TSUKUBA J. MATH.
Vol. 14 No. 1 (1990), 9–26

A CHARACTERIZATION OF A REAL HYPERSURFACE OF TYPE B

By

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Introduction.

A complex n-dimensional Kaehler manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a complex projective space P_nC, a complex Euclidean space C^n or a complex hyperbolic space H_nC according as $c>0$, $c=0$ or $c<0$.

In his study [12] of real hypersurfaces of P_nC, Takagi showed that all homogeneous hypersurfaces could be divided into six types. Namely, he proved the following

THEOREM A. *Let M be a homogeneous real hypersurface of P_nC. Then M is locally congruent to one of the following hypersurfaces*:

(A_1) *a geodesic hypersurface,*
(A_2) *a tube over a totally geodesic P_kC $(1\leqq k\leqq n-2)$,*
(B) *a tube over a complex quadric Q_{n-1},*
(C) *a tube over $P_1C\times P_{(n-1)/2}C$ and $n(\geqq 5)$ is odd,*
(D) *a tube over a complex Grassmann $G_{2,5}$ and $n=9$,*
(E) *a tube over a Hermitian symmetric space $SO(10)/U(5)$ and $n=15$.*

Moreover, Takagi [13] proved that if a real hypersurface of P_nC has two or three distinct constant principal curvatures, then M is locally congruent to the case of the homogeneous ones of type A_1, A_2 or B. In what follows the induced almost contact metric structure of the real hypersurface of $M_n(c)$ is denoted by (ϕ, g, ξ, η). The structure vector ξ is said to be *principal* if $A\xi=\alpha\xi$, where A is the shape operator in the direction of the unit normal C and $\alpha=\eta(A\xi)$. Real hypersurfaces of P_nC have been studied by many differential geometers ([2], [4], [5], [6] and [7] etc.) and as one of them, Kimura [5] asserts

*) This research was partially supported by TGRC-KOSEF.
Received January 17, 1989.

that M has constant principal curvatures and ξ is principal if and only if M is locally congruent to a homogeneous real hypersurface.

On the other hand, real hypersurfaces of H_nC have also been investigated by many authors ([1], [3], [4], [8], [9] and [11] etc.) In particular, Berndt [1] proved recently the following interesting result.

THEOREM B. *Let M be a real hypersurface of H_nC, $n\geqq 2$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following*:

(A$_0$) *a horosphere in H_nC,*

(A$_1$) *a geodesic hypersurface or a tube over a complex hyperbolic hyperplane $H_{n-1}C$,*

(A$_2$) *a tube over a totally geodesic submanifold H_kC $(1\leqq k\leqq n-2)$,*

(B) *a tube over a totally real hypersurface $H_n\boldsymbol{R}$.*

According to Takagi's classification theorem and Berndt's one, the principal curvatures and their multiplicitites of homogeneous real hypersurfaces of $M_n(c)$ are given.

Now, in [10] Maeda estimated the norm of the second fundamental form and proved the following

THEOREM C. *Let M be a real hypersurface of P_nC. Then*

$$|\nabla A|^2\geqq(n-1)c^2/4\,,$$

where the equality holds if and only if M is locally congruent to one of the real hypersurfaces of type A$_1$ *and* A$_2$.

For the above minimum of the norm of the covariant derivative of the second fundamental form, it seems to be interested to investigate whether or not there is the next estimation of the norm which gives a characterization of real hypersurfaces of P_nC different from type A$_1$ and A$_2$. The purpose of this paper is to obtain an estimation under certain condition and to give the following characterization of real hypersurfaces of type B.

THEOREM. *Let M be a real hypersurface of P_nC, $n\geqq 3$, on which the structure vector ξ is principal with principal curvature α. If $\alpha\ \mathrm{Tr}\,A\leqq\alpha^2-(n-1)c$, then*

$$|\nabla A|^2\geqq(n-1)c(7c+6\alpha^2)/4\,,$$

where the equality holds if and only if M is locally congruent to a real hyper-

surface of type B.

1. Preliminaries.

We begin with recalling basic properties of real hypersurfaces of a complex space form. Let M be a real hypersurface of an $n(\geqq 2)$-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c(\neq 0)$ and let C be a unit normal field on a neighborhood of a point x in M. We denote by J the almost complex structure of $M_n(c)$. For a local vector field X on a neighborhood of x in M, the images of X and C under the transformation J can be represented as

$$JX=\phi X+\eta(X)\xi, \qquad JC=-\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on the neighborhood of x in M, respectively. Moreover, it is seen that $g(\xi, X)=\eta(X)$, where g denotes the induced Riemannian metric on M. By properties of the almost complex structure J, the set (ϕ, ξ, η, g) of tensors satisfies then

$$\phi^2=-I+\eta\otimes\xi, \quad \phi\xi=0, \quad \eta(\phi X)=0, \quad \eta(\xi)=1, \tag{1.1}$$

where I denotes the identity transformation. Accordingly, the set defines the almost contact metric structure on M. Furthermore, the covariant derivatives of the structure tensors are given by

$$\nabla_X\phi(Y)=\eta(Y)AX-g(AX, Y)\xi, \qquad \nabla_X\xi=\phi AX, \tag{1.2}$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to C on M.

Since the ambient space is of constant holomorphic sectional curcature c, the equations of Gauss and Codazzi are respectively given as follows:

$$\begin{aligned} R(X, Y)Z=c\{&g(Y, Z)X-g(X, Z)Y+ \\ &+g(\phi Y, Z)\phi X-g(\phi X, Z)\phi Y-2g(\phi X, Y)\phi Z\}/4 \\ &+g(AY, Z)AX-g(AX, Z)AY, \end{aligned} \tag{1.3}$$

$$\nabla_X A(Y)-\nabla_Y A(X)=c\{\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X, Y)\xi\}/4, \tag{1.4}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

The Ricci tensor S' of M is the tensor of type (0, 2) given by $S'(X, Y)=\mathrm{tr}\{Z\rightarrow R(Z, X)Y\}$. But it may be also regarded as the tensor of type (1,1) and denoted by $S: TM \to TM$; it satisfies $S'(X, Y)=g(SX, Y)$. By the Gauss equation, (1.1) and (1.2) the Ricci tensor S is given by

$$S=c\{(2n+1)I-3\eta\otimes\xi\}/4+hA-A^2, \tag{1.5}$$

where h is the trace of the shape operator A. Recently, in order to give another characterization of homogeneous hypersurfaces of type A_1, A_2 and B in P_nC, Kimura and Maeda [6] introduced the notion of a η-parallel second fundamental form, which was defined by $g(\nabla_X A(Y), Z)=0$ for any vector fields X, Y and Z orthogonal to ξ. Now, we prepare the followings:

THEOREM D ([6]). *Let M be a real hypersurface of P_nC. Then the second fundamental form is η-parallel and ξ is principal if and only if M is locally congruent to one of the homogeneous real hypersurfaces of type* A_1, A_2 *and* B.

THEOREM E ([11]). *Let M be a real hypersurface of H_nC. Then the second fundamental form is η-parallel and ξ is principal if and only if M is locally congruent to one of real hypersurfaces of type* $A_0 \sim$ B.

PROPOSITION F ([4]). *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$. If the structure vector ξ is principal, then the corresponding principal curvature α is locally constant.*

In the sequel, assume that the structure vector ξ is principal and denote by α the corresponding principal curvature. Namely, $A\xi=\alpha\xi$ is assumed. It follows from (1.4) that we have

$$2A\phi A=c\phi/2+\alpha(A\phi+\phi A) \tag{1.7}$$

and therefore, if $AX=\lambda X$ for any vector field X orthogonal to ξ, then we get

$$(2\lambda-\alpha)A\phi X=(\alpha\lambda+c/2)\phi X. \tag{1.8}$$

Accordingly, it turns out that in the case where $\alpha^2+c\neq 0$, ϕX is also a principal vector with principal curvature $\mu=(\alpha\lambda+c/2)/(2\lambda-\alpha)$, namely, we have

$$\begin{aligned} &A\phi X=\mu\phi X, \\ &2\lambda-\alpha\neq 0, \qquad \mu=(\alpha\lambda+c/2)/(2\lambda-\alpha). \end{aligned} \tag{1.9}$$

2. Real hypersurfaces of type B.

This section is devoted to the investigation of a characterization of real hypersurfaces of type B in a complex space form. First of all, we shall consider a property of hypersurfaces of a real space form. An m-dimensional Riemannian manifold of constant curvature c is called a *real space form*, which is denoted by $N^{m}(c)$. Let N be a hypersurface of $N^{m+1}(c)$ and let A' be the shape operator and g' (resp. ∇') be the induced Riemannian metric tensor (resp. the Riemannian connection) of N.

LEMMA 2.1. *Let N be a hypersurface of an $(m+1)$-dimensional real space form $N^{m+1}(c)$ of constant curvature c and let D' be a distribution defined by the kernel of the operator $A'^2+aA'+bI$ for some constants a and b. If $a^2\neq 4b$, then we have*

$$g'(\nabla'_X A'(Y),\ Z)=0 \tag{2.1}$$

for any vector fields X, Y and Z in D'.

PROOF. Differentiating the equation $(A'^2+aA'+bI)Y=0$ covariantly, we get

$$\nabla'_X A'(A'Y)+A'\nabla'_X A'(Y)+a\nabla'_X A'(Y)+(A'^2+aA'+bI)\nabla'_X Y=0 \tag{2,2}$$

for any vector field X. Taking account of the fact that A' and A'^2 are self-adjoint, we have

$$g'(\nabla'_X A'(A'Y)+A'\nabla'_X A'(Y)+a\nabla'_X A'(Y),\ Z)=0 \tag{2.3}$$

for any vector fields Y and Z in D'. Substituting $A'X$ into X in (2.3), we have

$$g'(\nabla'_{A'X} A'(A'Y)+A'\nabla'_{A'X} A'(Y)+a\nabla'_{A'X} A'(Y),\ Z)=0$$

because Z belongs to D'. Since $g'(\nabla'_X A'(Y),\ Z)$ is symmetric with respect to X, Y and Z by means of the Codazzi equation for hypersurfaces of a real space form and since the distribution D' is A'-invaiant, the first term of the above equation can be deformed for any vector fields X, Y and Z in D' as follows:

$$\begin{aligned} g'(\nabla'_Z A'(A'X),\ A'Y) &= -g'(A'\nabla'_Z A'(X)+a\nabla'_Z A'(X),\ A'Y) \\ &= g'(\nabla'_Z A'(X),\ aA'Y+bY)-ag'(\nabla'_Z A'(X),\ A'Y) \\ &= bg'(\nabla'_Z A'(X),\ Y), \end{aligned}$$

where the definition of the distribution and (2.3) are used. It turns out that we have $2bg'(\nabla'_Z A'(X),\ Y)+ag'(\nabla'_Y A'(Z),\ A'X)=0$ for any X, Y and Z in D'. If $a=0$, then it is trivial, so unless it is seen that $g'(\nabla'_Y A'(Z),\ A'X)$ is symmetric with respect to X, Y and Z, from which together with (2.3) it follows that $2g'(\nabla'_X A'(Y),\ A'Z)+ag'(\nabla'_X A'(Y),\ Z)=0$. By the last two equations we have

$(a^2-4b)g'(\nabla'_X A'(Y), Z)=0$ for any vector fields X, Y and Z in D'. q. e. d.

REMARK 2.1. In [9], Montiel and Romero proved that in a Lorentz hypersurface N_1^m of an anti-de Sitter space H_1^{m+1}, if the shape operator A' satisfies a polynomial $p(x)=x^2-ax+1$ for some constant a such that $a^2\neq 4$, then A' is parallel. The proof of Lemma 2.1 is essentially similar to that of their result.

When the shape operator A' satisfies (2.1), it is said to be *D'-parallel.*

Combing back to the subject of real hypersurfaces of a complex space form, let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 3$, on which the structrue vector ξ is principal. That is,

$$A\xi=\alpha\xi\,, \tag{2.4}$$

where α is constant by Proposition F. The covariant derivative of (2.4) gives $\nabla_X A(\xi)=\alpha\phi AX-A\phi AX$ for any vector field X, where we have used the second equation of (1.2), from which together with (1.7) and the above equation it follows that

$$\nabla_X A(\xi)=-c\phi X/4-\alpha(A\phi-\phi A)X/2\,. \tag{2.5}$$

On the other hand, the Codazzi equation (1.4) coupled with (2.5) implies

$$\nabla_\xi A(X)=-\alpha(A\phi-\phi A)X/2\,. \tag{2.6}$$

By $X^\perp$ we denote an $\xi^\perp$-component of a vector field X. Namely, let $X=X^\perp+\eta(X)\xi$. Then the vector field $\nabla_X A(Y)$ can be decomposed into three terms as follows;

$$\nabla_X A(Y)=\nabla_{X^\perp}A(Y^\perp)+\eta(X)\nabla_\xi A(Y^\perp)+\eta(Y)\nabla_{X^\perp}A(\xi) \tag{2.7}$$

because of $\nabla_\xi A(\xi)=0$ by means of (2.4) and (2.6). Taking account of (2.5) and (2.6), we seen that (2.7) is reformed as

$$\begin{aligned}\nabla_X A(Y)=&(\nabla_{X^\perp}A(Y^\perp))^\perp-c\{\eta(Y)\phi X+g(\phi X, Y)\xi\}/4\\&-\alpha\{\eta(X)(A\phi-\phi A)Y+\eta(Y)(A\phi-\phi A)X\\&+g((A\phi-\phi A)X, Y)\xi\}/2\,.\end{aligned} \tag{2.8}$$

In particular, assume that M is of type B. Then the shape operator A satisfies

$$A\phi+\phi A=k\phi\,,\qquad k=-c/\alpha\,. \tag{2.9}$$

In this case, by using (2.9), the equation (2.5) is equivalent to

$$\nabla_X A(\xi)=-\alpha(A\phi-3\phi A)X/4\,, \tag{2.10}$$

which together with (2.6) and (2.7) yields

$$\nabla_X A(Y)=-\alpha\{2\eta(X)(A\phi-\phi A)Y+\eta(Y)(A\phi-3\phi A)X + g((A\phi-3\phi A)X,\ Y)\xi\}/4 \tag{2.11}$$

because the shape operator A of M is η-parallel by Theorems D and E.

Now, let S^{2n+1} be a $(2n+1)$-dimensional unit sphere in a $(2n+2)$-dimensional Euclidean space $\boldsymbol{R}^{2n+2}=\boldsymbol{C}^{n+1}$ and let (Φ, E, ω, G) be the Sasakian structure induced from the natural almost complex structure on $\boldsymbol{C}^{n+1}$. Then the Hopf fibration $\pi: S^{2n+1}\to P_n\boldsymbol{C}$ is a principal circle bundle over $P_n\boldsymbol{C}$, in which ω is a connection and the orbits of E are fibers. Let * be a horizontal lift with respect to this connection ω. The usual Riemannian structure on $P_n\boldsymbol{C}$ is characterized by the fact that π is a Riemannian submersion. Let M be a real hypersurface of a complex projective space $P_n\boldsymbol{C}$. Then the principal circle bundle N over M is a hypersurface of S^{2n+1} and the natural immersion i' of N into S^{2n+1} respects the submersion π. That is, N is the hypersurface of S^{2n+1} tangent to E and for the Hopf fibration $\pi: S^{n+1}\to P_n\boldsymbol{C}$ there is a fibration $\pi: N\to M$, where M is a real hypersurface of $P_n\boldsymbol{C}$ such that the diagram

$$\begin{array}{ccc} N & \xrightarrow{\ i'\ } & S^{2n+1} \\ \pi\downarrow & & \downarrow\pi \\ M & \xrightarrow[\ i\ ]{} & P_n\boldsymbol{C} \end{array}$$

is commutative and the immersion i' of N into S^{2n+1} is a diffeomorphism of the fibers. By g' and g the induced Riemannian metric tensors of N and M are denoted, respectively. Let ∇' and ∇ be also the Riemannian connections of N and M, and the shape operators are denoted by A' and A. Then it is seen that

$$g'(\nabla'_{X^*}A'(Y^*),\ Z^*)=\{g(\nabla_X A(Y),\ Z)+\eta(Y)g(\phi X,\ Z)+\eta(Z)g(\phi X,\ Y)\}^* \tag{2.12}$$

for any vector fields X, Y and Z on M.

Next, for positive numbers r and s and an integer m $(2\leqq m\leqq n-1)$ we denote by $N_0(2n, r)$ and $N(2n, m, s)$ real hypersurfaces of S^{2n+1} defined by $\{z=(z_1, \cdots, z_{n+1})\in S^{2n+1}\subset \boldsymbol{C}^{n+1}: \sum_{j=1}^{n}|z_j|^2=r|z_{n+1}|^2\}$ and $\{z=(z_1, \cdots, z_{n+1})\in S^{2n+1}\subset \boldsymbol{C}^{n+1}: \sum_{j=1}^{m}|z_j|^2=s\sum_{k=m+1}^{n+1}|z_k|^2\}$, respectively. Then $N_0(2n, r)$ and $N(2n, m, s)$ have two distinct constant principal curvatures $\cot\theta$ and $-\tan\theta$ with multiplicities $2n-1$, 1 and $m-1$, $2n-m+1$, respectively, and the angle θ is given by $\cos^2\theta=1/(r+1)$, $\sin^2\theta=(m-1)/(n-1)$ and $s=(m-1)/(n-m)$. Accordingly the shape operator A' is parallel by Lemma 2.1. On the other hand, for a number t $(0<t<1)$ we denote by $N(2n, t)$ a hypersurface of S^{2n+1} defined by $\{z=(z_1, \cdots, z_{n+1})\in S^{2n+1}\subset C^{n+1}: |\sum_{j=1}^{n+1}z_j^2|^2=t\}$. Then $N(2n, t)$ has four distinct constant principal curvatures $\cot(\theta-\pi/4)$, $\cot\theta$, $\cot(\theta+\pi/4)=-\tan(\theta-\pi/4)$, $\cot(\theta+\pi/2)=-\tan\theta$ with multiplicities $n-1$, 1, $n-1$ and 1, respectively and the angle θ is given by $t=\sin^2 2\theta$ $(0<\theta<\pi/4)$. By D' a distribution over $N(2n, t)$ of the direct sum of eigenspaces corresponding to principal curvatures $\cot(\theta-\pi/4)$ and $-\tan(\theta-\pi/4)$ is denoted. Then Lemma 2.1 yields that A' is D'-parallel, but it is not parallel by the property of isoparametric hypersurfaces in a sphere. Thus, in real hypersurfaces $M_0(2n-1, r)=\pi(N_0(2n, r))$ of type A_1, $M(2n-1, m, s)=\pi(N(2n, m, s))$ of type A_2 and $M(2n-1, t)=\pi(N(2n, t))$ of type B, it is seen that the shape operator A is η-parallel because of (2.12). Moreover, $\nabla_\xi A(Y)$ vanishes identically in the former two hypersurfaces, but it is not so in the last case. This means that the equation (2.11) gives a characterization by which real hypersurfaces of type B of P_nC are distinguished from ones of type A_1 and A_2.

For real hypersurfaces of a complex hyperbolic space H_nC the same situation as above is considered.

In the sequel, let M be a real hypersurface of a complex space form $M_n(c)$, $c\neq 0$, $n\geqq 3$. The shape operator A in the direction of C is said to be *pseudo-parallel* if A satisfies

$$\nabla_X A(Y)=a\{2\eta(X)(A\phi-\phi A)Y+\eta(Y)(A\phi-3\phi A)X +g((A\phi-3\phi A)X, Y)\xi\} \tag{2.13}$$

for any tangent vector fields X, Y and $a\in \boldsymbol{R}$ and to be *pseudo-η-parallel* if (2.13) holds for any vector fields X and Y orthogonal to ξ and $a\in\boldsymbol{R}$. The hypersurface M is said to be *of pseudo-(η-)parallel* if A is pseudo-(η-)parallel, respectively. For these hypersurfaces, the following property is first asserted.

LEMMA 2.2. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 3$. If M is of pseudo-parallel, then ξ is principal.*

PROOF. Suppose that M is of pseudo-parallel and $a=0$. Then the shape operator A is parallel. However, it is well known that there exist no real hypersurfaces with parallel shape operator of $M_n(c)$, $c\neq 0$, $n\geqq 3$. Thus we may assume that a is nonzero. From (2.13) it is seen that

$$\nabla_X A(Y)-\nabla_Y A(X)=a\{\eta(X)(A\phi+\phi A)Y-\eta(Y)(A\phi+\phi A)X -2g((A\phi+\phi A)X, Y)\xi\}.$$

On the other hand, it follows from the Codazzi equation (1.4) that

$$\nabla_X A(Y)-\nabla_Y A(X)=c\{\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X, Y)\xi\}/4,$$

which combined with the last equation turns out to be

$$\eta(X)BY-\eta(Y)BX-2g(BX, Y)\xi=0 \tag{2.14}$$

for any tangent vector fields X and Y, where we have put $B=a(A\phi+\phi A)-c\phi/4$ which is the skew-symmetric transformation. It is clear from (2.14) that $g(BX, Y)=0$ for any tangent vector fields X and Y orthogonal to ξ. Putting $Y=\xi$ in (2.14), we obtain $-BX-2g(BX, \xi)\xi=0$ for any vector field X orthogonal to ξ and hence the above two equations means that $BX=0$ for any vector field X orthogonal to ξ.

Moreover, making use of (1.1), we find easily that $B\xi=a\phi A\xi$. Replacing Y by $B\xi$ and putting $X=\xi$ in (2.14), we obtain $B^2\xi-2g(B\xi, B\xi)\xi=0$, where we have used the fact that $B\xi=a\phi A\xi$. Thus $B\xi=0$, which means, coupled with the fact $BX=0$ for any vector field X orthogonal to ξ, that the operator B vanishes identically on M. That is, $a(A\phi+\phi A)=c\phi/4$. Because of $a\neq 0$, we get $A\phi+\phi A=k\phi$, where $k=c/4a\in \boldsymbol{R}-\{0\}$. Hence ξ is principal. q. e. d.

From the above lemma, we reach a theorem which gives a characterization of real hypersurfaces of type B of $M_n(c)$.

THEOREM 2.3. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 3$. Then M is of pseudo-parallel if and only if M is locally congruent to one of real hypersurfaces of type* A_0, A_1 *and* B.

PROOF. It is enough to show that the "only if" part is true. Let M be of pseudo-parallel. Then ξ is principal and M is of η-parallel. So, by using Theorems D and E, M is locally congruent to one of real hypersurfaces of type A and B (type A means A_1 or A_2 when $c>0$ and A_0, A_1 or A_2 when $c<0$). Therefore, the proof is complete, since the equation (2.13) is not realized for a a real hypersurface of type A_2. q. e. d.

Moreover, taking account of the proof above, we have

COROLLARY 2.4. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 3$. If ξ is principal, then M is of pseudo-η-parallel if and only if M is locally congruent to a real hypersurface of type A_0, A_1 and B.*

Now, in order to prove the main theorem mentioned in the introduction we consider a tensor H of type (1,2) defined by

$$H(X, Y)=\nabla_X A(Y)+\alpha\{2\eta(X)(A\phi-\phi A)Y+\eta(Y)(A\phi-3\phi A)X$$
$$+g((A\phi-3\phi A)X, Y)\xi\}/4.$$

If H vanishes identically on M, then M is of pseudo-parallel and hence M is of type A_0, A_1 or B by means of Theorem 2.3. Galculating the norm of H and using (1.1), (1.7), (2.5) and (2.6), we have

$$|\nabla A|^2 \geqq \{10\alpha^2 h_2-2(7\alpha^2+2c)\alpha h+4\alpha^4-(7n-11)\alpha^2 c\}/8, \tag{2.15}$$

where h_2 denotes the trace of the transformation A^2 and $|\nabla A|$ is the norm of ∇A. The equality holds if and only if the tensor H vanishes identically and therefore M is of type A_0, A_1 and B. Now, let M be a real hypersurface of type B in $M_n(c)$, $c\neq 0$, $n\geqq 3$. Then the restriction of the shape operator A to the orthogonal complement $\xi^\perp$ satisfies $A^2-kA-cI/4=0$, where $k=-c/\alpha$, from which the equation $h_2-\alpha^2-k(h-\alpha)-(n-1)c/2=0$ is derived. Since the shape operator A of M has also three distinct principal curvatures $\alpha=\sqrt{c}\cot 2\theta$, $(\sqrt{c}/2)\cot(\theta-\pi/4)$ and $-(\sqrt{c}/2)\tan(\theta-\pi/4)$ if $c>0$ or $\alpha=\sqrt{-c}\tanh 2\theta$, $(\sqrt{-c}/2)\coth(\theta-\pi/4)$ and $-(\sqrt{-c}/2)\tanh(\theta-\pi/4)$ if $c<0$, with multiplicities 1, $n-1$, $n-1$, respectively, which gives us $h=\alpha+(n-1)k$. Combining (2.15) and making use of above properties, we find

$$|\nabla A|^2=(n-1)(7c^2+6\alpha^2 c)/4. \tag{2.16}$$

REMARK 2.2. Let M be a real hypersurface of type A of $P_n C$. Then it is seen in [7] that the norm of ∇A satisfies $|\nabla A|^2=(n-1)c^2/4$. In particular, note here that in a real hypersurface of type A_0 in $H_n C$, (2.16) holds because of $\alpha^2+c=0$.

Next, we calculate the norm of the tensor $A\phi+\phi A-k\phi$, where $k=-c/\alpha$. Under the assumption that ξ is principal, we get

$$2\alpha^2 h_2-4\alpha^2(\alpha^2+c)+(\alpha^2+2c)\{(n-1)c+2\alpha h\}\geqq 0, \tag{2.17}$$

where the equality holds if and only if M is of type A_0, A_1 and B, from which together with (2.15) it follows that

$$|\nabla A|^2\geqq -\{12(\alpha^2+c)\alpha h+5(n-1)c^2+6(n-3)c\alpha^2-12\alpha^4\}/4. \tag{2.18}$$

Suppose that $(\alpha^2+c)\{\alpha h-\alpha^2+(n-1)c\}\leqq 0$. Then the inequality $|\nabla A|^2\geqq (n-1)(7c^2+6\alpha^2 c)/4$ is derived from (2.18). Thus, by Theorem 2.3 we oetain the following theorem. The main theorem is its direct consequence.

THEOREM 2.5. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 3$, on which*

ξ *is principal. If* $(\alpha^2+c)\{\alpha h-\alpha^2+(n-1)c\}\leqq 0$, *then we have*

$$|\nabla A|^2 \geqq (n-1)(7c^2+6\alpha^2 c)/4\,,$$

where the equality holds if and only if M is locally congruent to one of real hypersurfaces of type A_0 *and B, the former case arising only when* $c<0$.

REMARK 2.3. Let M be a real hypersurface of type A_2 in a complex hyperbolic space $H_n\boldsymbol{C}$. Then the principal curvatures are given by $\alpha=\sqrt{-c}\coth 2\theta$, $\sqrt{-c}/2\coth\theta$ and $\sqrt{-c}/2\tanh\theta$ with multiplicities 1, p and q, respectively, and hence we have

$$\alpha h-\alpha^2+(n-1)c=-c(p\coth^2\theta+q\tanh^2\theta)/4+(n-1)c/2\,.$$

For any integers p and q such that $p\geqq q$, the right hand side is monotonously decreasing and bounded from below by 0 and hence there is some θ such that $0<\alpha h-\alpha^2+(n-1)c<\varepsilon$ for any sufficiently small positive ε. While $\alpha^2+c=c(1-\coth^2 2\theta)$ is positive, we have $(\alpha^2+c)\{\alpha h-\alpha^2+(n-1)c\}>0$. This means that the assumption of $(\alpha^2+c)\{\alpha h-\alpha^2+(n-1)c\}\leqq 0$ in the above theorem can not be omitted in $H_n\boldsymbol{C}$.

REMARK 2.4. Let M be a real hypersurface of type A_1 or A_2 in $P_n\boldsymbol{C}$. Then we have $|\nabla A|^2=(n-1)c^2/4$. On the other hand, the principal curvatures are given by $\alpha=\sqrt{c}\cot 2\theta$, $\sqrt{c}/2\cot\theta$ and $-\sqrt{c}/2\tan\theta$ with multiplicities 1, p and q, respectively, and hence we have

$$\begin{aligned}\alpha h-\alpha^2+(n-1)c&=c(p\cot^2\theta+q\tan^2\theta)/4+(n-1)c/2\\&\geqq (n-1)c/2\,.\end{aligned}$$

Last, we give another characterization of a real hypersurface of type B in $P_n\boldsymbol{C}$. The inequality (2.17) holds under the assumption that the structure vector ξ is principal. Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 3$, on which ξ is not assumed to be principal. For a vector field $U=\nabla_\xi\xi$ there is a 1-form ω associated with it, which is given by $\omega(X)=g(U, X)$. The codifferential $\delta\theta$ of the 1-form ω is given by $\delta\omega=-\sum_{j=1}^{2n-1}g(E_j, \nabla_{E_j}U)$, where $\{E_j\}$ is an orthonormal frame such that $E_{2n-1}=\xi$. Then, taking account of two equations of (1.2) and the Codazzi equation (1.4), we have

$$2h_2-2(\alpha_2+\alpha^2+2c)+(1+2c/\alpha^2)\{(n-1)c+2\alpha h\}\geqq 2\delta\omega\,,$$

where $\alpha=g(A\xi, \xi)$ and $\alpha_2=g(A\xi, A\xi)$, in place of (2.17). Accordingly, by the well known theorem due to Green and by Yano and Kon's theorem [14], one finds

THEOREM 2.6. *Let M be a compact orientable real hypersurface of P_nC. If the shape operator A satisfies*

$$2\,\mathrm{Tr}\,A^2 \leqq 2(\alpha_2+\alpha^2+2c)-(1+2c/\alpha^2)\{(n-1)c+2\alpha\,\mathrm{Tr}\,A\}, \qquad \alpha\neq 0,$$

then M is congruent to one of the real hypersurfaces of type A_1 *and* B.

REMARK 2.5. In [10], Okumura proved that in a compact orientable real hypersurface M of P_nC if the shape operator A of M satisfies

$$\mathrm{Tr}\,A^2 \leqq \alpha\,\mathrm{Tr}\,A+(n-1)c/2,$$

then M is of type A_1 or A_2.

3. Real hypersurfaces of type A and B.

Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 3$. The shape operator A is said to be *cyclic-pseudo-parallel* if $\mathfrak{S}g(\nabla_X A(Y), Z)=3aT(X, Y, Z)$ for any vector fields X, Y and Z, where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z and we have put $T(X, Y, Z)=\eta(X)g((A\phi-\phi A)Y, Z)$ and $a\in\boldsymbol{R}$. We say that M is *of cyclic-pseudo-parallel* if A is cyclic-pseudo-parallel. We note here that a real hypersurface of $M_n(c)$ is said to be of *cyclic-parallel* if a is zero (cf. [3]).

REMARK 3.1. It was proved in [3] and [9] that a real hypersurface of $M_n(c)$, $c\neq 0$, is of cyclic-parallel if and only if $A\phi=\phi A$. Therefore it is of type A.

REMARK 3.2. By Theorems D and B and (2.8) or also by (2.9) real hypersurfaces of type B in $M_n(c)$, $c\neq 0$, are cyclic-pseudo-parallel.

From now on, we suppose that M is of cyclic-pseudo-parallel. Then it follows from the Codazzi equation (1.4) that

$$\begin{aligned}(3.1)\qquad \nabla_X A(Y)=&-c\{\eta(Y)\phi X+g(\phi X, Y)\xi\}/4+a\{\eta(X)(A\phi-\phi A)Y\\ &+\eta(Y)(A\phi-\phi A)X+g((A\phi-\phi A)X, Y)\xi\}.\end{aligned}$$

It is easily seen that the fact M is cyclic-pseudo-parallel is equivalent to (3.1), and in the case where the shape operator A is η-parallel. Accordingly, by Theorems D and E the structure of M is determined if the structure vector ξ is principal. So, the property that ξ is principal is investigated. Putting $Y=\xi$ in (3.1) and using (1.1), we get

$$(3.2)\qquad \nabla_X A(\xi)=-c\phi X/4+a\{-\eta(X)\phi A\xi+(A\phi-\phi A)X-g(X, \phi A\xi)\xi\},$$

which joined with the second formula of (1.2) implies

$$\nabla_{\xi}A(\xi)=-2aU\,, \tag{3.3}$$

where $U=\nabla_{\xi}\xi$. For any point x on M we can choose an othonormal basis $\{E_1, \cdots, E_{2n-1}\}$ for the tangent space T_xM such that $\nabla_{E_i}E_j=0$ $(i, j=1, \cdots, 2n-1)$. Then, differentiating (3.1) covariantly along M and making use of (1.2), we have

$$\begin{aligned}\nabla_W\nabla_X A(Y)=&-c\{g(\phi AW, Y)\phi X+g(\phi X, Y)\phi AW+\eta(X)\eta(Y)AW\\&+\eta(X)g(AW, Y)\xi-2\eta(Y)g(AW, X)\xi\}/4\\&+a[g(\phi AW, X)(A\phi-\phi A)Y+g(\phi AW, Y)(A\phi-\phi A)X\\&+g((A\phi-\phi A)X, Y)\phi AW-\eta(X)\{\eta(AY)AW+g(AW, Y)A\xi\\&-2g(A^2W, Y)\xi+\phi\nabla_W A(Y)-\nabla_W A(\phi Y)\}-\eta(Y)\{\eta(AX)AW\\&+g(AW, X)A\xi-2\eta(X)A^2W+\phi\nabla_W A(X)-\nabla_W A(\phi X)\}\\&-\{\eta(AX)g(AW, Y)+\eta(AY)g(AW, X)-2\eta(Y)g(A^2W, X)\\&-g(\nabla_W A(X), \phi Y)-g(\nabla_W A(\phi X), Y)\}\xi]\,,\end{aligned} \tag{3.4}$$

which combined with the Ricci formula for the shape operator A gives forth

$$\begin{aligned}&R(W, X)AY-A(R(W, X)Y)\\&=-c\{g(\phi AW, Y)\phi X-g(\phi AX, Y)\phi W+g(\phi X, Y)\phi AW-g(\phi W, Y)\phi AX\\&\quad+\eta(X)\eta(Y)AW-\eta(W)\eta(Y)AX+\eta(X)g(AW, Y)\xi-\eta(W)g(AX, Y)\xi\}/4\\&\quad+a[g((\phi A+A\phi)W, X)(A\phi-\phi A)Y\\&\quad+g(\phi AW, Y)(A\phi-\phi A)X-g(\phi AX, Y)(A\phi-\phi A)W\\&\quad+g((A\phi-\phi A)X, Y)\phi AW-g((A\phi-\phi A)W, Y)\phi AX\\&\quad-\eta(X)\{\eta(AY)AW+g(AW, Y)A\xi-2g(A^2W, Y)\xi+\phi\nabla_W A(Y)-\nabla_W A(\phi Y)\}\\&\quad+\eta(W)\{\eta(AY)AX+g(AX, Y)A\xi-2g(A^2X, Y)\xi+\phi\nabla_X A(Y)-\nabla_X A(\phi Y)\}\\&\quad-\eta(Y)\{\eta(AX)AW-\eta(AW)AX-2\eta(X)A^2W+2\eta(W)A^2X+\phi\nabla_W A(X)\\&\quad-\phi\nabla_X A(W)-\nabla_W A(\phi X)+\nabla_X A(\phi W)\}\\&\quad-\{\eta(AX)g(AW, Y)-\eta(AW)g(AX, Y)-g(\nabla_W A(X), \phi Y)+g(\nabla_X A(W), \phi Y)\\&\quad-g(\nabla_W A(\phi X), Y+g(\nabla_X A(\phi W), Y)\}\xi]\,.\end{aligned} \tag{3.5}$$

We substitute (1.3) into (3.5) and then put $X=Y=\xi$ in this result. Finally, if we take the inner product of the last equation and ξ, then we obtain

$$(\alpha+2a)A^2\xi=(\beta+a\alpha-2a^2)A\xi-a(\alpha^2-2\beta-2a\alpha)\xi\,, \tag{3.6}$$

where we have defined $\beta=\eta(A^2\xi)$ and used (1.1) and (3.1).

On the other hand, putting $W=E_j$ in (3.5), taking the inner product of this result and E_j and summing up with respect to j ($j=1, \cdots, 2n-1$), we find

$$\begin{aligned} &hA^2X+\{c(n+1)/2-h_2-2a\alpha\}AX-c(2a+h)X/4 \\ &=-c\phi A\phi X+(c/2+ah)\eta(X)A\xi+(c/2+ah+4a^2)\eta(AX)\xi \\ &-[ch/4-a\{(2n-1)c/2-4a\alpha-2h_2\}]\eta(X)\xi \\ &-a\{A\phi A\phi X-\phi A\phi AX-2\phi A^2\phi X+2\eta(AX)A\xi \\ &-\eta(X)A^2\xi+\eta(A^2X)\xi\}, \end{aligned} \tag{3.7}$$

where we have used (1.1), (1.4), (1.5) and (3.1). If we set $X=\xi$ in (3.7) and if we use (1.1), then we attain

$$\begin{aligned} &hA^2\xi+(cn/2-h_2)A\xi-c\alpha\xi/2 \\ &=a[\phi AU+A^2\xi-hA\xi-\{c(n-1)+\beta-2h_2+\alpha h\}\xi]\,. \end{aligned} \tag{3.8}$$

By means of (3.1), (3.2) and the definition of h it is easily seen that we have $dh(Y)=-2ag(U, Y)=g(\nabla_Y A(\xi), \xi)$ and hence we get $\nabla_W\nabla_Y h=\nabla_Y\nabla_W h$, which implies that $\nabla_W g(\nabla_Y A(\xi), \xi)=\nabla_Y g(\nabla_W A(\xi), \xi)$. From (3.2) we bring out

$$\begin{aligned} \nabla_W g(\nabla_Y A(\xi), \xi)=2a\{&-\alpha g(Y, AW)+\eta(Y)\eta(A^2W) \\ &+g(\nabla_W A(\xi), \phi Y)+g(A\phi Y, \phi AW)\} \end{aligned}$$

and hence, taking the skew-symmetric part of this and the above two equations we have

$$\begin{aligned} 2a\{&\eta(Y)\eta(A^2W)-\eta(W)\eta(A^2Y)+g(\nabla_W A(\xi), \phi Y)-g(\nabla_Y A(\xi), \phi W) \\ &+g(A\phi Y, \phi AW)-g(A\phi W, \phi AY)\}=0\,. \end{aligned}$$

Thus, putting $Y=\xi$ in the last equation and making use of (1.1) and (3.3), we get

$$a\phi AU=-a\{A^2\xi+2aA\xi-(2a\alpha+\beta)\xi\}, \tag{3.9}$$

which together with (3.8) leads to

$$\begin{aligned} &hA^2\xi+(cn/2-h_2)A\xi-c\alpha\xi/2 \\ &=-a[(h+2a)A\xi+\{c(n-1)-2h_2-2a\alpha+\alpha h\}\xi]\,. \end{aligned} \tag{3.10}$$

And this, combined with (3.6), gives rise to

$$\{h(\beta+a\alpha-2a^2)+(\alpha+2a)(cn/2-h_2)\}A\xi$$
$$-\{ah(\alpha^2-2\beta-2a\alpha)+c\alpha^2/2+ca\alpha\}\xi$$
$$=-a(\alpha+2a)[(h+2a)A\xi+\{c(n-1)-2h_2-2a\alpha+\alpha h\}\xi].$$

Moreover, the inner product of (3.10) and ξ produces

$$h\beta=(\alpha+2a)\{h_2-c(n-1)/2\}-2a\alpha h. \tag{3.11}$$

Thus, combing with the last two equations, we have

$$(a^2+c/4)(\alpha+2a)(A\xi-\alpha\xi)=0.$$

In the rest of this section, we will make efforts in order to reach our goal —proving that ξ is principal. Let M_0 be a set of points of M at which the function $\beta-\alpha^2$ does not vanish. Suppose that M_0 is not empty. Then $(\alpha+2a)(a^2+c/4)=0$ because of the fact that $|A\xi-\alpha\xi|^2=\beta-\alpha^2$. If $\alpha+2a=0$ at some point of M_0, then it follows from (3.6) that $\beta=4a^2=\alpha^2$ at that point, which is a contradiction. Therefore we gain $a^2+c/4=0$ on M_0. Now, taking the covariant derivative of $\alpha=\eta(A\xi)$ and using (3.2), we have

$$AU=-aU-\operatorname{grad}\alpha/2. \tag{3.12}$$

From (3.9) the inner product of (3.12) and ϕX gives us

$$-d\alpha(\phi X)/2=\eta(A^2X)+3a\eta(AX)-(\beta+3a\alpha)\eta(X),$$

on M_0, which connected with (3.6) yields

$$-(\alpha+2a)d\alpha(\phi X)/2=(\beta+4a^2+4a\alpha)\{\eta(AX)-\alpha\eta(X)\}.$$

Moreover, replacing X by ϕX into the last equation and making use of (1.1), it is clear that

$$-(\alpha+2a)\operatorname{grad}\alpha/2=(\beta+4a^2+4a\alpha)U, \tag{3.13}$$

where we used the fact $d\alpha(\xi)=0$ which is derived from (3.12). Substituting (3.13) into (3.12), we have

$$AU=FU, \tag{3.14}$$

where $F=(\beta+2a^2+3a\alpha)/(\alpha+2a)$. Now, differentiating (3.14) covariantly along M, we obtain $\nabla_Y A(U)+A\nabla_Y U=dF(Y)U+F\nabla_Y U$. If we take the inner product of this and U and if we use (3.14), then we see that $dF(Y)g(U,U)=0$ for any vector field Y because of $g(\nabla_Y A(U),U)=0$ which is obtained by (3.1). Thus $dF=0$ on M_0, that is, $F=(\beta+2a^2+3a\alpha)/(\alpha+2a)$ is constant on M_0. Accordingly we have $(\alpha+2a)\operatorname{grad}\beta=(\beta-4a^2)\operatorname{grad}\alpha$, which linked with (3.13) gives rise to

$$(3.15)\qquad (\alpha+2a)^2\operatorname{grad}\beta=-2(\beta-4a^2)(\beta+4a^2-4a\alpha)U\,.$$

On the other hand, since we have put $h=\operatorname{tr} A$ and $h_2=\operatorname{tr} A^2$, we have $\nabla_X h_2=2\sum_{j=1}^{2n-1}g(\nabla_X A(E_j), AE_j)$ and $\nabla_X h=\sum_{j=1}^{2n-1}g(\nabla_X A(E_j), E_j)$. So, by the straightforward calculation it follows from (3.1), (3.6), (3.12) and (3.13) that we get $\nabla_X h=-2ag(U, X)$ and $\nabla_X h_2=(c+8a^2)U$. Consequently, by combining with the fact that $c=-4a^2$, it gives us that

$$(3.16)\qquad \nabla_X h=-2ag(U, X) \quad\text{and}\quad \nabla_X h_2=4a^2U \qquad \text{on } M_0\,.$$

Finally, if we differentiate (3.11), then from (3.16) we get

$$h\operatorname{grad}\beta=\{h_2-c(n-1)/2-2ah\}\operatorname{grad}\alpha+2a\{\beta+4a^2+4a\alpha\}U$$

on M_0, which together with (3.11), (3.13) and (3.15) implies

$$(\alpha+2a)\{\beta-\alpha^2+(2a+\alpha)^2\}U=0$$

and hence $\{\beta-\alpha^2+(2a+\alpha)^2\}|U|^2=0$ on M_0. Therefore, it follows that $\{|U|^2+(2a+\alpha)^2\}|U|^2=0$ on M_0. This contradicts the definition of M_0 and consequently M_0 is empty.

Summing up, we have

LEMMA 3.1. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$. If M is of cyclic-pseudo-parallel, then ξ is principal.*

From (3.1) we find that the shape operator A of cyclic-pseudo-parallel real hypersurfaces is η-parallel and hence it is seen by Theorems D and E that M is locally congruent to one of real hypersurfaces of type A and B. Moreover, we obtain

$$\nabla_X A(\xi)=-c\phi X/4+a(A\phi-\phi A)X\,,$$

which, combined together with (2.5), yields that $\alpha=-2a$ or $A\phi=\phi A$ because α and a are both constant. Conversely, by Remarks 3.1 and 3.2, a real hypersurface of type A or B in $M_n(c)$, $c\neq 0$, is of cyclic-pseudo-parallel. Thus we find the following

THEOREM 3.2. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$. Then M is of cyclic-pseudo-parallel if and only if M is locally congruent to one of the homogeneous hypersurfaces of type* A_1, A_2 *and* B *when $c>0$ and of type* A_0, A_1, A_2 *and* B *when $c<0$.*

REMARK 3.3. We shall again estimate the norm of the covariant derivative of the second fundamental form of a real hypersurface M of a complex space form. Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 3$, on which the structure vector ξ is principal. It is derived from (2.8) that

$$|\nabla A|^2\geqq(n-1)c^2/4+3\alpha^2|\eta\otimes(A\phi-\phi A)|^2/4\,. \tag{3.17}$$

The second term of the right hand side in the above equation is equal to $3\alpha^2\{h_2-\alpha h-(n-1)c/2\}/2$. Thus we have

$$|\nabla A|^2\geqq(n-1)c^2/4+3\alpha^2\{h_2-\alpha h-(n-1)c/2\}/2\,,$$

where the equality holds if and only if the shape operator A satisfies

$$\begin{aligned}\nabla_X A(Y)=&-c\{\eta(Y)\phi X+g(\phi X, Y)\xi\}/4\\&-\alpha\{\eta(X)(A\phi-\phi A)Y+\eta(Y)(A\phi-\phi A)X+g((A\phi-\phi A)X, Y)\xi\}/2.\end{aligned}$$

Namely, A is η-parallel and hence M is locally congruent to one of real hypersurfaces of type A and B by Theorems D and E.

REMARK 3.4. By means of (3.17) it is shown that M is of type A if and only if thc square of the norm of the convariant derivative of A is equal to $|\nabla A|^2=(n-1)c^2/4$. However, this result holds without the assumption that ξ is principal (cf. Maeda [7]). In fact, the square of the norm of $\nabla_X A(Y)+c\{\eta(Y)\phi X+g(\phi X, Y)\xi\}/4$ is equal to $|\nabla A|^2-(n-1)c^2/4$.

REMARK 3.5. By taking account of the square of the norm $|A\phi-\phi A|$, Okumura's theorem stated in Remark 2.5 is proved. This is a simple and direct proof different from his.

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