K$.

Suppose that the functor Φ is defined on $\mathbf{Smp}_m$, where $m>n$. Now we define $\Phi(\sigma_{m+1})$ for an $(m+1)$-simplex σ_{m+1}. Let $F=\{\sigma_{m+1}\to\sigma_m\}$ be the family of all surjective simplicial maps. By $\dot{F}$ we denote the family $\{g|\sigma^{(m)}_{m+1}:\ \sigma^{(m)}_{m+1}\to \sigma_m|g\in F\}$. By Δf we denote the diagonal product

$$\Phi_1(\sigma^{(m)}_{m+1}) \xrightarrow{\ \Delta f\ } \pi\{\Phi_1(\sigma_m)|f\in\Phi_1(\dot{F})\},$$

here by $\Phi_1(L)$ and $\Phi_2(L)$ we denote the domain and range of the map $\Phi(L)$, respectively. For each map α: $L\to K$, $\Phi(\alpha)$ means the following commutative diagram:

$$\begin{array}{ccc} \Phi_1(L) & \xrightarrow{\Phi_1(\alpha)} & \Phi_1(K) \\ \Phi(L)\downarrow & & \downarrow\Phi(K) \\ \Phi_2(L) & \xrightarrow[\Phi_2(\alpha)]{} & \Phi_2(K)\,. \end{array}$$

The diagram

$$\begin{array}{ccc} \Phi_1(\sigma_{m+1}^{(m)}) & \xrightarrow{\Delta f} & \pi\Phi_1(\sigma_m) \\ \Phi(\sigma_{m+1}^{(m)})\downarrow & & \downarrow \\ |\sigma_{m+1}^{(m)}| & \xrightarrow[c]{} & pt \end{array}$$

induces the map

$$\Phi(\sigma_{m+1}) : M(\Delta f) \longrightarrow M(c) = |\sigma_{m+1}|\,.$$

Any isomorphism $\alpha : \sigma_{m+1} \to \sigma_{m+1}$ induces a bijective correspondence A: $\Phi_1(\dot{F}) \to \Phi_1(\dot{F})$. The correspondence A induces a homeomorphism

$$a : \pi\{\Phi_1(\sigma_m) \mid f \in \Phi_1(\dot{F})\} \to \pi\{\Phi_1(\sigma_m) \mid f \in \Phi_1(\dot{F})\}$$

which commutes the following diagram:

$$\begin{array}{ccc} \pi\Phi_1(\sigma_m) & \xrightarrow{a} & \pi\Phi_1(\sigma_m) \\ \Delta f\downarrow & & \downarrow\Delta f \\ \Phi_1(\sigma_{m+1}^{(m)}) & \xrightarrow[\Phi_1(\alpha')]{} & \Phi_1(\sigma_{m+1}^{(m)}) \end{array}$$

where $\alpha' = \alpha \mid \sigma_{m+1}^{(m)}$. A commutative diagram

$$\begin{array}{ccc} \Delta f & \xrightarrow{(\Phi_1(\alpha'),\, a)} & \Delta f \\ \downarrow & & \downarrow \\ c & \xrightarrow[(|\alpha|,\, id)]{} & c \end{array}$$

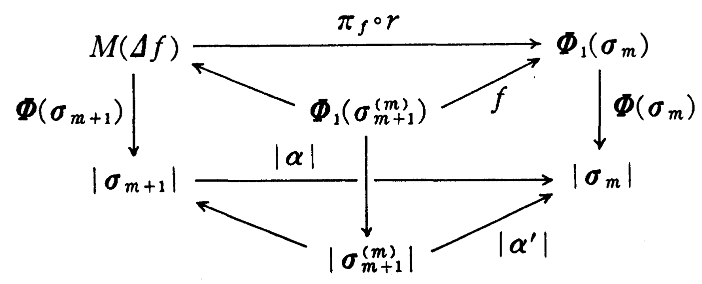
induces the diagram

$$\begin{array}{ccc} M(\Delta f) & \xrightarrow{M(\Phi_1(\alpha'),\, a)} & M(\Delta f) \\ \Phi(\sigma_{m+1})\downarrow & & \downarrow\Phi(\sigma_{m+1}) \\ |\sigma_{m+1}| & \xrightarrow[|\alpha|]{} & |\sigma_{m+1}|\,. \end{array}$$

We define

$$\Phi(\alpha) = (M(\Phi_1(\alpha'),\, a),\, |\alpha|) : \Phi(\sigma_{m+1}) \longrightarrow \Phi(\sigma_{m+1})\,.$$

For an arbitrary surjective morphism $\alpha : \sigma_{m+1} \to \sigma_m$, there is a commutative diagram:



where r is the deformation retraction of the mapping cylinder $M(\Delta f)$ onto the image $\pi\Phi_1(\sigma_m)$ and $\pi_f: \pi\Phi_1(\sigma_m)\to\Phi_1(\sigma_m)$ is the projection on the factor $\Phi_1(\sigma_m)$ with an index $f=\Phi_1(\alpha')$. Hence we define

$$\Phi(\alpha)=(\pi_f\circ r,\ |\alpha|): \Phi(\sigma_{m+1}) \longrightarrow \Phi(\sigma_m).$$

For any morphism $\alpha: \sigma_{m+1}\to\sigma_k$, where $k<m$, we define

$$\Phi(\alpha)=\Phi(\beta)\circ\Phi(\alpha'): \Phi(\sigma_{m+1}) \longrightarrow \Phi(\sigma_k),$$

where $\alpha=\beta\circ\alpha'$ and $\alpha': \sigma_{m+1}\to\sigma_m$ is a surjective simplicial map. It is easy to check that $\Phi(\alpha)$ does not depend on a decomposition of α into $\beta\circ\alpha'$.

For an arbitrary $(m+1)$-dimensional simplicial complex K, the map $\Phi(K)$ is defined by pasting corresponding maps $\Phi(\sigma)$ for all $\sigma\in K$.

REMARK 1. If $j: S^n\to E$ induces an epimorphism of n-dimensional cohomology groups, then the following statement is true:

3) *the inclusion* $\Phi(\sigma)^{-1}(|\partial\sigma|)\hookrightarrow\Phi(\sigma)^{+1}(|\sigma|)$ *induces an epimorphism of n-dimensional cohomology groups.*

PROOF OF LEMMA 3. We regard that E_n is a locally finite countable CW-complex for every n. By Lemma 5, the inclusion $E_n^{(n+1)}\times\{0\}\hookrightarrow E_n\times l_2$ is a symmetric embedding. In view of Remark 1, $\Phi(L)$ is an E_n-modification of L.

REMARK 2. In the case that E_n is not a countable CW-complex one must consider, in Lemma 5, Y-manifold instead of an l_2-manifold with some suitable Y, for example $Y=\mathrm{R}^\infty$.

§ 2. Cohomological dimension of strongly infinite-dimensional compacta.

A space X is *strongly infinite-dimensional* if it contains an essentaial family $\{(A_i, B_i)|i=1, 2, \cdots\}$: this means that each (A_i, B_i) is a disjoint pair of closed subsets in X and if C_i is a closed subset of X separating A_i and B_i, $i=1, 2, \cdots$, then $\bigcap_{i=1}^{\infty} C_i\neq\emptyset$. In other words, there exists an essential map $f: X\to I^\infty$ from

X onto the Hilbert cube I^∞: this means that for each n, the relative map

$$w_n \circ f : (X, f^{-1}(w_n^{-1}(\partial \mathrm{I}^n))) \longrightarrow (\mathrm{I}^n, \partial \mathrm{I}^n)$$

is essential, where $w_n : \mathrm{I}^\infty \to \mathrm{I}^n = \underbrace{\mathrm{I} \times \cdots \times \mathrm{I}}_{n \text{ times}}$ is the projection.

DEFINITION 4. Let $f : X \to L$ be a map and let L be a polyhedron with a triangulation τ. We shall write $c\text{-}\dim_{\mathrm{Z}}(f, \tau) \leqq n$ if the following condition is satisfied: for every subpolyhedron $A \subset Y$ with respect to τ, the image of the homomorphism $(f \mid f^{-1}(A))^* : \check{H}^n(A ; \mathrm{Z}) \to \check{H}^n(f^{-1}(A) ; \mathrm{Z})$ is contained in the image of the homomorphism $j^* : \check{H}^n(X ; \mathrm{Z}) \to \check{H}^n(f^{-1}(A) ; \mathrm{Z})$, where $j : f^{-1}(A) \hookrightarrow X$ is the inclusion.

LEMMA 6 [10]. *Let $X \subset \mathrm{I}^\infty$ be a compactum in the Hilbert cube I^∞ and suppose that $X = \varprojlim \{X_i, p_i^{i+1}\}$, where each $X_i \subset \mathrm{I}^\infty$ is a compact polyhedron with a fixed triangulation τ_i, such that $\varinjlim_i$ mesh $(\tau_i) = 0$ and $\varinjlim_i p_i^\infty = id_X$. Assume that $c\text{-}\dim_{\mathrm{Z}}(p_i^{i+1}, \tau_i) \leqq n$ for infinitely many i. Then also $c\text{-}\dim_{\mathrm{Z}} X \leqq n$.*

The following is actually proved in [10, 12]:

LEMMA 7. *For any prime number p, a finite simplicial complex K and given finite nonzero elements $\alpha_i \in K_C^*(|K|, |L_i| ; \mathrm{Z}_p)$, where L_i is a subcomplex of K, there exists a compact polyhedron M and a map $f : M \to |K|$ such that*

1) *$c\text{-}\dim_{\mathrm{Z}}(f, K) \leqq 3$,*
2) *$f^*(\alpha_i) \neq 0$ for all i.*

THEOREM 3. *There exists a strongly infinite-dimensional compactum X with $c\text{-}\dim_{\mathrm{Z}} X = 3$.*

PROOF. Let $\boldsymbol{S} = \{I^i, w_i^{i+1}\}$ be the standard inverse sequence for I^∞. We shall construct an inverse sequence $\boldsymbol{S}' = \{X_i, q_i^{i+1}\}$, where each X_i is a compact polyhedron with a triangulation τ_i, and a morphism between inverse sequences $\{f_i\} : \boldsymbol{S}' \to \boldsymbol{S}$.

Define $X_1 = I$ and $f_1 = id_{\mathrm{I}}$.

Suppose that the following part of the sequences $\boldsymbol{S}'$ is constructed: $X_1 \xleftarrow{q_1^2} X_2 \xleftarrow{q_2^3} \cdots \longleftarrow X_n$, and a family $\{f_i : X_i \to \mathrm{I}^i\}_{i=1,2,\dots,n}$ is defined such that

1) $c\text{-}\dim_{\mathrm{Z}}(q_i^{i+1}, \tau_i) \leqq 3$,
2) $(q_i^n)^* \circ f_i^*(\mu_i) \neq 0$ for $i = 1, 2, \cdots, n$, where $\mu_i \in K_C^*(\mathrm{I}^i, \partial \mathrm{I}^i ; \mathrm{Z}_p)$ is a generator,
3) $X_i \subset \mathrm{I}_1^\infty \times \cdots \times \mathrm{I}_1^\infty \times \{0\} \times \{0\} \times \cdots \subset \prod_{i=1}^\infty \mathrm{I}_i^\infty = \mathrm{I}^\infty$, and
4) $mesh(\tau_i) < 1/i$ with respect to a fixed metric on I^∞.

Lemma 7 for τ_n and $(q_i^n)^* \circ f_i^*(\mu_i) \in K_C^*(X_n, (q_i^n)^{-1}(f_i^{-1}(\partial I^i); \boldsymbol{Z}_p)$, $i=1, 2, \cdots, n$, gives us a compact polyhedron Y_n and a map $g_n: Y_n \to |\tau_n| = X_n$ with the properties: $c\text{-dim}_Z(g_n, \tau_n) \leqq 3$ and $g_n^* \circ (q_i^n)^* \circ f_i^*(\mu_i) \neq 0$ for all $i=1, 2, \cdots, n$. Define

$$X_{n+1} = Y_n \times I, \ f_{n+1} = (f_n \circ g_n) \times id_I, \quad \text{and} \quad q_n^{n+1} = g_n \circ v_n,$$

where $v_n: Y_n \times I \to Y_n$ is the projection on the first factor. Embed X_{n+1} into $I_1^\infty \times \cdots \times I_{n+1}^\infty \times \{0\} \times \cdots$ as the graph of q_n^{n+1}. Then it is easy to see that $c\text{-dim}_Z (q_n^{n+1}, \tau_n) \leqq 3$.

We choose a triangulation τ_{n+1} of X_{n+1} with $mesh(\tau_{n+1}) < 1/n+1$.

Künneth formula implies that $f_{n+1}^*(\mu_n) \neq 0$. Since v_n^* is an isomorphism, $(q_i^{n+1})^* \circ f_i^*(\mu_i) = v_n^* \circ g_n^* \circ (q_i^n)^* \circ f_i^*(\mu_i) \neq 0$, $i=1, 2, \cdots, n$.

Define $X = \varprojlim \boldsymbol{S}' \subset I^\infty$ and $f = \varprojlim \{f_i\}: X \to I^\infty$. For each $n \geqq 1$, the homomorphism $(w_n^\infty \circ f)^*: K_C^*(I^n, \partial I^n; Z_p) \to K_C^*(X, (w_n^\infty \circ f)^{-1}(\partial I^n); Z_p)$ is not trivial because $(w_n^\infty \circ f)^*(\mu_n) = (q_n^\infty)^* \circ f_n^*(\mu_n) \neq 0$. Hence the map f is essential, and thereby X is strongly infinite-dimensional. On the other hand, by Lemma 6, we have that $c\text{-dim}_Z X \leqq 3$. The converse inequality $c\text{-dim}_Z X \geqq 3$ is trivial by the construction of Lemma 7.

THEOREM 4. *There exists a compactum X with the dimensions $c\text{-dim}_Z X = 3$ and $c\text{-dim}_S X = \infty$.*

PROOF. In [10, 12], for any n, a compactum X_n with $c\text{-dim}_Z X_n = 3$ and a map $f_n: X_n \to S^n$ with $f_n^* \neq 0$, where $f_n^*: \tilde{K}_C^*(S^n; Z_p) \to \tilde{K}_C^*(X_n; Z_p)$ is induced by f_n. Then the suspension map $\Sigma^k f_n$ is essential for every k. Hence $\pi_s^n(X_n) \neq 0$. Then $c\text{-dim}_S X_n \geqq n$. Therefore $X = \alpha(\bigoplus_n X_n)$ is desired compactum, where $\alpha(\bigoplus X_n)$ is the one point compactification of the topological sum $\bigoplus_n X_n$.

PROBLEM 1. Does there exist a strongly infinite-dimensional compactum X with finite stable cohomotopy dimension $c\text{-dim}_S X < \infty$?

PROBLEM 2 (S. Nowak). Is it true that $\dim X = c\text{-dim}_S X$ for every compactum X?

§ 3. The dimension $\dim_\alpha$.

LEMMA 8. *Let X be a compactum and let K be a CW-complex. Then the condition $X\tau K$ implies the condition $X\tau\Sigma K$.*

PROOF. Let $A \subset X$ be a closed subset and let $\varphi: A \to \Sigma K$ be a map. By the definition, $\Sigma K = \text{con}_1 K \cup \text{con}_2 K$, where $\text{con}_1 K \cap \text{con}_2 K = K$ and each $\text{con}_i K$,

$i=1, 2$, is a cone over K. Denote $A_i=\varphi^{-1}(\mathrm{con}_i K)$, $i=1, 2$. Then there is a map $f: A\to \mathrm{R}$ with the properties:

$$f(A_1)\subset \mathrm{R}^+,\ f(A_2)\subset \mathrm{R}^-, \quad \text{and} \quad f^{-1}(0)=A_1\cap A_2.$$

Let $\tilde{f}: X\to \mathrm{R}$ be a continuous extension of f. Define $F_1=\tilde{f}^{-1}(\mathrm{R}^+)$ and $F_2=\tilde{f}^{-1}(\mathrm{R}^-)$. From the condition $X\tau K$, there exists a map $\psi: F_1\cap F_2\to K$ such that $\psi|A_1\cap A_2=\varphi|A_1\cap A_2$. Since $\mathrm{con}_i K$ is an absolute extensor for compacta, there exists a continuous extension $\psi_i: F_i\to \mathrm{con}_i K$ of $\psi\cup\varphi|A_i$, $i=1, 2$. Then the union $\psi_1\cup\psi_2$ is a desired extension of φ over X. Thus, $X\tau\Sigma K$.

Let $\alpha\in\pi_{n+k}(S^n)$ is an element of the $(n+k)$-dimensional homotopy group of the n-sphere. Define

$$B_\alpha=S^n\cup_f D^{n+k+1},$$

as a complex obtained from the n-sphere S^n by attaching an $(n+k+1)$-cell D^{n+k+1} using an attaching map $f: \partial D^{n+k+1}\to S^n$ from the homotopy class α. It is easy to see that if $f\simeq f': \partial D^{n+k+1}\to S^n$, the spaces $S^n\cup_f D^{n+k+1}$ and $S^n\cup_{f'} D^{n+k+1}$ are homotopy equivalent. Moreover, it can be seen that the spaces $B_{\Sigma\alpha}$ and $\Sigma(B_\alpha)$ are homotopy equivalent.

DEFINITION 5. The dimension $\dim_\alpha X$ of a compactum X is defined as follows:

$$\dim_\alpha X\leqq m \Longleftrightarrow \begin{cases} \dim X\leqq m & \text{if } m<n \\ X\tau\Sigma^{m-n}B_\alpha & \text{if } m\geqq n. \end{cases}$$

In view of Lemma 8, the definition is correct.

THEOREM 5. *$\dim_\alpha X=\dim X$ for every compactum X.*

PROOF. The implication $\dim X\leqq m\Rightarrow\dim_\alpha X\leqq m$ is trivial.

Suppose that $\dim_\alpha X\leqq m$ and $m\geqq n$. It is sufficient to prove the inequality $\dim X\leqq m$. Let $\varphi: A\to S^m$ be an arbitrary map from a closed subset A of X to the m-sphere S^m. By the condition $\dim_\alpha X\leqq m$, there exists a continuous extension $\varphi_1: X\to S^m\cup_{\Sigma^{m-n}f} D^{m+k+1}$ of φ, where $f\in\alpha\in\pi_{n+k}(S^n)$. Choose an $(m+k+1)$-cell $B^{m+k+1}\subset \mathrm{Int}\, D^{m+k+1}$ and define a retraction

$$r_1: S^m\cup_{\Sigma^{m-n}f} D^{m+k+1}-\mathrm{Int}\, B^{m+k+1}\longrightarrow S^m.$$

Denote $A_1=\varphi_1^{-1}(\partial B^{m+k+1})$. From the condition $X\tau\Sigma^{m-n+k}B_\alpha$, $\varphi_1|A_1$ has a continuous extension

$$\varphi_2: \varphi_1^{-1}(B^{m+k+1})\longrightarrow S^{m+k}\cup_{\Sigma^{m-n+k}} D^{m+2k+1}$$

Choose an $(m+2k+1)$-cell $B^{m+2k+1}\subset \mathrm{Int}\, D^{m+2k+1}$ and define a retraction r_2 and

A_2 and so on.

By the construction, $r_i|\partial B^{m+ik+1}\simeq\Sigma^{m-n+(i-1)k}f$, $i=1, 2, \cdots$. By the *nipotency theorem* for the ring π_*^S[22], there exists a number l such that $\xi=r_1\circ r_2\circ\cdots\circ r_l|\partial B^{m+lk+1}$ is null-homotopic. Consequently $\xi\circ\varphi_l|A_l$ has a continuous extension

$$\eta:\ \varphi_l^{-1}(B^{m+lk+1})\longrightarrow S^m.$$

The finite stratificatification $Z_l\subset Z_{l-1}\subset\cdots\subset Z_1\subset Z_0=X$ is defined by $Z_i=\varphi_i^{-1}(B^{m+ik+1})$ for $i=1, 2, \cdots, l$. Define

$$\bar\varphi|Z_0-Z_1=r_1\circ\varphi_1,$$

$$\bar\varphi|Z_1-Z_2=r_1\circ r_2\circ\varphi_2,$$

....

$$\bar\varphi|Z_{l-1}-Z_l=r_1\circ r_2\circ\cdots\circ r_l\circ\varphi_l,$$

$$\bar\varphi|Z_l=\eta.$$

Then it is easy to see that $\bar\varphi$: $X\rightarrow S^m$ is well-defined and continuous, and $\bar\varphi$ is an extension of φ. Thus, $\dim X\leqq m$.

REMARK 3. The sequence of spaces

$$E_m=\begin{cases} pt & \text{if}\ \ m<0 \\ S^m & \text{if}\ \ 0\leqq m<n \\ \Sigma^{m-n}B_\alpha & \text{if}\ \ m\geqq n \end{cases}$$

defines a spectrum $\boldsymbol{E}(\alpha)$. Then Theorems 1 and 5 imply the inequality $c\text{-}\dim_{E(\alpha)}X\leqq\dim_\alpha X$ for every compactum X.

PROBLEM 3. Let K be a compact $(n-1)$-connected polyhedron with $\pi_n(K)=Z$. Then one can define the dimension $\dim_K$ as the above. Is it true that $\dim_K X=\dim X$ for every compactum X?

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VANISHING OF HOCHSCHILD'S COHOMOLOGIES $H^i(A\otimes A)$ AND GRADABILITY OF A LOCAL COMMUTATIVE ALGEBRA A

By

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0. Introduction.

In [8] Nakayama conjectured that a finite dimensional algebra R with an infinite dominant dimension is selfinjective. As such an algebra R is isomorphic to an endomorphism ring of a generator-cogenerator over an algebra A, Tachikawa [10] has shown that the Nakayama's conjecture is reduced to the following conjectures (i) and (ii): For a finite dimensional algebra A over a field K,

(i) A is selfinjective if Hochschild's cohomological groups $H^i(A\otimes_K A)\cong \mathrm{Ext}_A^i(D(A), A)=0$ for $i\geqq 1$, where $D(A)=\mathrm{Hom}_K(A, K)$.

(ii) An A-module X is projective if A is selfinjective and if $\mathrm{Ext}_A^i(X, X)=0$ for $i\geqq 1$.

It is to be noted here that the Nakayama's conjecture is true if and only if both the conjectures (i) and (ii) are true.

For the conjecture (ii) there have been already several interesting results by Hoshino [6] and Schulz [9]. In [7] Hoshino applied Wilson's therem to settle the conjecture (i) for algebras A's with cube zero radicals, because in this case both A's and the corresponding endomorphism rings R's are positively $\mathbf{Z}$-graded.

This paper concerns with the conjecture (i) for local commutative algebras. In §1 we provide a theorem that for a local (not necessarily commutative) algebra A, $R=\mathrm{End}_A(A\oplus D(A))$ is positively $\mathbf{Z}$-graded if and only if so is A. It is proved in §2 that local algebras with quartic zero radicals such that they are homomorphic images of polynomial ring $K[x, y]$ over an algebraically closed field K are positively $\mathbf{Z}$-graded, and applying Wilson's theorem we can prove that conjecture (i) is true for such algebras. In §3 we shall give, however, a not positively $\mathbf{Z}$-graded commutative local algebra, which is a homomorphic image of the polynomial ring $K[x, y, z]$ with quartic zero radical.

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1. Preliminary

Let R be an finite dimensional algebra over a field K. Let

$$0 \longrightarrow R \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_n \longrightarrow \cdots \tag{1}$$

be a minimal injective resolution of the right R-module R.

In [2] Auslander and Reiten introduced the generalized Nakayama conjecture: Every simple R-module appears as a submodule of some E_n in (1). We shall say dom dim $R_R \geqq n$ (resp. $=\infty$) if E_j are projective R-modules for all $j<n+1$ (resp. all $j>0$) in (1).

In [8] Nakayama conjectured that R is selfinjective if dom dim $R_R=\infty$. The Nakayama conjecture is true if the generalized Nakayama conjecture is true, because the injective envelope of any simple right R-module S is projective, if dom dim $R_R=\infty$.

In [11] Wilson proved that the generalized Nakayama conjecture is true for positively **Z**-graded algebras.

Suppose dom dim $R_R \geqq 2$. It is well known that there exists a minimal faithful left R-module which is a projective and injective left ideal Re for an idempotent e. Further $R \cong \mathrm{End}_{eRe} Re$ and Re is a generator-cogenerator as a right eRe-module. Cf. [10]. Conversely for any algebra A and for a generator-cogenrator X_A, dom dim $\mathrm{End}_A X \geqq 2$. This connection between A and $\mathrm{End}_A X$ plays an important role in this paper. In our context $\mathrm{End}_A X$ is selfinjective iff A is selfinjective.

A *graded* algebra is an algebra A together with a vector space decomposition $A=\bigoplus_{k \in \mathbf{Z}} A_k$ such that $A_i A_j \subset A_{i+j}$.

Since A is a finite dimensional algebra, $A_k=0$ for $|k| \gg 0$. We will consider positively **Z**-*graded* algebras, that is, graded algebras with $A_k=0$ if $k<0$. We will further assume rad $A=\bigoplus_{k \geq 1} A_k$. Thus we will write $A=\bigoplus_{k \geq 0} A_k$.

A *graded right A-module* is a module M together with a vector space decomposition $M=\bigoplus_{k \in \mathbf{Z}} M_k$ such that $M_i A_j \subset M_{i+j}$. Notice that we are allowing negative gradings on our modules. If $L=\bigoplus_{k \in \mathbf{Z}} L_k$ is another graded A-module, we define a degree i morphism to be an A-homomorphism $f: M \rightarrow L$ such that $f(M_k) \subset L_{i+k}$. It is to be noted that for a graded A-module M the degrees of morphisms make $\mathrm{End}_A M$ be a (not necessarily positively) **Z**-graded algebra (see [4, §2]).

The i-th shift $\sigma(i)(M)$ of $M=\bigoplus_{k \in \mathbf{Z}} M_k$ is defined to be a graded A-module $L=\bigoplus_{k \in \mathbf{Z}} L_k$ such that $L_k=M_{k-i}$.

THEOREM 1.1. *Let A be a local algebra, $D(A)=\mathrm{Hom}_K(A, K)$ the injective* *cogenerator as a right A-module and R the endomorphism ring of* $A\oplus D(A)$. *Then R is positively* $\mathbf{Z}$*-graded iff so is A. Here it is to be noted that the grading of A is one induced from the grading of R and the grading of R is one induced by the degrees of morphisms in* $\mathrm{End}_A(A\oplus D(A))$.

PROOF. "Only if" part. Let $R=\bigoplus_{k=0}^{n} R_k$ and e a projection: $A\oplus D(A)\to A$. Since $\mathrm{rad}\ R=\bigoplus_{k=1}^{n} R_k$, there is an idempotent f of R such that $f\cong e$ and we have that $A\cong eRe$ is isomorphic to a positively $\mathbf{Z}$-graded algebra $fRf=\bigoplus_{k=0}^{n}(fRf)_k$ with $(fRf)_k=fR_kf$.

"If" part. Let $A=\bigoplus_{k=0}^{n} A$. Then $D(A)$ is gradable such that $D(A)_{-k}=D(A_k)$ for $n\geqq k\geqq 0$. If A is selfinjective, i.e. $A\cong D(A)$, then $R\cong\begin{pmatrix} A & A\\ A & A\end{pmatrix}$ and R has a grading with $R_k\cong\begin{pmatrix} A_k & A_k\\ A_k & A_k\end{pmatrix}$. So we may assume that A is not selfinjective.

By using the n-th shift $\sigma(n)$ we obtain a new grading of $D(A)$ such that $D(A)=(D(A))_0\oplus(D(A))_1\oplus\cdots\oplus(D(A))_n$, where $(D(A))_i\cong D(A_{n-i})$, $0\leqq i\leqq n$.

Now

$$R\cong\mathrm{End}_A(A\oplus D(A))\cong\begin{pmatrix} \mathrm{Hom}_A(A, A) & \mathrm{Hom}_A(D(A), A)\\ \mathrm{Hom}_A(A, D(A)) & \mathrm{Hom}_A(D(A), D(A))\end{pmatrix}$$

and it is clear that $\mathrm{Hom}_A(A, A)\cong\mathrm{Hom}_A(D(A), D(A))\cong A$, $\mathrm{Hom}_A(A, D(A))\cong D(A)$ and degrees of morphisms define naturally non-negative $\mathbf{Z}$-gradings of $\mathrm{Hom}_A(A, A)$, $\mathrm{Hom}_A(D(A), D(A))$ and $\mathrm{Hom}_A(A, D(A))$ which are respectively identical with A, A and $D(A)$.

Next for the $\mathbf{Z}$-grading $\mathrm{Hom}_A(D(A), A)=\bigoplus_{i\in\mathbf{Z}}\mathrm{Hom}_{\mathrm{gr}A}(D(A), \sigma(-i)A)$, we want to notice here that the degree of any morphism from $D(A)$ to A is at least one. This fact will be proved by induction on n as follows: If $n=1$, then $A=A_0\oplus A_1$ and $(D(A))_0\cong D(A_1)$, $(D(A))_1\cong D(A_0)$. Hence it is clear that $-1\leqq$degree of $\phi\leqq 1$ for $\phi\in\mathrm{Hom}_A(D(A), A)$. But since $D(A_0)$ is the socle of $D(A)$ and A is not selfinjective, ϕ is not a monomorphism and the degree ϕ must be 1. Assume that for any grading $B=B_0\oplus B_1\oplus\cdots\oplus B_r$, $r<n$, the degree of $\varphi\geqq 1$ for $\varphi\in\mathrm{Hom}_B(D(B), B)$ and suppose the degree of $\phi=i\leqq 0$ for $\phi\in\mathrm{Hom}_A(D(A), A)$. In the case $i=0$, $0\neq\phi(D(A)_0)\subset A_0$ and A_0 is considered to be a division algebra. Hence $\phi(D(A)_n)\supset\phi(D(A)_0A_n)=A_0A_n=A_n$ and ϕ must be a monomorphism. Then similarly as in $n=1$ this contradicts to that A is not selfinjective. Next assume $i\leqq -1$. Then $0=\phi(D(A)_0)=\phi(D(A_n))$. Hence ϕ is considered to be a homomorphism of $D(A_{n-1}\oplus A_{n-2}\oplus\cdots\oplus A_0)$ to A and $A_{n-1}\oplus A_{n-2}\oplus\cdots\oplus A_0$ can be cosidered as a grading of A/A_n. Let ρ: $D(A/A_n)\to A/A_n$ be the composition of ϕ and the canonical homomorphism from A to A/A_n. Then we know that the degree of $\rho\leqq -1$ but this contradicts to the assumption

of induction.

Let us denote the gradings of $\mathrm{Hom}_A(A, A)$, $\mathrm{Hom}_A(D(A), D(A))$, $\mathrm{Hom}_A(A, D(A))$ and $\mathrm{Hom}_A(D(A), A)$ by

$$\mathrm{Hom}_A(A, A)=\bigoplus_{i=0}^{n} E_i^{(1,1)}, \qquad \mathrm{Hom}_A(D(A), D(A))=\bigoplus_{i=0}^{n} E_i^{(2,2)},$$

$$\mathrm{Hom}_A(A, D(A))=\bigoplus_{i=0}^{n} E_i^{(2,1)} \quad \text{and} \quad \mathrm{Hom}_A(D(A), A)=\bigoplus_{i=0}^{n} E_i^{(1,2)}.$$

Now we can introduce a positive **Z**-grading of R by

$$R_{2k}=\begin{pmatrix} E_k^{(1,1)} & 0 \\ 0 & E_k^{(2,2)} \end{pmatrix}, \qquad R_{2k+1}=\begin{pmatrix} 0 & E_{k+1}^{(1,2)} \\ E_k^{(2,1)} & 0 \end{pmatrix}.$$

Because

$$R_{2k+1}R_{2j+1}=\begin{pmatrix} E_{k+1}^{(1,2)}E_j^{(2,1)} & 0 \\ 0 & E_k^{(2,1)}E_{j+1}^{(1,2)} \end{pmatrix} \subset \begin{pmatrix} E_{k+j+1}^{(1,1)} & 0 \\ 0 & E_{k+j+1}^{(2,2)} \end{pmatrix} = R_{2(k+j+1)},$$

$$R_{2k}R_{2j+1}=\begin{pmatrix} 0 & E_k^{(1,1)}E_{j+1}^{(1,2)} \\ E_k^{(2,2)}E_j^{(2,1)} & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & E_{k+j+1}^{(1,2)} \\ E_{k+j}^{(2,1)} & 0 \end{pmatrix} = R_{2(k+j)+1}$$

and

$$R_{2k+1}R_{2j}=\begin{pmatrix} 0 & E_{k+1}^{(1,2)}E_j^{(2,2)} \\ E_k^{(2,1)}E_j^{(1,1)} & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & E_{k+j+1}^{(1,2)} \\ E_{k+j}^{(2,1)} & 0 \end{pmatrix} = R_{2(k+j)+1}.$$

Since a commutative algebra is a direct sum of local algebras we have immediately

COROLLARY 1.2. *Let A be a commutative algebra. Then $\mathrm{End}_A(A\oplus D(A))$ is positively* **Z***-graded if and only if so is A.*

THEOREM 1.3. *Let A be a positively* **Z***-graded local algebra. If $\mathrm{Ext}_A^i(D(A), A)=0$ for all $i\geqq 1$, then A is selfinjective.*

PROOF. Suppose that A_A is not selfinjective and $\mathrm{Ext}_A^i(D(A), A)=0$ for all $i\geqq 1$. Let

$$0 \longrightarrow A\oplus D(A) \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n \longrightarrow \cdots$$

be a minimal injective resolution of $A\oplus D(A)$ as a right A-module. Denote $\mathrm{End}_A(A\oplus D(A))$ by R. Since $E_i\in \mathrm{Add}\text{-}D(A)$, $D(A)$ is a direct summand of $A\oplus D(A)$ and since $\mathrm{Ext}_A^i(D(A), A)=0$ for all $i\geqq 1$, we have the following injective resolution of R_R:

$$0 \longrightarrow R \longrightarrow H_0 \longrightarrow H_1 \longrightarrow \cdots \longrightarrow H_n \longrightarrow \cdots,$$

where $H_i\cong \mathrm{Hom}_A({}_R(A\oplus D(A))_A, E_i)$ and H_i are projective and injective right R-modules.

On the other hand, by Theorem 1.1 R is positively $\mathbf{Z}$-graded. Hence by Wilson's theorem R is selfinjective. However this implies that A is selfinjective and a contradiction.

PROPOSITION 1.4. *Let A be a positively $\mathbf{Z}$-graded local algebra and R the endomorphism ring of right A-module $A\oplus D(A)$. Then Nakayama conjecture is true for R.*

§ 2. Local Commutative Graded Algebras

Throughout this section K is assumed to be an algebraically closes field of characteristic zero. The following Lemma 2.1 and Proposition 2.2 are well known, cf. [1] and [4, V. 3.9.5], but for the sake of reader's convenience, we shall write elementary proofs.

LEMMA 2.1. *A commutative K-algebra A is local if and only if A is a homomorphic image of $K[x_1, x_2, \cdots, x_m]/I^n$, where I is the ideal of the polynomial ring $K[x_1, x_2, \cdots, x_m]$ of variables $x_1, x_2, \cdots, x_m$, which is generated by $x_1, x_2, \cdots, x_m$.*

PROOF. Let J be the radical of a local commutative algebra A and $J^n=0$. Then there are ring-homomorphisms $\alpha: K[X_1, X_2, \cdots, X_m]\to A$ and $\beta: A\to A/J \cong K$. Put $\beta\alpha(X_i)=a_i$. Then $\beta\alpha(X_i-a_i)=0$ and hence $\alpha(X_i-a_i)\in J$. Therefore $\alpha((X_i-a_i)^n)=(\alpha(X_i-a_i))^n=0$ and hence $(X_i-a_i)^n\in\operatorname{Ker}\alpha$. Now we can take $x_i=X_i-a_i$.

For $f(x, y)\in K[x, y]$ we shall denote by $f_t(x, y)$ the homogeneous term of $f(x, y)$ of degree t.

PROPOSITION 2.2. *Let $f(x, y)$ be a polynomial in $K[x, y]$ such that $f(x, y)=\sum_{t\geq 2} f_t(x, y)$ with the non-zero homogeneous term $f_2(x, y)=ax^2+bxy+cy^2$ of degree 2, $I=(x, y)$ and $A=K[x, y]/(I^n, f(x, y))$, $n\geqq 3$. Then A is isomorphic to a local algebra $K[X, Y]/(L^n, g(X, Y))$ such that $L=(X, Y)$ and $g(X, Y)=XY$ or X^2-Y^p, $p>2$.*

PROOF. Assume $a\neq 0$. Then $ax^2+bxy+cy^2=a(x-\alpha y)(x-\beta y)$ for some α, β $\in K$.

Case (1): $\alpha\neq\beta$. As we can consider $x-\alpha y$ and $x-\beta y$ as new parametess of $K[x, y]$ we can take $f_2(x, y)=xy$. On the other hand, in the case (2): $\alpha=\beta$, by replacing $x-\alpha y$ with x we can take $f_2(x, y)=x^2$. Further it is easily seen that the above context for $f_2(x, y)$ are valid even if $a=0$.

At first we shall proceed the proof for the Case (1) by induction on n: we can replace xy with $f(x, y)-xy$ mod I^n and after repetitions of such rearrangements we obtain an expression of $f(x, y)$ which excludes terms x^iy^j, $i, j\geqq 1$ and $ij>1$. So if $n=4$ we may assume that $f(x, y)=xy+ax^3+by^3$. Put $X=x+by^2$ and $Y=y+ax^2$. Then $XY=xy+ax^3+by^3$ mod I^4. Since X and $Y\in \mathrm{rad}A\setminus\mathrm{rad}^2A$, we can take X and Y as new parameters and we have $A\cong K[X, Y]/((X, Y)^4, XY)$.

Assume $n>4$. Applying the assumption of induction to $K[x, y]/(I^{n-1}, f(x, y))$ we can take $f(x, y)\equiv xy+ax^{n-1}+by^{n-1}$ mod I^n. Similarly as in the case $n=4$, putting $X=x+by^{n-2}$ and $Y=y+ax^{n-2}$ we can take X an Y as new parameters and we conclude $A\cong K[X, Y]/(L^n, XY)$.

Now we shall begin the proof of the Case (2). First we can replace x^2 with $f(x, y)-x^2$ mod I^n, which is a sum of homogeneous terms of degrees >2. And by repetitions of such rearrangements we may assume that terms x^iy^j, $i>1$, $j>0$ do not appear in $f(x, y)$. Hence if $n=4$, $f(x, y)\equiv x^2+ay^3+bxy^2$ mod I^4. Then $f(x, y)\equiv(x+(1/2)by^2)^2+ay^3$ mod I^4. So replacing parameters x and y with $X=x+(1/2)by^2$ and $Y=-a^{1/3}y$ respectively, we have $A\cong K[X, Y]/((X, Y)^4, X^2-Y^3)$.

Assume $n>4$. Applying the assumption of induction to $K[x, y]/(I^{n-1}, f(x, y))$ we can take $f(x, y)=x^2-y^p+ay^{n-1}+bxy^{n-2}$, $3\leqq p<n$. Then $f(x, y)\equiv (x+(1/2)by^{n-2})^2-(y-(1/p)ay^{n-p})^p$ mod I^n and we can replace parameters x and y with $X=x+(1/2)by^{n-2}$ and $Y=y-(1/p)ay^{n-p}$ respectively. Therefore $A\cong K[X, Y]/(L^n, X^2-Y^p)$. This completes the proof.

It should be noted that $K[x, y]/((x, y)^n, xy)$, $n\geqq 3$, is biserial in the sense of Fuller [3]. On the other hand, $K[x, y]/((x, y)^4, x^2-y^3)$ has a unique maximal serial ideal, i. e., a serial ideal which contains every non-simple serial ideal.

PROPOSITION 2.3. *Let A be a local commutative algebra as in Proposition 2.2. Then A is positively* **Z***-graded.*

PROOF. Denote by $\bar{u}$ the residue class of $K[x, y]/((x, y)^n, xy)$ (resp. $K[x, y]/((x, y)^n, x^2-y^p)$ which contains $u\in K[x, y]$. It is easily seen that $K[x, y]/((x, y)^n, xy)=\bigoplus_{i=0}^{n-1}A_i$, where $A_0=\overline{K}$ and $A_i=\overline{Kx^i+Ky^i}$, $i>0$, gives a positive **Z**-grading. On the other hand, according to $p(p<n)$ is odd or even we have the following positive **Z**-gradings of $K[x, y]/((x, y)^n, x^2-y^p))$ respectively:

$$K[x, y]/((x, y)^n, x^2-y^p))=B_0\oplus\bigoplus_{i=0}^{n-2}B_{p+2i}\oplus\bigoplus_{i=0}^{n-1}B_{2j},$$

where $B_0=\overline{K}$, $B_{p+2i}=\overline{Kxy^i}$ and $B_{2j}=\overline{Ky^j}$, and

$$K[x,\ y]/((x,\ y)^n,\ x^2-y^p))=B_0\oplus\oplus_{i=0}^{q-1}B_i\oplus\oplus_{j=0}^{n-q-1}B_{q+j}\oplus\oplus_{k=n-q}^{n-2}B_k\ ,$$

where $p=2q$, $B_0=\overline{K}$, $B_i=\overline{Ky^i}$, $B_{q+j}=\overline{Kxy^j+Ky^{q+j}}$ and $B_k=\overline{Ky^k}$.

If $p\geqq n$, $K[x,\ y]/((x,\ y)^n,\ x^2)=\oplus_{i=0}^{n-1}C_i$ where $C_0=\overline{K}$ and $C_i=\overline{Kxy^{i-1}+Ky^i}$, $i>0$, gives a positively **Z**-grading.

COROLLARY 2.4. *A homomorphic image of* $K[x,\ y]/(x,\ y)^4$ *is positively* **Z**-*graded.*

PROOF. $K[x,\ y]/(x,\ y)^4=\overline{K}\oplus\overline{(Kx+Ky)}\oplus\overline{(Kx+Ky)^2}\oplus\overline{(Kx+Ky)^3}$ is a positive **Z**-grading of $K[x,\ y[/(x,\ y)^4$. If $g(x,\ y)=\sum_{i=1}^3 g_i(x,\ y)$ with $g_1(x,\ y)\neq 0$, then $K[x,\ y]/((x,\ y)^4,\ g(x,\ y))$ is uniserial and clearly its homomorphic image is positively **Z**-graded. Therefore by Proposition 2.3 it is enough to consider homomorphic images of $K[x,\ y]/((x,\ y)^4,\ f(x,\ y))$, where $f_0(x,\ y)=f_1(x,\ y)=0$ and $f_2(x,\ y)+f_3(x,\ y)=xy,\ x^2-y^3$ or x^2. However if $b\neq 0$ in the below, the ideal of $K[x,\ y]/((x,\ y)^4,\ x^2-y^3)$ (resp. $K[x,\ y]/((x,\ y)^4,\ x^2))$ generated by $\overline{(axy+by^2+cxy^2+dy^3)}$ contains $\overline{(x,\ y)^3}=\text{rad}^3(K[x,\ y]/((x,\ y)^4,\ x^2-y^3))$ (resp. $\text{rad}^3(K[x,\ y]/((x,\ y)^4,\ x^2)))$. Hence $K[x,\ y]/((x,\ y)^4,\ x^2-y^3,\ axy+by^2+cxy^2+dy^3)$ (resp. $K[x,\ y]/((x,\ y)^4,\ x^2,\ axy+by^2+cxy^2+dy^3))$ with $b\neq 0$ has a cube zero radical and consequently is positively **Z**-graded. Similarly, if $ab\neq 0$, the ideal generated by $\overline{ax^2+by^2+cx^3+dy^3}$ contains $\overline{(x,\ y)^3}=\text{rad}^3(K[x,\ y]/(x,\ y)^4,\ xy))$. Hence $K[x,\ y]/((x,\ y)^4,\ xy,\ ax^2+by^2+cx^3+dy^3)$, with $ab\neq 0$, has a cube zero radical and consequently positively **Z**-graded. Further positive **Z**-gradings of $K[x,\ y]/((x,\ y)^4,\ xy,\ ax^i+by^j)$, $3\geqq i\geqq 2$, $3\geqq j\geqq 2$, are induced by one of $K[x,\ y]/((x,\ y)^4,\ xy)$, if $ab=0$. Also a positive **Z**-grading of $K[x,\ y]/((x,\ y)^4,\ xy,\ ax^3+by^3)$, $ab\neq 0$, is induced by one of $K[x,\ y]/((x,\ y)^4,\ xy)$. Since both $K[x,\ y]/((x,\ y)^4,\ xy,\ ax^2+cx^3+dy^3)$ and $K[x,\ y]/((x,\ y)^4,\ xy,\ by^2+cx^3+dy^3)$ are isomorphic to $K[x,\ y]/((x,\ y)^4,\ x^2-y^3,\ a'xy)$, $a'\in K$, we return to check the positive **Z**-gradability of $K[x,\ y]/((x,\ y)^4,\ x^2-y^3,\ axy^i-by^3)$, $2\geqq i\geqq 1$. But in the case $i=1$ and $ab\neq 0$, it is isomorphic to $K[x,\ y]/((x,\ y)^4,\ b'xy,\ x^2-y^3)$ with $b'(\neq 0)\in K$ because $axy-by^3=(ax-by^2)y$ and we can take $ax-by^2$ and y as new parameters. So the grading is induced by one of $K[x,\ y]/((x,\ y)^4,\ x^2-y^3)$. Further in the case $i=2$ and $ab=0$, the grading is induced by $K[x,\ y]/((x,\ y)^4,\ x^2-y^3)$. For $K[x,\ y]/((x,\ y)^4,\ x^2-y^3,\ axy^2-by^3)$ with $ab\neq 0$, by taking $X=x-(a/2b)y^2$ and $Y=y$ as new parameters we have $K[x,\ y]/((x,\ y)^4,\ x^2-y^3,\ axy^2-by^3)\cong K[X,\ Y]/((X,\ Y)^4,\ X^2,\ aXY^2-bY^3)$ and so the grading of A is induced by $K[X,\ Y]/((X,\ Y)^4,\ X^2)=\overline{K}\oplus\overline{KX+KY}\oplus\overline{KXY+KY^2}\oplus\overline{KXY^2+KY^3}$. For homomorphic images of $K[x,\ y]/((x,\ y)^4,\ x^2)$ it remains to check the positively **Z**-gradability of $K[x,\ y]/((x,\ y)^4,\ axy-by^3)$ with $b\neq 0$, but it is isomorphic to a positively $\mathbf{Z}$-graded algebra $K[x, y]/((x, y)^4, x^2, axy)$. Now by the analogous discussion we know that the grading of any homomorphic image of all local algebra considered above is induced by one of $K[x, y]/((x, y)^4, xy)$, $K[x, y]/((x, y)^4, x^2-y^3)$ or $K[x, y]/((x, y)^4, x^2)$. This completes the proof.

By Theorem 1.3 we have immediately

THEOREM 2.5. *The conjecture* (i) *is true for homomorphic images of* $K[x, y]$ *with quartic zero radicals.*

Similarly we know that the conjecture (i) is true for local algebras $K[x, y]/((x, y)^n, f(x, y))$, where $n\geqq 4$ and $f(x, y)=ax^2+bxy+cy^2+dx^3+\cdots$, provided at least one of a, b, c is nonzero. It seems to be of interest that those local algebras correspond to Arnol'd's normal forms A_t, $t<n$, of functions in the neighborhood of a simple critical point. We are indebted to Drs. K. Watanabe and M. Tomari for drawing our attention to these facts. (cf. [1]).

§ 3. Example of Local Commutative Algebra Which Is Not Gradable

As our proof in § 2 is effective for positively $\mathbf{Z}$-graded local algebras it is important to assure the existence of a local commutative algebra which is not positively $\mathbf{Z}$-graded. The following Proposition provides the example.

PROPOSITION 3.1. *Let* $A=K[x_1, x_2, x_3]/((x_1, x_2, x_3)^4, x_1x_2-x_3^3, x_2x_3-x_1^3, x_3x_1-x_2^3)$. *Then* A *is not positively* $\mathbf{Z}$*-graded.*

PROOF. Suppose that $A=A_0\oplus A_1\oplus A_2\oplus\cdots\oplus A_q$ is a positive $\mathbf{Z}$-grading such that $\operatorname{rad} A=A_1\oplus A_2\oplus\cdots\oplus A_q$. Let us denote by $\overline{f(x, y, z)}$ an element of A which is the residue class containing $f(x, y, z)\in K[x, y, z]$. Then $A=K\bar{1}=K\bar{x}_1+K\bar{x}_2+K\bar{x}_3+K\bar{x}_1^2+K\bar{x}_2^2+K\bar{x}_3^2+K\bar{x}_1^3+K\bar{x}_2^3+K\bar{x}_3^3$, $\operatorname{rad}A=K\bar{x}_1+K\bar{x}_2+K\bar{x}_3+K\bar{x}_1^2+K\bar{x}_2^2+K\bar{x}_3^2+K\bar{x}_1^3+K\bar{x}_2^3+K\bar{x}_3^3$, $\operatorname{rad}^2A = K\bar{x}_1^2+K\bar{x}_2^2+K\bar{x}_3^2+K\bar{x}_1^3+K\bar{x}_2^3+K\bar{x}_3^3$ and $\operatorname{rad}^3A=\operatorname{soc}A=K\bar{x}_1^3\oplus K\bar{x}_2^3\oplus K\bar{x}_3^3$. Since $\dim_K(\operatorname{rad}A\setminus\operatorname{rad}^2A)=3$, there exists $\alpha_i\in\operatorname{rad}A\setminus\operatorname{rad}^2A$, $i=1, 2, 3$ and positive integers n_1, n_2, n_3 such that $\alpha_1\in A_{n_1}$, $\alpha_2\in A_{n_2}$, $\alpha_3\in A_{n_3}$ with $n_1\leqq n_2\leqq n_3$ and α_i, $i=1, 2, 3$, are K-linearly independent. For the simplicity we shall abbreviate from now $\bar{x}_i$ to x_i, $i=1, 2, 3$.

Then we have

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}=(a_{ij})\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}+(b_{ij})\begin{pmatrix}x_1^2\\ x_2^2\\ x_3^2\end{pmatrix}+(c_{ij})\begin{pmatrix}x_1^3\\ x_2^3\\ x_3^3\end{pmatrix},\ a_{ij}, b_{ij}, c_{ij}\in K,$$

$i, j=1, 2, 3$ and $\Delta=\det(a_{ij})\neq 0$.

At first we shall notice that $n_1=n_2=n_3$ is impossible. Let $n=n_1$ and $\begin{pmatrix}\alpha'_1\\ \alpha'_2\\ \alpha'_3\end{pmatrix}=(a_{ij})^{-1}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}$. Then $\alpha'_i\in A_n$, $i=1, 2, 3$, and $\alpha'_i=x_i+\alpha''_i$ with $\alpha''_i\in\mathrm{rad}^2 A$, $i=1, 2, 3$. Therefore $0\neq\alpha'_1\alpha'_2=x_1x_2+cx_1^3+dx_2^3$ with $c, d\in K$. However $\alpha'_1\alpha'_2\in A_{2n}$ but $x_1x_2+cx_1^3+dx_2^3=x_3^3+cx_1^3+dx_2^3\in A_{3n}$ because $\alpha_i'^3=x_i^3$, $i=1, 2, 3$, this is a contradiction.

Now assume that $n_1<n_2<n_3$. It is clear that $0\neq\alpha_i^3\in A_{n_i}^3\subset\mathrm{soc}\,A$, $i=1, 2, 3$. Since $\dim_K\ \mathrm{soc}\,A=3$ and $A_{n_i}^3\subset A_{3n_i}$, $i=1, 2, 3$, $A_{n_1}^3\oplus A_{n_2}^3\oplus A_{n_3}^3=\mathrm{soc}\,A$. By the assumption it holds that $3n_1<n_1+2n_3<n_2+2n_3<3n_3$ and $3n_1<3n_2<n_2+2n_3<3n_3$.

Further we make an assumption (a): $n_1+2n_3\neq 3n_2$. Since $A_{n_1}A_{n_3}^2\subset A_{n_1+2n_3}\cap\mathrm{soc}\,A$ and $A_{n_2}A_{n_3}^2\subset A_{n_2+2n_3}\cap\mathrm{soc}\,A$, $\alpha_1\alpha_3^2=\sum_{i=1}^3 a_{1i}a_{3i}^2x_i^3\in A_{n_1}A_{n_2}^2=0$ and $\alpha_2\alpha_3^2=\sum_{i=1}^3 a_{2i}a_{3i}^2x_i^3\in A_{n_2}A_{n_3}^2=0$. It follows that $a_{1i}a_{3i}^2=a_{2i}a_{3i}^2=0$, $i=1, 2, 3$. Further from $3n_1<2n_1+n_2<3n_2$ we similarly obtain $\alpha_1^2\alpha_2\in A_{n_1}^2A_{n_2}=0$ and consequently $a_{1i}^2a_{2i}=0$, $i=1, 2, 3$. Therefore $a_{1i}\neq 0$ implies $a_{2i}=a_{3i}=0$. Also $a_{2i}\neq 0$ implies $a_{3i}=0$. Then (a_{ij}) must be a monomial matrix because $\Delta=\det(a_{ij})\neq 0$. So we have $0\neq\alpha_1\alpha_2=c\alpha_3^3\in\mathrm{soc}\,A$ for some $c\in K$. But this implies $0\neq A_{n_1}A_{n_2}\cap A_{n_3}^3\subset A_{n_1+n_2}\cap A_{3n_3}$. But $n_1+n_2=3n_3$ contradicts to $n_1<n_2<n_3$.

Now we make another assumption (b): $n_1+2n_3=3n_2$. In this case it holds that $3n_1<2n_1+n_2<2n_1+n_3<n_2+2n_3<3n_3$, $2n_1+n_2<3n_2<n_2+2n_3<3n_3$ and $2n_1+n_3\neq 3n_2$ because $n_1+2n_3=3n_2$ and $n_1<n_2<n_3$. Then $A_{n_1}^3A_{n_2}\subset A_{2n_1+n_2}\cup\mathrm{soc}\,A$, $A_{n_1}A_{n_3}\subset A_{2n_1+n_3}\cap\mathrm{soc}\,A$, and $A_{n_2}A_{n_3}^2\subset A_{n_2+2n_3}\cap\mathrm{soc}\,A$ and they induce $\alpha_1^2\alpha_2=\alpha_1^2\alpha_2=\alpha_1^2\alpha_3=a_2a_3^2=0$. Hence we have $a_{1i}^2a_{2i}=a_{1i}^2a_{3i}=a_{2i}a_{3i}^2=0$, $i=1, 2, 3$, and we arrive at the same contradiction as in the case (a).

Assume now that $n_1<n_2=n_3$. And at first assume further α_2^3 and α_3^3 are K-linearly independent. Then it holds that $3n_1<2n_1+n_1+2n_2<3n_2$ and $n_{n_1}^3\oplus A_{n_1}^2A_{n_2}\oplus A_{n_1}A_{n_3}^2\oplus A_{n_2}^3\subset\mathrm{soc}\,A$. So it follows that $A_{n_1}^2A_{n_2}=A_{n_1}A_{n_3}^2=0$ and hence $a_{1i}^2a_{2i}=a_{1i}a_{3i}^2=0$, $i=1, 2, 3$. If $a_{11}\neq 0$, then $a_{21}=a_{31}=0$.

Further suppose one of a_{12} or a_{13} is nonzero. Then $\Delta=0$. Therefore $a_{21}=a_{31}=a_{12}=a_1{}^s=0$ and $\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}\neq 0$.

So we have

$$\alpha_1=a_{11}x_1+b_{11}x_1^2+b_{12}x_2^2+b_{13}x_3^2+\beta'_1,$$

$$\alpha_2=a_{22}x_2+a_{23}x_3+b_{21}x_1^2+b_{22}x_2^2+b_{23}x_3^2+\beta'_2$$

and

$$\alpha_3=a_{32}x_2+a_{33}x_3+b_{31}x_1^2+b_{32}x_2^2+b_{33}x_3^2+\beta'_3,$$

where $\beta'_i\in\mathrm{rad}^3A$, $i=1, 2, 3$.

Then we have $x_2=d_{22}\alpha_2+d_{23}\alpha_3+x_2'$ and $x_3=d_{32}\alpha_2+d_{33}\alpha_3+x_3'$ with $x_2', x_3'\in \mathrm{rad}^2A$ and $(d_{ij})=\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{pmatrix}^{-1}$, $i, j=2, 3$ and hence $0\neq A_{n_1}^3\cap A_{n_2}^2$ and $3n_1=2n_2$ because $a_{11}^{-3}\alpha_1^3=x_2x_3\in A_{n_1}^3$ and $(d_{22}\alpha_2+d_{23}\alpha_3)(d_{32}\alpha_2+d_{33}\alpha_3)\in A_{n_2}^2$ and $x_2'(d_{22}\alpha_2+d_{23}\alpha_3)+(d_{32}\alpha_2+d_{33}\alpha_3)x_3'\in Kx_2^3\oplus Kx_3^3=A_{n_2}^3$.

However either $\alpha_1\alpha_2=a_{11}b_{21}x_1^3+(b_{12}a_{22}a_{11}a_{23})x_2^3+(a_{11}a_{22}+b_{13}a_{23})x_3^3$ or $\alpha_1\alpha_3=a_{11}b_{31}x_1^3+(b_{12}a_{32}+a_{11}a_{33})x_2^3+(a_{11}a_{32}+b_{13}a_{33})x_3^3$ is nonzero, for otherwise $a_{11}=0$ and it contradicts to our assumption. As they belong to both $A_{n_1}A_{n_2}$ and soc $A=A_{n_1}^3\oplus A_{n_2}^3$, we have $n_1+n_2=3n_1$, i. e. $n_2=2n_1$. But this is also impossible because $3n_1=2n_2$. As in the case where $a_{12}\neq 0$ or $a_{13}\neq 0$ we arrive at a similar contradiction. We can proceed our proof to the next case where α_2^3 and α_3^3 are K-linearly dependent. Then since $a_{21}^3/a_{31}^3=a_{22}^3/a_{32}^3=a_{23}^3/a_{33}^3$, we have $a_{31}=\omega_1a_{21}$, $a_{32}=\omega_2a_{22}$ and $a_{33}=\omega_3a_{23}$, where ω_i, $i=1, 2, 3$, are cube roots of unit. (It is to be noted that this case does not occur if the characteristic of K is 3).

Now the inequality $3n_1<2n_1+n_2<n_1+2n_2<3n_2$ induces either $A_{n_1}^2A_{n_2}=0$ or $A_{n_1}A_{n_2}^2=0$. Then according to them we have either $a_{11}^2a_{21}=a_{12}^2a_{22}=a_{13}^2a_{23}=0$ and $\mathrm{soc}A=A_{n_1}A_{n_2}^2\oplus A_{n_1}^3\oplus A_{n_2}^3$, or $a_{11}a_{21}^2=a_{12}a_{22}^2=a_{13}a_{23}^2=0$ and $\mathrm{soc}A=A_{n_1}^2A_{n_2}\oplus A_{n_1}^3\oplus A_{n_2}^3$.

Assume $a_{11}\neq 0$. Then $a_{21}=0$. And both a_{12} and $a_{13}=0$; otherwise, $a_{12}\neq 0$ or $a_{13}\neq 0$ implies $\Delta=0$. Therefore we have $\alpha_1=a_{11}x_1+\gamma_1'$, $\alpha_2=a_{22}x_2+a_{23}x_3+\gamma_3'$ and $\alpha_3=\omega_2a_{22}x_2+\omega_3a_{23}x_3+\gamma_2'$, where $\gamma_i'\in\mathrm{rad}^2A$, $i=1, 2, 3$. Then similarly as in the preceding case, from the assumption $a_{11}\neq 0$ and $\begin{vmatrix} a_{22} & a_{23} \\ \omega_2a_{22} & \omega_3a_{23}\end{vmatrix}\neq 0$ we have $2n_2=3n_1$, and either $\alpha_1\alpha_2\neq 0$ or $\alpha_1\alpha_3\neq 0$. The later fact induces that $A_{n_1}A_{n_2}\cap(A_{n_1}A_{n_2}^2\oplus A_{n_1}^3\oplus A_{n_2}^3)\neq 0$ or $A_{n_1}A_{n_2}\cap(A_{n_1}^2A_{n_2}\oplus A_{n_1}^3\oplus A_{n_2}^3)\neq 0$, and it follows that $n_1+n_2=3n_1$, but this contradicts to $2n_2=3n_1$. In the case where $a_{12}\neq 0$ or $a_{13}\neq 0$, we also arrive at a similar contradiction.

Now it remains to prove thae $n_1=n_2<n_3$ does not occur. In this case $A_{n_1}^3\oplus A_{n_3}^3=\mathrm{soc}\,A$ and α_1^3 and α_2^3 are K-linearly independent, and the inequality $3n_1<2n_1+n_3<n_1+2n_3<3n_3$ implies $A_{n_1}^2A_{n_3}=0$ and $A_{n_1}A_{n_3}^2=0$. Thus we have $a_{11}^2a_{31}=a_{12}^2a_{32}=a_{13}^2a_{33}=0$ and $a_{11}a_{31}^2=a_{12}a_{32}^2=a_{13}a_{33}^2=0$. Then similarly as in the case where $n_1<n_2=n_3$ and α_2^3 and α_3^3 are K-linearly independent, we arrive at a similar contradiction.

It is to be noted that for this example our conjecture (i) is true.

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