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KILLING VECTOR FIELDS ON SEMIRIEMANNIAN MANIFOLDS

By

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Abstract It is well known that a Killing vector field on a riemannian compact manifold is holonomic (Kostant (4)). In other words, the A_X operator ($A_X=L_X-\nabla_X=-\nabla X$) lies in the holomony algebra of the manifold.

The covariant derivative of A_X gives us a curvature transformation. This fact and the Ambrose-Singer theorem show that the A_X operator lies infinitesimally in the holonomy algebra $\boldsymbol{h}$.

$$(\text{i.e. } \forall Y,\ \nabla_Y A_X=R_{XY}\in\boldsymbol{h}) \qquad (*)$$

The subject of our study is the holonomicity of a Killing vector field on a semiriemannian compact manifold. We remark the validity of (*) on semiriemannian manifolds.

In order to simplify its study, we constrain it to Lorentz locally strictly weakly irreducible manifolds (l. SWI). We remark that Berger (1) showed that the holonomy algebra of a Lorentz manifold which is irreducible and non locally symmetric is the whole $po(n, 1)$. Therefore, we can leave out this case.

Strictly weakly irreducible manifolds, defined by H. Wu (5, 6) in 1963 are the cornerstones of this study. Among these we have found examples of compact manifolds with a non holonomic Killing vector field.

0. Preliminaries.

Let M be a semiriemannian manifold of dimension n and signature s and take $p\in M$. Any loop σ with base point p provides us with an isometry of T_pM. The set of isometries can be structured as a Lie group: the holonomy group with base-point p, $G_p(M)$. When we consider only nulhomotopes loops,

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we obtain $G_p^0(M)$, the restricted holonomy group. Both are Lie groups. Their algebra $\boldsymbol{h}$ is the holonomy algebra of M; it is a subalgebra of $\boldsymbol{po}(n, s)$.

The $G_p(M)$-action on T_pM is strictly weakly irreducible (SWI) if there is some degenerate subspace of T_pM invariant by the G_p-action and there are no invariant and nondegenerate subspaces.

EXAMPLE 1. Let $e_0, e_1, \cdots, e_n$ be a basis of R^{n+1}. In this basis we define an inner product $\langle\cdot, \cdot\rangle$ on R^{n+1} by the matrix

$$\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & Id \end{pmatrix}$$

Let G be the group of isometries of $(R^{n+1}, \langle\cdot, \cdot\rangle)$ which have e_0 as an eigenvector. Then $(R^{n+1}, \langle\cdot, \cdot\rangle)$ is SWI by the G-action.

PROOF. Clearly e_0 spans an invariant degenerate subspace. If U is a G-invariant and nondegenerate subspace of R^{n+1}, then

$$R^{n+1}=U\oplus U^{\perp} \tag{0.0}$$

and U and $U^{\perp}$ are invariant and nondegenerate.

The eigenvector e_0 lies on U or on $U^{\perp}$. Suppose $e_0\in U$. Then

$$U^{\perp}\subseteqq\{e_0\}^{\perp}=\langle e_0, e_2, \cdots, e_n\rangle.$$

If $v\in U^{\perp}$, we can find an isometry $\varphi\in G$ such that

$$\varphi(v)=e_0+v$$

Then $e_0\in G(v)$ because $v\in G(v)$ and this is impossible by (0.0). (Q. E. D.)

REMARKS. i) Whenever we take into consideration the G_p^0-action instead of the G_p-one, we will add the word "locally" to the other abjectives.

ii) A vector space S is G_p^0-invariant if and only if it is $\boldsymbol{h}$-invariant.

PROPOSITION 2. *Let M be a SWI manifold. Then, there is an isotropic subspace of T_pM, invariant by the G_p-action.*

PROOF. The SWI condition provides us with a G_p-invariant degenerate subspace V of T_pM. Take $w\in V$ in such a way that $\langle v, w\rangle=0$ $\forall v\in V$. The subspace $W=G_p(w)$ is G_p-invariant and isotropic. (Q. E. D.)

COROLLARY 3. *Let M be a Lorentz SWI manifold. Then,*

i) *there is a G-invariant totally geodesic distribution of dimension one on M,*
ii) *if* $\dim M>2$, *that distribution is unique.*

PROPOSITION 4. *Let M be a Lorentz manifold with* $\dim M>2$. *If M is locally SWI, then M is SWI.*

PROOF. Let W_q be the G_p^0-invariant subspace of Corollary 3. If τ is a path, by the uniqueness of the distribution (3. ii)

$$\tau(W_{\tau(0)})=W_{\tau(1)}$$

Where $\tau(W)$ means the parallel transport of W along τ. If σ is a loop, we have

$$W_{\sigma(0)}=\sigma_0^{1/2}(W_{\sigma(0)})=W_{\sigma(1/2)}=\sigma_{1/2}^1(W_{\sigma(1/2)})=\sigma_0^1(W_{\sigma(0)})$$

tus W_q is G_p-invariant. (Q. E. D.)

LEMMA 5. *Take a basis* $e_0, e_1, \cdots, e_n$ *of the Lorentz space* L_{n+1}. *Suppose that the inner product matrix is, in such basis,*

$$\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & Id \end{pmatrix} \tag{0.1}$$

The matrix of an isometry Ψ leaving e_0 invariant looks like

$$\begin{pmatrix} \lambda & a & {}^t w \\ 0 & \lambda^{-1} & 0 \\ 0 & -\dfrac{Aw}{\lambda} & A \end{pmatrix} \tag{0.2}$$

where $\lambda\in R-\{0\}$, $w\in R^{n-1}$, $A\in O(n-1)$, $a=(-\langle w, w\rangle/2\lambda)\in R$, $v=\lambda^{-1}(-Aw)\in R^{n-1}$.

Let M be a time orientable Lorentz manifold, locally SWI. Let D be the distribution of Corollary 3. We can take a global vector field V_0 that generates D and, locally, a frame $V_0, V_1, \cdots, V_n$ in such a way that the matrix of the inner product is (0, 1). If necessary, the field V_1 could be global.

DEFINITION. The set of isometries of lemma 5 is a group J which is isomorphic to $R\times R^{n-1}\times O(n-1)$ with the product rule:

$$(\lambda, {}^t w, A)\cdot(\mu, {}^t v, B)=(\lambda\mu, \lambda{}^t v+{}^t wB, AB)$$

where $(\lambda, {}^t w, A)$ refers to the matrix (0.2).

The group J is a Lie group. Its algebra $\boldsymbol{J}$ is isomorphic to $R\times R^{n-1}\times \boldsymbol{o}(n-1)$

with the bracket rule:

$$[(a, w, A), (b, v, B)]=(0, b^t w-a^t v+{}^t vA-{}^t wB, [A, B])$$

where (a, w, A) refers to the matrix:

$$\begin{pmatrix} a & 0 & -{}^t w \\ 0 & -a & 0 \\ 0 & w & A \end{pmatrix}$$

In order to reduce the Levi-Civita connection we are going to define a fibre bundle on M. Let D be the distribution of Corollary 3. If $\pi: L(M)\to M$ is the bundle of linear frames on M, we define $B(M)$ by:

i) $u\in L(M)$ is an isometry between L_{n+1} and $T_{\pi(u)}M$.

ii) $u\in B(M)$ if and only if $u(e_0)\in D$ and the inner product matrix related to the basis $\{u(e_i)\}$ is (0.1). ($\{e_i\}$ basis as in lemma 5).

PROPOSITION 6. *$B(M)$ is a principal fibre bundle on M with structural group J.*

PROOF. The J-action on $B(M)$ is free; on the other hand, $B(M)/J\cong M$ and $B(M)$ is locally trivial because so is $L(M)$. (Q. E. D.)

PROPOSITION 7. *The Levi Civita connection of M is reducible to a connection on $B(M)$.*

PROOF. Let $s(t)$ be a curve on M and $\tilde{s}(t)$ one lift of $s(t)$ on $L(M)$. In a trivializing neighborhood we have

$$\tilde{s}(t)=(s(t), W_0(t), W_1(t), \cdots, W_n(t)]$$

It is sufficient to prove that if $\tilde{s}(o)\in B(M)$, then $\tilde{s}(t)\in B(M)$. Assume $\tilde{s}(0)\in B(M)$. Then $W_0(0)\in D$. Hence $W_0(t)\in D$, since D is parallel. And the linner product matrix is (0.1) because it is in $\tilde{s}(0)$ and the parallel transport is an isometry. (Q. E. D.)

COROLLARY 8. *If $\boldsymbol{h}$ is the holonomy algebra of M, then*

$$\dim \boldsymbol{h}\leqq 1+\frac{n(n-1)}{2}.$$

1. First Aproach.

THEOREM 9. *Let M be a Lorentz SWI manifold. If $\boldsymbol{J}$ is the Lie algebra of J, then any Killing vector field on M satisfies $A_X \in \boldsymbol{J}$.*

PROOF. We can take a frame $V_0, V_1, \cdots, V_n$ such that the subspace subspace spanned by V_0 is D (Corollary 3) and the inner product is expressed in this basis by the matrix (0.1). A skew symmetric matrix takes for form:

$$\begin{pmatrix} a & 0 & -{}^t u \\ 0 & -a & -{}^t w \\ w & u & A \end{pmatrix} \tag{1.1}$$

where $a \in R$, u and $v \in R^{n-1}$, $A \in \boldsymbol{o}(n-1)$.

The elements of the holonomy algebra have the forme:

$$\begin{pmatrix} b & 0 & -{}^t v \\ 0 & -b & 0 \\ 0 & v & B \end{pmatrix}$$

where $b \in R$, $v \in R^{n-1}$, $B \in \boldsymbol{o}(n-1)$.

Let (1.1) be the A_X operator matrix. Since $[A_X, \boldsymbol{h}] \subset \boldsymbol{h}$, we have

$$B \cdot w - b \cdot w = 0$$

hence $w=0$ or $b=0$.

If $w=0$ the proof is finished. If this is not the case, it must be $b=0$ and $Bw=0$ for any $(b, v, B) \in \boldsymbol{h}$. In order to have $[A_X, \boldsymbol{h}] \subset \boldsymbol{h}$, it must be $w \cdot {}^t v = 0$. But then the vector $(0, 0, w)$ would be $\boldsymbol{h}$-invariant and this is impossible because M is locally SWI. Q.E.D.

REMARK. An interesting and simple case occurs when $R_{XY} D \equiv 0 \ \forall X, Y$. Then we can choose a frame as $V_0, V_1, \cdots, V_n$ satisfying that V_0 is parallel. In this case the (b, v, B) elements of the holonomy algebra have $b=0$.

Theorem 9 is not enough for this case. We also need $A_X V_0 = 0$. This This happens when the parallel vector field is globally defined and M is compact. The following example shows how indispensable the compactness of M is.

EXAMPLE 10. Let $\alpha_0, \alpha_1, \alpha_2$ be coordinates of R^3. In the associated frame

$$\partial_1 = \frac{\partial}{\partial \alpha_1} \qquad i=0, 1, 2.$$

the following matrix defines an inner product

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & g \\ 0 & g & h^2 \end{pmatrix} \tag{1.2}$$

where $g=g(\alpha_1, \alpha_2)$ and $h=h(\alpha_1, \alpha_2)$ are R-valued functions and $h\neq 0$ everywhere.

Changing the frame to

$$V_0=\partial_0, \qquad V_1=\partial_1, \qquad V_2=h^{-1}(-g\partial_0+\partial_2),$$

it is easy to check that:

$$[V_0, V_1]=[V_0, V_2]=0 \quad \text{and} \quad [V_1, V_2]=-h^{-1}[(\partial_1 g)V_0+(\partial_1 h)V_2]$$

The matrices of the endomorphisms of TM, ∇V_0, ∇V_1, ∇V_2, using the basis V_0, V_1, V_2 act on the left and are given by:

$$\nabla V_0\equiv 0 \quad \nabla V_1\equiv\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \dfrac{\partial_1 g}{h} & \dfrac{\partial_1 h}{h} \end{bmatrix} \quad \nabla V_2\equiv\begin{bmatrix} 0 & \dfrac{-\partial_1 g}{h} & \dfrac{-\partial_1 h}{h} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Finally

$$R_{V_1V_2}V_1=h^{-2}[h(\partial_1\partial_1 h)-(\partial_2\partial_1 g)+h^{-1}(\partial_1 g)(\partial_2 h)]V_2.$$

Then V_0 is parallel and $\dim \boldsymbol{h}\leqq 1$. If $\dim \boldsymbol{h}=1$ (i.e. $R_{V_1V_2}V_1\neq 0$), then the holonomy algebra $\boldsymbol{h}$ is spanned by

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{(in the basis } V_0, V_1, V_2\text{)}.$$

Note that the inner product matrix is (0.1).

A Killing vector field

$$X=x_0V_0+x_1V_1+x_2V_2$$

such that $A_XV_0=aV_0$ must satisfy:

$$\begin{aligned} &x_0=-a\alpha_0+F(\alpha_1, \alpha_2) \\ &x_1=a\alpha_1+K \qquad (K=cst) \\ &x_2=x_2(\alpha_1, \alpha_2) \end{aligned} \tag{1.3}$$

and

$$\begin{gathered}\partial_1 F - x_2 h^{-1}(\partial_1 g) = 0 \\ -(\partial_1 x_2 + (a\alpha_1 + K)h^{-1}(\partial_1 g)) = h^{-1}(\partial_2 F - x_2 \partial_1 h) \\ \partial_2 x_2 + (a\alpha_1 + K)\partial_1 h = 0\,.\end{gathered} \tag{1.4}$$

We are interested in a solution with $x_2=0$. Then (1.4) becomes:

$$\begin{gathered}\partial_1 F = 0 \\ -(a\alpha_1 + K)h^{-1}(\partial_1 g) = h^{-1}(\partial_2 F) \\ (a\alpha_1 + K)(\partial_1 h) = 0\end{gathered} \tag{1.5}$$

Hence $\partial_1 h=0$. Finally,

$$F = G(\alpha_2)\,, \qquad g = (-\partial_2 G)\log(a\alpha_1 + K)\,, \qquad h = 1 \tag{1.6}$$

satisfies (1.5).

Summarizing the example. In the subspace of R^3 defined by $a\alpha_1+K>0$, we consider the inner product

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & g \\ 0 & g & 1 \end{pmatrix}$$

where g is given by (1.6). This manifold is locally SWI. The vector field ∂_0 is parallel and $\dim \boldsymbol{h}=1$. The vector field

$$X = (-a\alpha_0 + G)V_0 + (a\alpha_1 + K)V_1$$

is a non holonomic Killing vector field since $A_X V_0 = aV_0$ and $\forall h \in \boldsymbol{h},\ h(V_0)=0$.

Corollary 11. *Let (M, g) be a compact Lorentz locally SWI manifold. Let $\boldsymbol{h}$ be the holonomy algebra and assume that there is on M a global parallel light-like vector field V_0. Let X be a Killing vector field. Then $A_X V_0=0$.*

Proof. It is easy to check that

$$\operatorname{grad}(g(V_0, X)) = A_X V_0\,.$$

By Theorem 9, $A_X V_0 = aV_0$. Actually a is a constant. In fact, for every vector field Y,

$$\begin{aligned}0 = R_{XY}V_0 = (\nabla_Y A_X)V_0 &= (\nabla_Y A_X)V_0 + A_X(\nabla_Y V_0) = \nabla_Y(A_X V_0) \\ &= \nabla_Y(aV_0) = (Ya)V_0 + a(\nabla_Y V_0) = (Ya)V_0\end{aligned}$$

because V_0 is parallel. Hence $Ya\equiv 0$ and a is constant. Taking a frame V_0,

$V_1, \cdots, V_n$ where the inner product is expressed by (0.1), it is easy to verify that

$$a = V_1 g(V_0, X)$$

Since M is compact, $g(V_0, X)$ reaches a maximum (minimum), On this point

$$a = V_1 g(V_0, X) = 0$$

so $A_X V_0 = 0$. (Q. E. D.)

2. General case.

DEFINITION. Let φ, ψ be endomorphisms of T_pM. We define

$$\Phi(\varphi, \psi) = -\text{trance}\,(\varphi \circ \psi)$$

This is a bilinear form called the Cartan-Killing form.

THEOREM 12. (2) *Let M be a semiriemannian compact manifold, X a Killing vector field on M. If Φ is nondegenerate on the holonomy algebra, then the A_X-operator decompose in the form*

$$A_X = h + B_X$$

where $h \in \boldsymbol{h}$, $B_X \boldsymbol{h}^\perp$ and $\Phi(B_X, B_X) = 0$. This decomposition is unique.

REMARK. On Lorentz surfaces the Cartan-Killing form is negative definite. A Lorentz surface which is not flat is locally SWI.

COROLLARY 13. *Let M be a compact Lorentz surface. If X is a Killing vector field on M then $A_X \in \boldsymbol{h}$.*

THEOREM 14. *Let M be a Lorentz SWI manifold, $\boldsymbol{h}$ its holonomy algebra, V_0 a light-like vector field in the direction of the parallel* 1*-distribution D and $\boldsymbol{r}$ the radical of the trace form on $\boldsymbol{h}$.*

Let X be a Killing vector field on M, we have

i) *If* $\dim M = 3$
   - a) $\dim \boldsymbol{h} = 2$ *implies X is holonomic.*
   - b) $\dim \boldsymbol{h} = 1$ *implies* $\dim \boldsymbol{r} = 1$.
   - c) *If* $\dim \boldsymbol{h} = 1$, *M is compact and V_0 is global, then X is holonomic (See* 10 *for the noncompact case).*

ii) *If* $\dim M = 4$ *and* $\boldsymbol{h}(V_0) \neq 0$, *then*
   - a) $\dim \boldsymbol{h} \leqq 4$
   - b) *If* $\dim \boldsymbol{h} \neq 3$, *then X is holonomic. (See* 22 *for* $\dim \boldsymbol{h} = 3$).

iii) *If* $\dim M = 4$ *and* $\boldsymbol{h}(V_0) = 0$, *then*

a) $\dim \boldsymbol{h} \leqq 3$

b) *If M is compact, V_0 is global and* $\dim \boldsymbol{h}=3$, *then X is holonomic.*

PROOF. i) In an adequate basis, the holonomy algebra $\boldsymbol{h}$ is generated by

$$\text{a)}\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{c)}\quad \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Theorem 9 implies a).

The SWI character of M implies b) and Corollary 11 implies c).

ii) In this case the discussion is longer but the tools are the same as in i) plus the fact that $[A_X, \boldsymbol{h}] \subset \boldsymbol{h}$.

iii) In an adequate basis, the elements of the holonomy algebra can be written as

$$a\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

so that $\dim \boldsymbol{h} \geqq 3$. Hence Corollary 11 gives iii-b). (Q. E. D.)

3. Lorentz nondegenerate case.

THEOREM 15. *Let M be a compact Lorentz locally SWI manifold, $\boldsymbol{h}$ its holonomy algebra ond Φ the Cartan-Killing from. Assume that Φ is nondegenerate on $\boldsymbol{h}$. Then if X is a Xilling vector field, we have $A_X \in \boldsymbol{h}$.*

To prove the Theorem we use the following lemmas:

LEMMA 16.

$$\Phi(A, [B, C]) = \Phi([A, B], C).$$

LEMMA 17. *Let V be a K-vector space. Assume that $A, B \in \mathrm{End}(V)$ and $[A, B]=0$. Then $\forall p \in K[x]$,* Ker $p(\Phi)$ *is B-invariant.*

LEMMA 18. *Let V be a Lorentz vector space. Take a basis where the inner product is given by* (0.1) *and an endomorphism A which has in this basis the form:*

$$A=\begin{pmatrix} b & 0 & -{}^t v \\ 0 & -b & 0 \\ 0 & v & \Psi \end{pmatrix}$$

where $b\in R$, $v\in R^{n-1}$ and $\Psi\in \boldsymbol{o}(n-1)$. If $b\neq 0$ or $\Psi\neq 0$, then there is a subspace of V which is A-invariant and nondegenerate by the Lorentz inner product.

PROOF. Let $e_0, e_1, \cdots, e_n$ be our basis. Since $\Psi\in \boldsymbol{o}(n-1)$, there exists an orthonormal basis $u_2, \cdots, u_n$ of $\langle e_0, e_1\rangle^{\perp}$ in such a way that Ψ is given by the matrix

$$B=\begin{pmatrix} 0 & -a_1 & & & & & & & & & & \\ a_1 & 0 & & & & & & & & & & \\ & & \ddots & & & & & & & & & \\ & & & 0 & -a_1 & & & & & & & \\ & & & a_1 & 0 & & & & & & & \\ & & & & & \ddots & & & & & & \\ & & & & & & 0 & -a_r & & & & \\ & & & & & & a_r & 0 & & & & \\ & & & & & & & & \ddots & & & \\ & & & & & & & & & 0 & -a_r & \\ & & & & & & & & & a_r & 0 & \\ & & & & & & & & & & & 0 \\ & & & & & & & & & & & & \ddots \\ & & & & & & & & & & & & & 0 \end{pmatrix}.$$

Related to the basis $e_0, e_1, u_2, \cdots, u_n$ the endomorphism A is

$$\begin{pmatrix} b & 0 & -{}^t v \\ 0 & -b & 0 \\ 0 & v & B \end{pmatrix}$$

If $b\neq 0$ or $\Psi\neq 0$, then $b^2+a_i^2\neq 0$ for some a_i. The subspace $\operatorname{Ker}(A^2+a_i^2 I)$ is A-invariant and nondegenerate by the Lorentz inner product. This is the primary component associated to the eigen-value a_i. (Q. E. D.)

PRROF OF THEOREM 15. Theorem 12 allows us to decompose

$$A_X=K+B_X$$

where $K\in \boldsymbol{h}$, $B_X\in \boldsymbol{h}^{\perp}$ and $\Phi(B_X, B_X)=0$.

It is easily verified that

$$[B_X, h]=0 \qquad \forall h\in \boldsymbol{h} \tag{3.1}$$

In fact, from Lemma 16

$$\Phi([B_X, h], 1)=\Phi(B_X, [h, 1])=0\,.$$

Then

$$\Phi([B_X, h], 1)=0 \qquad \forall h\in \boldsymbol{h}.$$

But $[B_X, h]\in \boldsymbol{h}$ and Φ is nondegenerate on $\boldsymbol{h}$, hence (3.1) holds.

By theorem 9, there exists a frame $V_0, V_1, \cdots, V_n$ where B_X is expressed by

$$\begin{pmatrix} b & 0 & -{}^t v \\ 0 & -b & 0 \\ 0 & v & B \end{pmatrix}$$

where $b\in R$, $v\in R^{n-1}$, $B\in \boldsymbol{o}(n-1)$ and $b^2=\Phi(B, B)$.

We must consider two cases

a) $b\neq 0$

By Lemma 18 there is a nondegenerate subspace of TM which is B_X invariant. By Lemma 17 this subspace is $\boldsymbol{h}$-invariant. Then M will not be locally SWI.

b) $b=0$. Consequently $B_X\equiv 0$.

An element of $\boldsymbol{h}$ can be written as

$$\begin{pmatrix} a & 0 & -{}^t w \\ 0 & -a & 0 \\ 0 & w & H \end{pmatrix}.$$

Then, since A_X lies in the normalizer of $\boldsymbol{h}$ and Φ is nondegenerate on $\boldsymbol{h}$, it must be

$$Hv+av=0$$

$\forall(H, a)$ such that $H\in \boldsymbol{o}(n-1)$, $a\in R$ and $\exists w\in R^{n-1}$ such that

$$\begin{pmatrix} a & 0 & -w \\ 0 & -a & 0 \\ 0 & w & H \end{pmatrix}\in \boldsymbol{h}.$$

If $a\neq 0$ for some $H\in \boldsymbol{h}$, it must be $v=0$. Otherwise, if $v\neq 0$, a frame such as $V_0, V_1, \cdots, V_{n-1}, V_n=v/\|v\|$ could be taken. In such a frame, the elements of the holonomy algebra $\boldsymbol{h}$ are expressed by

$$\begin{pmatrix} a & 0 & -{}^t w & -w_{n-1} \\ 0 & -a & 0 & 0 \\ 0 & w & H & 0 \\ 0 & w_{n-1} & 0 & 0 \end{pmatrix}$$

where $w \in R^{n-2}$, $w_{n-1} \in R$ and $H \in \boldsymbol{o}(n-2)$. Since some w_{n-1} must be different from 0, we can choose an $\boldsymbol{h}$-basis

$$I_i = (0, w_i, 0, H_i) \quad i=1, \cdots, (r-1); \qquad I_r = (0, w_r, 1, H_r).$$

Let $\boldsymbol{J}$ be the ideal spanned by $I_1, \cdots, I_{r-1}$ and assume that

$$L = (0, w, \varepsilon, H)$$

is a generator of $\boldsymbol{J}^{\perp} \subset \boldsymbol{h}$. It is easily verified that

$$\Phi([L, I_i], I_j) = \Phi(L, [I_i, I_j]) = 0 \qquad \forall i, j \in \{1, \cdots, r\}. \tag{3.2}$$

By the nondegeneracy of Φ

$$[L, I_i] = 0. \tag{3.3}$$

The L matrix in the V's frame takes the form

$$\begin{pmatrix} 0 & 0 & -{}^t u \\ 0 & 0 & 0 \\ 0 & u & U \end{pmatrix}$$

where $u \in R^{n-1}$, $U \in \boldsymbol{o}(n-1)$, $U \neq o$.

Again by Lemma 18 there is a subspace of TM which is L-invariant and nondegenerate. Using (3.3) and Lemma 17, we see that it is $\boldsymbol{h}$-invariant. But this is impossible because M is locally SWI. Then $v=0$ implies $v=0$. (Q.E.D.)

Proposition 19. *Let M be a compact Lorentz locally SWI manifold. Let Φ, $\boldsymbol{h}$ and D be as above. Suppose that the Ricci tensor is negative semidefinite, $\boldsymbol{h}$ is nondegenerate by Φ and $\boldsymbol{h}(D)=0$. Then any Killing vector field X must lie in the distribution $D^{\perp}$.*

Proof. Since $\boldsymbol{h}(D)=0$, we can locally choose a vector field V_0 which is parallel and $RV_0 = D$. In a frame $V_0, V_1, \cdots, V_n$ where the inner product is given by (0.1), the elements of $\boldsymbol{h}$ can be expressed by

$$\begin{pmatrix} 0 & 0 & -{}^t v \\ 0 & 0 & 0 \\ 0 & v & B \end{pmatrix}.$$

Note that Φ is negative semidefinite on $\boldsymbol{h}$.

Since $A_X V_0 = 0$, from Theorem 15 and $\operatorname{grad}(g(V_0, X)) = A_X V_0$, we obtain that $g(X, V_0)$ is constant. If this constant were different from zero one could choose a frame V_0 '$X, V_2, \cdots, V_n$ in such a way that the inner product would be given by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2f & 0 \\ 0 & 0 & Id \end{pmatrix}.$$

It is well known that $\Delta f = -\text{trace}(A_X \circ A_X) - \text{Ricci}(X, X)$, which is positive or zero in this case. By integrating Δf on the compact manifold M,

$$0 = \int_M \Delta f . \tag{3.4}$$

Then $\Delta f = 0$ and $\text{trace}(A_X \circ A_X) = 0$. In the frame we have just defined, A_X is

$$\begin{pmatrix} 0 & 0 & -{}^t v \\ 0 & 0 & 0 \\ 0 & v & 0 \end{pmatrix}. \tag{3.5}$$

Now we could integrate

$$\frac{\Delta f^2}{2} = \Delta f \cdot f + g(\text{grad}\, f, \text{grad}\, f)$$

so as to obtain by (3.4)

$$0 = \int_M g(\text{grad}\, f, \text{grad}\, f) . \tag{3.6}$$

Since $\text{grad}\, f$ is spatial like, $\text{grad}\, f = 0$. But $\text{grad}\, f = A_X X$. Then f is constant and $A_X \equiv 0$. (See (3.3)). Consequently X is parallel and the subspace spanned by X and V_0 is invariant and nondegenerate. This is a contradiction. Hence $g(X, V_0) = 0$. (Q. E. D.)

THEOREM 20. *With the hypotheses of Proposition* 19, *the Killing vector field* X *is light-like and parallel.*

PROOF. By Theorem 14, $A_X \in \boldsymbol{h}$. If we take $f = (1/2) g(X, X)$, then $\Delta f = 0$ and

$$\Phi(A_X, A_X) = 0 \tag{3.6}$$

as in the last proposition.

In a frame $V_0, V_1, \cdots, V_n$ where the inner product is given by (0.1), the A_X matrix is

$$\begin{pmatrix} 0 & 0 & -{}^t v \\ 0 & 0 & 0 \\ 0 & v & B \end{pmatrix}$$

where $v\in R^{n-1}$ and $B\in\boldsymbol{o}(n-1)$. But $B\equiv 0$ by (3.6). Hence A_X is in the radical of $\Phi_{|\boldsymbol{h}\times\boldsymbol{h}}$. Then $A_X\equiv 0$ and X is parallel.

Finally since M is locally SWI, X must be light-like. By Proposition 19, $g(X, V_0)=0$. Then $X=kV_0$ and k is a constant. (Q. E. D.)

COROLLARY 21. *Let M be a compact locally SWI manifold. Assume that the Ricci tensor is negative semidefinite and the trace form Φ is nondegenerate on $\boldsymbol{h}$. If D and $\boldsymbol{h}(D)$ are as in Proposition 19, either there are no Killing vector fields on M or there is a parallel light-like Killing vector field X on M and any other Killing vector field is λX, where λ is a constant.*

4. Examples.

In this section we show that Theorem 12 cannot be improved and we complete Theorem 14. We will construct a compact Lorentz SWI manifold with a non holonomic Killing vector field X that cannot admit a decomposition like in Theorem 12.

EXAMPLE 22. Let S^1 be the unit circle included in the euclidean plane. We define:

$$U_1=S^1\setminus\{(1, 0)\} \qquad U_2=S^1\setminus\{(-1, 0)\}$$

$$U_{12}^+=\{(x, y)\in S^1 : y>0\} \qquad U_{12}^-=\{(x, y)\in S^1 : y<0\}$$

U_{12}^+, U_{12}^- are the path-components of $U_1\cap U_2$.

Let $\pi : M\to S^1$ be the bundle on S^1 such that

i) $\pi^{-1}(U_i)\cong S^1\times S^1\times S^1\times U_i \quad i=1, 2$

ii) The transition function $\varphi : U_1\cap U_2\to \mathrm{Aut}\,(S^1\times S^1\times S^1)$ is given by:

$$\varphi(x) : S^1\times S^1\times S^1 \longrightarrow S^1\times S^1\times S^1$$
$$(z_0, z_1, z_2)\longmapsto (z_0\cdot z_2^{-1}, z_1, z_2) \qquad \text{if } x\in U_{12}^+$$
$$\varphi(x) : S^1\times S^1\times S^1 \longrightarrow S^1\times S^1\times S^1$$
$$(z_0, z_1, z_2)\longmapsto (z_0\cdot z_2, z_1, z_2) \qquad \text{if } x\in U_{12}^- .$$

M is a fiber bundle on S^1 with the fibre isomorphic to $S^1\times S^1\times S^1$.

In order to define a metric tensor on M, consider a system of coordinates on $\pi^{-1}(U_1)$

$$I^4 \longrightarrow \pi^{-1}(U_1)$$
$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3)\longmapsto (e^{\pi i\alpha_0}, e^{2\pi i\alpha_1}, e^{2\pi i\alpha_2}, e^{2\pi i\alpha_3})$$

where $I=(0, 1)$.

We write $\partial_i=\partial/\partial\alpha_1$, $i=0, 1, 2, 3$. In this basis the inner product is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & h & 2\alpha_3 & 0 \\ 0 & 2\alpha_3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $h=h(\alpha_1, \alpha_2, \alpha_3)$ is a real C^∞ function well defined on $\pi^{-1}(U_1)$ such that

$$\lim_{\alpha_3\to 0,1} h=0 \tag{4.1}$$

and this also holds for the successive derivatives.

Analogously, on $\pi^{-1}(U_2)$ consider a system of coordinates

$$I^4 \longrightarrow \pi^{-1}(U_2)$$

$$(\alpha_0', \alpha_1', \alpha_2', \alpha_3') \longmapsto (e^{2\pi i\alpha_0'}, e^{2\pi i\alpha_1'}, e^{2\pi i\alpha_2'}, e^{2\pi i(\alpha_3'+(1/2))})$$

where $I=(0, 1)$.

We write $\partial_i'=\partial/\partial\alpha_1'$, $i=0, 1, 2, 3$. In this basis the inner product is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & h' & 2\alpha_3' & 0 \\ 0 & 2\alpha_3' & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $h'=h'(\alpha_0', \alpha_2', \alpha_3')$ is a real C^∞ function well defined on $\pi^{-1}(U_2)$ such that

$$h'_{|\pi^{-1}(U_1\cap U_2)}\equiv h_{|\pi^{-1}(U_1\cap U_2)} \quad \text{and} \quad h'_{|\pi^{-1}(\{(1,0)\})}\equiv 0\,.$$

This inner product is well defined on M and has signature one.

One can check, for instance on $\pi^{-1}(U_1)$ that, in the ∂_1 basis,

$$\nabla\partial_0=\begin{pmatrix} 0 & \dfrac{\partial_0 h}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.2}$$

$$\nabla\partial_1=\begin{pmatrix} \frac{\partial_0 h}{2} & (-4t_3^2+h)\frac{\partial_0 h}{2}+t_3\partial_2 h & \frac{\partial_2 h}{2} & \frac{\partial_3 h}{2}-2t_3 \\ 0 & -\frac{\partial_0 h}{2} & 0 & 0 \\ 0 & t_3\partial_0 h-\frac{\partial_2 h}{2} & 0 & 1 \\ 0 & -\frac{\partial_3 h}{2} & -1 & 0 \end{pmatrix} \tag{4.3}$$

$$\nabla\partial_2=\begin{pmatrix} 0 & \frac{\partial_2 h}{2} & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{4.4}$$

$$\nabla\partial_3=\begin{pmatrix} 0 & \frac{\partial_3 h}{2}-2t_3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.5}$$

$$R_{\partial_0\partial_1}=\begin{pmatrix} \frac{\partial_0\partial_0 h}{2} & (-4t_3^2+h)\frac{\partial_0\partial_0 h}{2}+t_3\partial_0\partial_2 h & \frac{\partial_0\partial_2 h}{2} & \frac{\partial_0\partial_3 h}{2} \\ 0 & -\frac{\partial_0\partial_0 h}{2} & 0 & 0 \\ 0 & t_3\partial_0\partial_0 h-\frac{\partial_0\partial_2 h}{2} & 0 & 0 \\ 0 & -\frac{\partial_0\partial_3 h}{2} & 0 & 0 \end{pmatrix} \tag{4.6}$$

$$R_{\partial_0\partial_2}\equiv 0 \tag{4.7}$$

$$R_{\partial_0\partial_3}\equiv 0 \tag{4.8}$$

$$R_{\partial_0\partial_3}=\begin{pmatrix} -\frac{\partial_2\partial_0 h}{2} & (4t_3^2-h)\frac{\partial_2\partial_0 h}{2}-t_3\partial_2\partial_2 h-2t_3 & 1-\frac{\partial_0\partial_2 h}{2} & \frac{\partial_0 h}{2}-\frac{\partial_0\partial_3 h}{2} \\ 0 & \frac{\partial_2\partial_0 h}{2} & 0 & 0 \\ 0 & -t_3\partial_2\partial_0 h+\frac{\partial_2\partial_2 h}{2}-1 & 0 & 0 \\ 0 & \frac{\partial_2\partial_3 h}{2}-\frac{\partial_0 h}{2} & 0 & 0 \end{pmatrix} \tag{4.9}$$

$$R_{\partial_2\partial_3} \equiv 0 \tag{4.10}$$

$$R_{\partial_1\partial_3}=\begin{pmatrix} -\frac{\partial_3\partial_0 h}{2} & (4t_3^2-h)\frac{\partial_3\partial_0 h}{2}-t_3\partial_3\partial_2 h+t_3\partial_0 h & \frac{\partial_0 h-\partial_3\partial_2 h}{2} & 1-\frac{\partial_3\partial_3 h}{2} \\ 0 & \frac{\partial_3\partial_0 h}{2} & 0 & 0 \\ 0 & -t_3\partial_3\partial_0 h+\frac{\partial_3\partial_2 h}{2}-\frac{\partial_0 h}{2} & 0 & 0 \\ 0 & \frac{\partial_3\partial_3 h}{2}-1 & 0 & 0 \end{pmatrix} \tag{4.11}$$

The knowledge of the holonomy algebra determines the existence of a non-holonomic Killing vector field. This is done in the following lemma.

LEMMA 23. *In the ∂_i basis, the holonomy algebra $\boldsymbol{h}$ is generated by*

$$h_1=\begin{pmatrix} 1 & -(4t_3^2-h) & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2t_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad h_2=\begin{pmatrix} 0 & -2t_3 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$h_3=\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \textit{and so } \dim \boldsymbol{h}=3.$$

PROOF. In the ∂_i basis, the skew-symmetric endomorphisms leaving ∂_0 invariant take the form:

$$\begin{pmatrix} a & -a(4t_3^2-h)-2t_3 b & -b & -c \\ 0 & -a & 0 & 0 \\ 0 & 2t_3 a+b & 0 & -d \\ 0 & 2t_3 d+c & d & 0 \end{pmatrix}$$

Then $\dim \leqq 4$ and (a, b, c, d) describes any of its elements.

By (4.7), $\cdots$, (4.11), the curvature transformations span a subalgebra included in the hyperplane $d=0$.

Assume $p\in\pi^{-1}(U_1)\cap\pi^{-1}(U_2)$. The holonomy alegebra $\boldsymbol{h}_p$ is spanned by all curvature transformations in p and those in any other point translated to p by parallel transport. If $q\in M$, we can assume that $q\in\pi^{-1}(U_1)$ and γ is a path joining p and q which also lies in $\pi^{-1}(U_1)$. Because of (4.2), $\cdots$, (4.5) we can assume that there exist functions

$$f,\ f_1,\ f_2,\ f_3 : I \longrightarrow R$$

satisfying the initial conditions

$$f(0)=1 \qquad f_1(0)=0 \qquad f_2(0)=1 \qquad f_3(0)=0$$

in such a way that the fields

$$f(t)\partial_0$$

$$f_1(t)\partial_0+f_2(t)\partial_2+f_3(t)\partial_3$$

are parallel along γ.

This fact and (4.6), $\cdots$, (4.11) show that

$$(\tau^{-1}R_{XY}\tau)\partial_0=\lambda\partial_0 \tag{4.12}$$

$$(\tau^{-1}R_{XY}\tau)\partial_2=\mu\partial_0 \qquad \forall X, Y \tag{4.13}$$

Hence the holonomy algebra $\boldsymbol{h}$ is included in the hyperplane $d=0$.

Finally, for a generic h, $\dim \boldsymbol{h}=3$, since curvature transformations (4.6), (4.9), and (4.11) are linearly independent. (Q. E. D.)

SUMMARZING THE EXAMPLE. From Example 22, M is a compact Lorentz SWI manifold. The vector $X=\partial_1$ on $\pi^{-1}(U_1)$ extends to $X=\partial_1'$ and it is a Killing vector field globally defined on M. It is non holonomic because A_X and h_1, h_2, h_3 are linearly independent and a decomposition like

$$A_X=h+B_X$$

where $h\in\boldsymbol{h}$, $B(B_X, B_X)=0$ and $B_X\in\boldsymbol{h}^\perp$ is impossible because $\Phi(B_X, B_X)\neq 0$.

It is not difficult to give an example like this in dimension n; for instance, by choosing an adequate inner product on $M\times S^1\times \underset{(n-4)}{\cdots\cdots} \times S^1$. A good metric tensor could be

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & \\ 1 & h & 2t_3 & 0 & \\ 0 & 2t_3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \\ \hline & & 0 & & Id \end{array}\right)$$

where $h=h(\alpha_0, \alpha_2, \alpha_i)$, $i=4, \cdots, n-1$.

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