TSUKUBA J. MATH.
Vol. 13 No. 2 (1989), 387—401

ALMOST-PRIMES IN ARITHMETIC PROGRESSIONS AND SHORT INTERVALS

By

Hiroshi MIKAWA

1. Introduction.

In 1936, P. Turán [9] showed, under the generalized Riemann hypothesis, there exists a prime p such that

$$p\equiv a \pmod q, \quad p\leqq q (\log q)^{2+\varepsilon}$$

for almost-all reduced classes a $(\bmod q)$. The terminology almost-all means that the number of exceptional reduced classes $\bmod q$ is $o(\phi(q))$ as $q\to\infty$. Y. Motohashi [6] considered the corresponding problem for almost-primes. Let P_2 denote integers with at most two prime factors, multiple factors being counted multiplicity. He proved that there exists a P_2 such that

$$P_2\equiv a \pmod q, \quad P_2\leqq q^{11/10}$$

for almost-all reduced classes a $\bmod q$. Moreover he remarked, assuming the q-analogue of Lindelöf hypothesis, the exponent 11/10 may be replaced by $1+\varepsilon$, $\varepsilon>0$.

It is the first purpose of this paper to make an improvement upon this result. Let $g(x)$ denote any positive function such that $g(x)\to\infty$ as $x\to\infty$.

THEOREM 1. *There exists a P_2 such that*

$$P_2\equiv a \pmod q, \quad P_2\leqq g(q)q(\log q)^5$$

for almost-all reduced classes a $\bmod q$.

In 1943, A. Selberg [8] showed, under the Riemann hypothesis, there exists a prime in the intervals

$$(n, n+g(n)(\log n)^2]$$

for almost-all n. Here almost-all means that the number of exceptional n's not exceeding x is $o(x)$ as $x\to\infty$.

Several authors considered the analogous problem for P_2. Thus D. R. Heath-

Received June 7, 1988.

Brown [3] proved that there exists a P_2 in the intervals

$$(n, n+n^{1/11+\varepsilon}]$$

for almost-all n. Motohashi [7] replaced the exponent $1/11+\varepsilon$ by ε, by a simple analytic trick. He noted that his method does not work on the above mentioned problem in arithmetic progressions. D. Wolke [10] reduced the length of intervals to the powers of $\log n$. By refining Wolke's argument, G. Harman [2] showed that there exists a P_2 in the intervals $(n, n+(\log n)^{7+\varepsilon}]$ for almost-all n.

Our second aim is to make a small improvement upon this result.

THEOREM 2. *There exists a* P_2 *in the intervals*

$$(n, n+g(n)(\log n)^5]$$

for almost-all n.

In contrast to [7] [10] [2], we appeal to Sieve method which is used in [6] [3]. Our treatment of the remainder terms from Sieve estimate is different from [6] [3]. We use C. Hooley's technique [4] to deal with a bilinear form for the remainder terms, which is due to H. Iwaniec [5].

We use the standard notation in number theory. Especially, $\bar{r}$, used in either $\frac{\bar{r}}{s}$ or congruence (mod s), means that $\bar{r}r\equiv 1 \pmod s$. $*$ in $\sum_{a=1}^{q} *$ stands for the restriction $(a, q)=1$. $n\sim N$ means $N\leqq N_1<n\leqq N_2\leqq 2N$ for some N_1 and N_2. c_1 and c_2 denote certain positive absolute constants. ε denotes a small positive constant and the constants implied in the symbols $\ll$ and O depend only on ε.

I would like to thank Professor S. Uchiyama for encouragement and careful reading the original manuscript. I would also thank the referee for making the paper easier to read.

2. Lemmas.

Firstly we state the inequality for the linear sieve. See [5] and [1]. Let $\mathcal{A}$ be a finite sequence of integers and $\mathcal{P}$ be a set of primes. Put, for $d\geqq 1$, $z>2$,

$$P(z)=\prod_{\substack{p<z\\ p\in\mathcal{P}}} p, \quad \mathcal{A}_d=\{n\in\mathcal{A}: n\equiv 0 \pmod d\}$$

and

$$S(\mathcal{A}, \mathcal{P}, z)=|\{n\in\mathcal{A}: (n, P(z))=1\}|.$$

Suppose that $|\mathcal{A}_d|$ has the approximation

$$|\mathcal{A}_d|=\frac{\omega(d)}{d}X+r(\mathcal{A}, d)$$

where X is some positive number independent of d, and $\omega(d)$ is multiplicative. Moreover we assume that, for any $2<w<z$,

$$\prod_{\substack{w\leqq p<z\\ p\in\mathcal{P}}}\left(1-\frac{\omega(p)}{p}\right)\leqq\frac{\log z}{\log w}\left(1+\frac{K}{\log w}\right),$$

and

$$\sum_{\substack{w\leqq p<z\\ p\in\mathcal{P}}}\sum_{\alpha\geqq 2}\frac{\omega(p^\alpha)}{p^\alpha}\leqq\frac{L}{\log 3w},$$

with some constants K, $L>1$. Write

$$V(z)=\prod_{\substack{p<z\\ p\in\mathcal{P}}}\left(1-\frac{\omega(p)}{p}\right).$$

LEMMA 1. *Let $z>2$, $D>2$. For any $\eta>0$ we have*

$$S(\mathcal{A}, \mathcal{P}, z)\leqq V(z)X\{F(s)+E\}+R^+$$

$$S(\mathcal{A}, \mathcal{P}, z)\geqq V(z)X\{f(s)-E\}-R^-$$

where $s=\log D/\log z$, $E=c\eta+0((\log D)^{-1/3})$ with some constant c. The functions $F(s)$ and $f(s)$ are the continuous solutions of some system of differential-difference equations. In particular,

$$sF(s)=2e^\gamma \qquad (0<s\leqq 3)$$

$$sf(s)=\begin{cases} 0 & (0<s\leqq 2)\\ 2e^\gamma\log(s-1) & (2<s\leqq 4)\end{cases}$$

where γ is the Euler constant. The remainder term $R^\pm$ has the form

$$R^\pm=\sum_{d\mid P(z)}\lambda_d^\pm(D, \eta)r(\mathcal{A}, d)$$

where the sequence $(\lambda_d^\pm)=(\lambda_d^\pm(D, \eta))$ has the properties:

$$\lambda_d^\pm=0 \qquad if\quad d\geqq D,$$

$$|\lambda_d^\pm|\leqq 1,$$

and, for any M, $N>1$, $MN=D$,

$$\lambda_d^\pm=\sum_{l<\exp(8\eta^{-3})}\sum_{\substack{m\leqq M\\ mn=d}}\sum_{m\leqq N}a_{m,l}^\pm(M, N, \eta)b_{n,l}^\pm(M, N, \eta)$$

with $|a_{m,l}^\pm|$, $|b_{n,l}^\pm|\leqq 1$.

In our simple cases, the above assumptions are of course satisfied. The following Lemma 2 is well known. Lemma 3 is the C. Hooley's version of bounds for incomplete Kloosterman sums. See [4] for both lemmas.

LEMMA 2. *Let* $\psi(t)=[t]-t+\frac{1}{2}$. *For* $H>2$, *we have*

$$\psi(t)=\sum_{0<|h|\leq H}\frac{e(ht)}{2\pi i}+O\left(\min\left(1,\frac{1}{H\|t\|}\right)\right)$$

where $e(x)=e^{2\pi ix}$ *and* $\|x\|=\min_{n\in\boldsymbol{Z}}|x-n|$. *Moreover,*

$$\min\left(1,\frac{1}{H\|t\|}\right)=\sum_{h\in\boldsymbol{Z}}C_h e(ht),$$

with

$$|C_h|\ll\min\left(\frac{\log H}{H},\frac{H}{|h|^2}\right).$$

LEMMA 3. *We have, for any* $\varepsilon>0$,

$$\sum_{\substack{m\sim M\\(m,cd)=1}}e\left(a\frac{\overline{m}}{d}\right)\ll\tau(c)(a,d)^{1/2}d^{1/2+\varepsilon}\left(1+\frac{M}{d}\right).$$

3. Proof of Theorem 1.

In this section we show the inequality (3.5) below, from which Theorem 1 follows by the routine argument. Our usage of the weighted sieve is standard and the necessary theory is found in [1, Chapter 9]. So we skipp.

For $(a,q)=1$, $q\leqq x$, put

$$\mathcal{A}=\{n:x<n\leqq 2x,\ n\equiv a(\operatorname{mod}q)\},\quad \mathcal{P}=\{p:p\nmid q\}$$

and

$$r(\mathcal{A},d)=|\mathcal{A}_d|-\frac{x}{qd}.$$

Let α, u, v be the parameters such that

$$\frac{1}{\alpha}<u<v,\quad \frac{2}{\alpha}\leqq v\leqq\frac{4}{\alpha},\quad u<3. \tag{3.1}$$

Write $D=MN=(2x)^\alpha$, $y=(2x)^u$, $z=(2x)^v$. By Lemma 1, we have

$$\Pi_2(x;q,a)=|\{P_2:x<P_2\leqq 2x,\ P_2\equiv a(\operatorname{mod}q)\}|$$

$$>\sum_{\substack{n\in\mathcal{A}\\(n,P(z))=1}}\left\{1-(3-u)^{-1}\sum_{z/2<P=2^k<y}\left(1-u\frac{\log P}{\log 2x}\right)\sum_{\substack{p\nmid q\\ \max(P,z)\leqq p<\min(2P,y)}}1\right\}-\sum_{z\leqq p<y}\sum_{\substack{n\in A\\ p^2|n}}1$$

$$=S(\mathcal{A},\mathcal{P},z)-(3-u)^{-1}\sum_{P}\left(1-u\frac{\log P}{\log 2x}\right)\sum_{p}S(\mathcal{A}_p,\mathcal{P},z)-\sum_{p}\sum_{n}1$$

$$>\prod_{\substack{p<z\\p\nmid q}}\left(1-\frac{1}{p}\right)\frac{x}{q}\left[\frac{2e^\gamma}{\alpha v}\left\{f(\alpha v)-(3-u)^{-1}\int_u^v F\left(v\left(\alpha-\frac{1}{t}\right)\right)\left(1-\frac{u}{t}\right)\frac{dt}{t}\right\}-E\right]$$

$$+\sum_{d\mid P(z)}\lambda_d^-(MN)r(\mathcal{A},\,d)-(3-u)^{-1}\sum_{z/2<P<y}\Big(1-u\frac{\log P}{\log 2x}\Big)\sum_{p}\sum_{e\mid P(z)}\lambda_e^+\Big(\frac{MN}{P}\Big)r(\mathcal{A},\,pe)$$

$$-\sum_{z\leq p<y}\sum_{\substack{n\in A\\ p^2\mid n}}1$$

$$=\frac{x}{\phi(q)}\frac{e^{-\gamma}}{\log z}(1+O((\log z)^{-1}))\{C(\alpha,\,u,\,v)-E\}+S_1-(3-u)^{-1}S_2-S_3,\ \text{say}.$$

By the factorization property of $\lambda_d^{\pm}(MN)$, the following lemma may be applied to the first remainder term S_1.

LEMMA 4. *Let* $(\lambda_d)=(\lambda_d(D))$ *denote any sequence such that*

$$\lambda_d=0 \qquad \text{if } d\geqq D,$$

$$|\lambda_d|\leqq\mu^2(d)$$

and, for $M\leqq x^{1/2-4\varepsilon}$, $N\leqq x^{1/20}$, $MN=D$, (λ_d) *has the decomposition*

$$\lambda_d=\sum_{l\leq(\log D)^2}\sum_{m\leq M}\sum_{\substack{n\leq N\\ mn=d}}a_m(l,\,M,\,N)b_n(l,\,M,\,N)$$

with $|a_m|,\ |b_n|\leqq 1$. *We have*

$$\sum_{a=1}^{q}{}^{*}\Big|\sum_{\substack{x<n<2x\\ n\equiv a\,(q)}}\Big(\sum_{\substack{d\mid n\\ (d,q)=1}}\lambda_d\Big)-\Big(\sum_{\substack{d\\ (d,q)=1}}\frac{\lambda_d}{d}\Big)\frac{x}{q}\Big|^2\ll x(\log x)^3+\Big(\frac{x}{q}\Big)x^{1-\varepsilon}.$$

In the second remainder term S_2, with $e=mn$, we interprete pm as one variable. Then, we may also apply Lemma 4 to S_2. We postpone the proof of Lemma 4 until section 5. As to S_3, in section 4, we shall show the following:

If $\Big(\frac{x}{q}\Big)<(\log x)^{100}$, *then we have*

$$S_4=\sum_{x^{2\varepsilon}\leq p<x^{1/2-\varepsilon}}\sum_{\substack{x<n\leq 2x\\ n\equiv a\,(q)}}1=O\Big(\frac{x^{1-\varepsilon}}{q}\Big)+E_1(x\,;\,q,\,a) \tag{3.2}$$

with

$$\sum_{a=1}^{q}{}^{*}|E_1(x\,;\,q,\,a)|^2\ll\Big(\frac{x}{q}\Big)x^{1-\varepsilon}+x\,.$$

Now we choose $\alpha=\frac{11}{20}-4\varepsilon$, $\frac{1}{u}=\frac{1}{2}-4\varepsilon$, $\frac{1}{v}=\frac{\alpha}{4}=\frac{11}{80}-\varepsilon$, then the conditions (3.1) are satisfied and $S_3\leqq S_4$. Put

$$E(x\,;\,q,\,a)=S_1-(3-u)^{-1}S_2-E_1\,.$$

Then, by the above argument, we see

$$\sum_{a=1}^{q}{}^{*}|E(x\,;\,q,\,a)|^2\ll x(\log x)^3+\Big(\frac{x}{q}\Big)x^{1-\varepsilon}. \tag{3.3}$$

Moreover,

$$ve^{-\gamma}\{C(\alpha,\ u,\ v)-E\}$$

$$=ve^{-\gamma}\frac{2e^{\gamma}}{\alpha v}\left\{\log(\alpha v-1)-\alpha u\log\frac{v}{u}+(\alpha u-1)\log\frac{\alpha v-1}{\alpha u-1}\right\}-c_1\varepsilon+O((\log x)^{-1/3})$$

$$>\frac{40}{11}\Big(\log 3-\frac{11}{10}\log\frac{40}{11}+\frac{1}{10}\log 30\Big)-c_2\varepsilon+O((\log x)^{1/3})>\frac{1}{200},$$

for sufficiently large x and small ε. Therefore we have, provided

$$\Big(\frac{x}{q}\Big)<(\log x)^{100}, \tag{3.4}$$

$$\Pi_2(x\,;\,q,\ a)>\frac{x}{\phi(q)\log 2x}(C+O((\log x)^{-1/3}))+E(x,\ q,\ a) \tag{3.5}$$

where C is a positive absolute constant and $E(x\,;\,q,\ a)$ satisfies (3.3).

We shall derive Theorem 1 from the above inequalities (3.5) and (3.3). Put $x=\frac{1}{2}g(q)q(\log q)^5$. Obviously we may assume $g(q)\leqq(\log q)$, so the condition (3.4) is satisfied. Let $\mathfrak{E}$ denote the exceptional set of reduced residue classes, namely,

$$\mathfrak{E}=\{a:{}^{1\leqq a\leqq q}_{(a,q)=1},\ \{P_2:{}^{P_2\leqq 2x=g(q)q(\log q)^5}_{P_2\equiv a\,(\mathrm{mod}\,q)}\}\ \text{is empty}\}.$$

By (3.5), we see $a\notin\mathfrak{E}$ unless

$$|E(x\,;\,q,\ a)|>\frac{C}{3}\frac{x}{\phi(q)\log 2x}.$$

Thus, by (3.3), we have that

$$\Big(\frac{C}{3}\frac{x}{\phi(q)\log 2x}\Big)^2|\mathfrak{E}|<\sum_{a\in\mathfrak{E}}|E(x\,;\,q,\ a)|^2$$

$$\leqq\sum_{a=1}^{q}{}^{*}|E(x\,;\,q,\ a)|^2$$

$$\ll x(\log x)^3+\Big(\frac{x}{q}\Big)x^{1-\varepsilon},$$

or

$$\phi(q)^{-1}|\mathfrak{E}|\ll\phi(q)(\log x)^5x^{-1}+\frac{\phi(q)}{q}(\log x)^2x^{-\varepsilon}$$

$$\ll g(q)^{-1}+q^{-\varepsilon/2},$$

as required.

4. Elementary treatment of remainder terms.

In this section we firstly reduce the proof of Lemma 4 to an estimation of R below. Next we prove (3.2).

Put

$$\Lambda(n)=\sum_{\substack{d\mid n\\(d,q)=1}}\lambda_d,\quad \Lambda=\sum_{\substack{d\\(d,q)=1}}\frac{\lambda_d}{d},$$

then,

$$E(x;q,a)=\sum_{\substack{x<n\leq 2x\\n\equiv a(q)}}\Lambda(n)-\frac{x}{q}\Lambda.$$

Now,

$$\sum_{a=1}^{q}{}^{*}|E(x;q,a)|^2\leqq\sum_{a=1}^{q}|E(x;q,a)|^2$$

$$=\sum_{x<n\leq 2x}\Lambda(n)^2+\sum_{\substack{x<n_1\neq n_2\leq 2x\\n_1\equiv n_2(q)}}\sum\Lambda(n_1)\Lambda(n_2)-2\frac{x}{q}\Lambda\sum_{x<n\leq 2x}\Lambda(n)+\left(\frac{x}{q}\right)^2\Lambda^2q$$

$$=O\Big(\sum_{x<n\leq 2x}\tau(n)^2\Big)+2\sum_{\substack{x<n_1<n_2\leq 2x\\n_1\equiv n_2(q)}}\sum\Lambda(n_1)\Lambda(n_2)-2\frac{x}{q}\Lambda\Big(\sum_{\substack{d\\(d,q)=1}}\lambda_d\sum_{\substack{x<n\leq 2x\\d\mid n}}1\Big)+\frac{x^2}{q}\Lambda^2$$

$$=2\sum_{0<l\leq x/q}\sum_{x<n\leq 2x-ql}\Lambda(n)\Lambda(n+ql)-\frac{x^2}{q}\Lambda^2+O\Big(x(\log x)^3+\frac{x}{q}D(\log D)\Big)$$

$$=D(x,q)-\frac{x^2}{q}\Lambda^2+O\Big(x(\log x)^3+\frac{x}{q}D(\log x)\Big),\text{ say.}\tag{4.1}$$

We proceed to consider $D(x,q)$.

$$D(x,q)=2\sum_{0<l\leq x/q}\sum_{\substack{d_1\\(d_1d_2,q)=1}}\sum_{d_2}\lambda_{d_1}\lambda_{d_2}\sum_{\substack{x<n\leq 2x+ql\\n\equiv 0(d_1)\\n+ql\equiv 0(d_2)}}1$$

$$=2\sum_{0<l\leq x/q}\sum_{\substack{d_1\\(d_1d_2,q)=1\\(d_1,d_2)\mid l}}\sum_{d_2}\lambda_{d_1}\lambda_{d_2}\sum_{\substack{x/d_1<m\leq(2x-ql)/d_1\\d_1^*m+ql/(d_1,d_2)\equiv 0(d_2^*)}}1$$

where $d_1^*=d_1/(d_1,d_2)$, $d_2^*=d_2/(d_1,d_2)$. Hence we see

$$D(x,q)=M+2R\tag{4.2}$$

where

$$M=M(x,q)=2\sum_{0<l\leq x/q}\sum_{\substack{d_1\\(d_1d_2,q)=1\\(d_1,d_2)\mid l}}\sum_{d_2}\lambda_{d_1}\lambda_{d_2}\frac{x-ql}{[d_1,d_2]}$$

$$R=R(x,q)=\sum_{0<l\leq x/q}\sum_{\substack{d_1\\(d_1d_2,q)=1\\(d_1,d_2)\mid l}}\sum_{d_2}\lambda_{d_1}\lambda_{d_2}\Big\{\psi\Big(\frac{2x-ql}{[d_1,d_2]}+\frac{l}{(d_1,d_2)}q\frac{\overline{d_1^*}}{d_2^*}\Big)$$

$$-\psi\Big(\frac{x}{[d_1,d_2]}+\frac{l}{(d_1,d_2)}q\frac{\overline{d_1^*}}{d_2^*}\Big)\Big\}\tag{4.3}$$

We now consider M. In the next section we shall estimate R.

$$M=\sum_{\substack{(d_1,d_2)\leq x/q\\(d_1d_2,q)=1}}\sum\frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]}2\sum_{\substack{0<l\leq x/q\\(d_1,d_2)\mid l}}(x-ql)$$

$$=\frac{x^2}{q}\sum_{\substack{d_1,d_2)\le x/q\\(d_1d_2,q)=1}}\sum\frac{\lambda_{d1}\lambda_{d2}}{[d_1,\ d_2]}\frac{1}{(d_1,\ d_2)}+O\Big(x\sum_{(d_1,d_2)\le x/q}\sum\frac{|\lambda_{d_1}||\lambda_{d_2}|}{[d_1,\ d_2]}\Big)$$

$$=\frac{x^2}{q}\Big\{\Big(\sum_{\substack{d\\(d,q)=1}}\frac{\lambda_d}{d}\Big)^2+O\Big(\sum_{(d_1,d_2)>x/q}\sum\frac{|\lambda_{d_1}||\lambda_{d_2}|}{d_1d_2}\Big)\Big\}+O(x(\log x)^3)$$

$$=\frac{x^2}{q}\Lambda^2+O\Big(\frac{x^2}{q}\cdot\frac{q}{x}(\log D)^2+x(\log x)^3\Big).$$

Thus, combining this with (4.1) (4.2), we have

$$\sum_{a=1}^{q}{}^*|E(x\,;q,\ a)|^2\leqq 2R+O\Big(x(\log x)^3+\frac{x}{q}D(\log x)\Big) \tag{4.4}$$

where R is given by (4.3).

Next we turn to the proof of (3.2). Put

$$\Lambda^*(n)=\sum_{\substack{p^2|n\\p\in I}}1,\quad \Lambda^*=\sum_{p\in I}\frac{1}{p^2}$$

where $I=\{p:\ x^{2\varepsilon}\leqq p\leqq x^{(1/2-\varepsilon)},\ p\nmid q\}$. Then, the left hand side of (3.2) is equal to

$$\sum_{\substack{x<n\le 2x\\n\equiv a(q)}}\Big\{\Big(\sum_{\substack{p^2|n\\p\in I}}1\Big)-\frac{x}{q}\Big(\sum_{p\in I}\frac{1}{p^2}\Big)\Big\}+\frac{x}{q}\Big(\sum_{p\in I}\frac{1}{p^2}\Big)\sum_{\substack{x<n\le 2x\\n\equiv a(q)}}1$$

$$=\sum_{\substack{x<n\le 2x\\n\equiv a(q)}}\Big\{\Lambda^*(n)-\frac{x}{q}\Lambda^*\Big\}+O\Big(\Big(\frac{x}{q}\Big)^2x^{-2\varepsilon}\Big)$$

$$=E_1(x\,;q,\ a)+O\Big(\frac{x^{1-\varepsilon}}{q}\Big),\ \text{say.}$$

Moreover we have

$$\sum_{a=1}^{q}{}^*|E_1(x\,;q,\ a)|^2\leqq\sum_{a=1}^{q}|E_1(x\,;q,\ a)|^2$$

$$=\sum_{\substack{x<n_1,n_2\le 2x\\n_1\equiv n_2(q)}}\sum\Big(\Lambda^*(n_1)-\frac{x}{q}\Lambda^*\Big)\Big(\Lambda^*(n_2)-\frac{x}{q}\Lambda^*\Big)$$

$$\ll\sum_{x<n\le 2x}\Big\{\Lambda^*(n)^2+\Big(\frac{x}{q}\Big)^2(\Lambda^*)^2\Big\}\frac{x}{q}$$

$$\ll\frac{x}{q}\Big(\sum_{p\in I}\sum_{\substack{x<n\le 2x\\p^2|n}}1+\sum_{p_1\ne p_2\in I}\sum\sum_{\substack{x<n\le 2x\\p_1^2p_2^2|n}}1\Big)+\Big(\frac{x}{q}\Big)^3x^{1-4\varepsilon}$$

$$\ll\frac{x}{q}\Big(\sum_{p\in I}\frac{x}{p^2}+\sum_{p_1\ne p_2\in I}\sum\Big(1+\frac{x}{p_1^2p_2^2}\Big)\Big)+x$$

$$\ll\frac{x}{q}x^{1-\varepsilon}+x,$$

as required.

5. Proof of Lemma 4.

In this section we estimate R by appealing to Lemma 2 and Lemma 3. Write $(d_1, d_2)=\delta$, $d_1^*=\nu_1$, $d_2^*=\nu_2$, $l=\delta k$, then

$$R=\sum_{0<\delta k\leq x/q}\sum \sum_{\substack{\nu_1 \\ (\nu_1,\nu_2)=1 \\ (\delta\nu_1\nu_2,q)=1}}\sum_{\nu_2} \lambda_{\delta\nu_1}\lambda_{\delta\nu_2}\Big\{\psi\Big(\frac{2x-\delta kq}{\delta\nu_1\nu_2}+kq\frac{\overline{\nu_1}}{\nu_2}\Big)-\psi\Big(\frac{x}{\delta\nu_1\nu_2}+kq\frac{\overline{\nu_1}}{\nu_2}\Big)\Big\}$$

We firstly decompose $\lambda_{\delta\nu}$. Since $\mu^2(\delta\nu)=1$, we may write

$$\lambda_{\nu\delta}=\sum_{l\leq(\log D)^2}\sum_{de=\delta}\sum_{\substack{mn=\nu \\ m\leq M/d,\, n\leq N/e \\ \mu^2(n)=1}} a_{dm}(l, M, N)b_{en}(l, M, N).$$

Thus,

$$R=\sum_{\substack{0<\delta k\leq x/q \\ (\delta,q)=1}}\sum \sum_{\substack{l_1,l_2\leq(\log D)^2 \\ d_1e_1=\delta \\ d_2e_2=\delta}}\sum \sum_{\substack{m_1\leq M/d_1 \\ m_2\leq M/d_2 \\ (m_1n_1,m_2n_2)=1 \\ (m_1n_1m_2n_2,q)=1 \\ \mu^2(n_1)=\mu^2(n_2)=1}}\sum_{\substack{n_1\leq N/e_1 \\ n_2\leq N/e_2}} a_{d_1m_1}(l_1, M, N)b_{e_1n_1}(l_1, M, N)\cdot$$

$$\cdot a_{d_2m_2}(l_2, M, N)b_{e_2n_2}(l_2, M, N)\Big\{\psi\Big(\frac{2x-\delta kq}{\delta m_1n_1m_2n_2}+kq\frac{\overline{m_1n_1}}{m_2n_2}\Big)$$

$$-\psi\Big(\frac{x}{\delta m_1n_1m_2n_2}+kq\frac{\overline{m_1n_1}}{m_2n_2}\Big)\Big\}$$

$$\ll \sum_{0<\delta k\leq x/q}\sum (\log D)^4\tau(\delta)^2\sum_{\substack{M_1,M_2\leq M \\ N_1,N_2\leq N}}\sup|R_1|,$$

where

$$R_1=R_1(\delta, k, q; M_1, M_2, N_1, N_2)$$

$$=\sum_{\substack{m_1\sim M_1 \\ m_2\sim M_2 \\ (m_1n_1,m_2n_2)=1 \\ (m_1n_1m_2n_2,q)=1 \\ \mu^2(n_1n_2)=1}}\sum \sum_{\substack{n_1\sim N_1 \\ n_2\sim N_2}}\sum \alpha_1(m_1)\alpha_2(m_2)\beta_1(n_1)\beta_2(n_2)\Big\{\psi\Big(\frac{2x-\delta kq}{\delta m_1n_1m_2n_2}+kq\frac{\overline{m_1n_1}}{m_2n_2}\Big)$$

$$-\psi\Big(\frac{x}{\delta m_1n_1m_2n_2}+kq\frac{\overline{m_1n_1}}{m_2n_2}\Big)\Big\},$$

M_1, M_2, N_1, N_2's run through the powers of 2, and the supremum is taken over all sequences α_1, α_2, β_1, β_2, such that $|\alpha_1|$, $|\alpha_2|$, $|\beta_1|$, $|\beta_2|\leqq 1$. Moreover,

$$R\ll \sum_{0<\delta k\leq x/q}\sum (\log D)^4\tau(\delta)^2\Big\{\sum_{M_1M_2N_1N_2>x^{1-2\varepsilon}}\sup|R_1|+\sum_{M_1M_2N_1N_2\leq x^{1-2\varepsilon}}(M_1M_2N_1N_2)\Big\}$$

$$\ll(\log D)^8\sum_{0<\delta k\leq x/q}\sum \tau(\delta)^2\sup_{\substack{\alpha_1,\alpha_2,\beta_1,\beta_2 \\ M_1,M_2\leq M,\,N_1,N_2\leq N \\ M_1M_2N_1N_2>x^{1-2\varepsilon}}}|R_1|+\frac{x}{q}x^{1-\varepsilon}. \tag{5.1}$$

Next we apply Lemma 2 to ψ-function in R_1. Thus, we see

$$R_1=R_2+R_3, \tag{5.2}$$

where

$$R_2=\sum_{\substack{m_1 \\ (m_1n_1,m_2n_2)=1 \\ (m_1n_1m_2n_2,q)=1}}\sum_{m_2}\sum_{n_1}\sum_{n_2}\frac{\alpha_1(m_1)\alpha_2(m_2)\beta_1(n_1)\beta_2(n_2)}{\delta m_1m_2n_1n_2}\sum_{0<|h|\le H}e\Big(hkq\frac{\overline{m_1n_1}}{m_2n_2}\Big)$$

$$\cdot\int_x^{2x-\delta kq}e\Big(\frac{ht}{\delta m_1m_2n_1n_2}\Big)dt$$

$$R_2=\sum_{\substack{m_1 \\ (m_1n_1,m_2n_2)=1 \\ (m_1n_1m_2n_2,q)=1}}\sum_{m_2}\sum_{n_1}\sum_{n_2}\min\Bigg(1,\ \frac{1}{H\Big\|\frac{x_j}{\delta m_1n_1m_2n_2}+kq\frac{\overline{m_1n_1}}{m_2n_2}\Big\|}\Bigg)$$

with $x_1=x$, $x_2=2x-\delta kq$.

Now we consider R_3. By Lemma 2, we have

$$R_3\ll\sum_{j=1,2}\sum_{\substack{l\sim M_1N_1 \\ (l,r)=1}}\sum_{\substack{r\sim M_2N_2 \\ (lr,q)=1}}\tau(l)\tau(r)\min\Bigg(1,\ \frac{1}{H\Big\|\frac{x_j}{\delta lr}+kq\frac{\bar{l}}{r}\Big\|}\Bigg)$$

$$\ll x^{\varepsilon}\sum_{j=1,2}\sum_{\substack{r\sim M_2N_2 \\ (r,q)=1}}\sum_{\substack{l\sim M_1N_1 \\ (l,r)=1}}\min\Bigg(1,\ \frac{1}{H\Big\|\frac{x_j}{\delta lr}+kq\frac{\bar{l}}{r}\Big\|}\Bigg)$$

$$\leqq x^{\varepsilon}\sum_{j=1,2}\sum_{h\in Z}|C_h||S(h)| \tag{5.3}$$

where

$$S(h)=S(h;M_1,M_2,N_1,N_2,j)$$

$$=\sum_{\substack{r\sim M_2N_2 \\ (r,q)=1}}\sum_{\substack{l\sim M_1N_1 \\ (l,r)=1}}e\Big(\frac{hx_j}{\delta lr}\Big)e\Big(hkq\frac{\bar{l}}{r}\Big).$$

We proceed to estimate $S(h)$. Trivially,

$$S(h)\ll M_1M_2N_1N_2. \tag{5.4}$$

For $h\neq0$, we get, by partial summation and Lemma 3,

$$S(h)\ll\sum_{\substack{r\sim M_2N_2 \\ (r,q)=1}}\Big(1+\frac{hx_j}{\delta M_1N_1r}\Big)\Big|\sum_{\substack{l\sim M_1N_1 \\ (l,r)=1}}e\Big(hkq\frac{\bar{l}}{r}\Big)\Big|$$

$$\ll\Big(1+\frac{hx}{M_1M_2N_1N_2}\Big)\sum_{\substack{r\sim M_2N_2 \\ (r,q)=1}}(hkq,r)^{1/2}r^{1/2+\varepsilon/2}\Big(1+\frac{M_1N_1}{r}\Big)$$

$$\ll\Big(1+\frac{hx}{M_1M_2N_1N_2}\Big)x^{\varepsilon/2}\Big(\sum_{r\sim M_2N_2}(hk,r)^{1/2}r^{1/2}+M_1N_1\sum_{r\sim M_2N_2}\frac{(hk,r)^{1/2}}{r^{1/2}}\Big)$$

$$\ll\Big(1+\frac{hx}{M_1M_2N_1N_2}\Big)x^{\varepsilon/2}\Big(\sum_{r\sim M_2N_2}\frac{(hk,r)}{r}\Big)^{1/2}\Big\{\Big(\sum_r r^2\Big)^{1/2}+M_1N_1\Big(\sum_r 1\Big)^{1/2}\Big\}$$

$$\ll\left(1+\frac{hx}{M_1M_2N_1N_2}\right)x^{\varepsilon}\{(M_2N_2)^{3/2}+M_1N_1(M_2N_2)^{1/2}\}, \tag{5.5}$$

since

$$\sum_{n\sim N}\frac{(m,\ n)}{n}\ll\tau(m)\log N.$$

In conjunction with (5.3) (5.4) (5.5), we have

$$R_3\ll x^{\varepsilon}M_1M_2N_1N_2\Big(|C_0|+\sum_{|h|>HM_2N_2}|C_h|\Big)$$

$$+x^{2\varepsilon}\{(M_2N_2)^{3/2}+M_1N_1(M_2N_2)^{1/2}\}\sum_{0<|h|\leq HM_2N_2}|C_h|\left(1+\frac{hx}{M_1M_2N_1N_2}\right)$$

$$\ll x^{\varepsilon}M_1M_2N_1N_2\left(\frac{\log H}{H}+\sum_{h>HM_2N_2}\frac{H}{h^2}\right)$$

$$+x^{2\varepsilon}((M_2N_2)^{3/2}+M_1N_1(M_2N_2)^{1/2})\Big\{\sum_{0<h\leq H}\frac{\log H}{H}\left(1+\frac{Hx}{M_1M_2N_1N_2}\right)$$

$$+\sum_{H<h\leq HM_2N_2}H\left(\frac{1}{h^2}+\frac{x}{hM_1M_2N_1N_2}\right)\Big\}$$

$$\ll x^{\varepsilon}M_1M_2N_1N_2\left(\frac{\log H}{H}+\frac{1}{M_2N_2}\right)+x^{2\varepsilon}((M_2N_2)^{3/2}$$

$$+M_1N_1(M_2N_2)^{1/2})(\log H)^2\left(1+\frac{Hx}{M_1M_2N_1N_2}\right).$$

Now, we choose

$$H=\frac{M_1M_2N_1N_2}{x^{1-4\varepsilon}}.$$

then we see $H>2$ since $M_1M_2N_1N_2>x^{1-2\varepsilon}$ in (5.1). Thus, we have

$$R_3\ll x^{\varepsilon}(x^{1-3\varepsilon}+M_1N_1)+x^{2\varepsilon}((M_2N_2)^{3/2}+M_1N_1(M_2N_2)^{1/2})(\log x)^2x^{4\varepsilon},$$

or

$$\sup|R_3|\ll x^{1-2\varepsilon}+x^{\varepsilon}MN+x^{7\varepsilon}(MN)^{3/2}\ll x^{1-2\varepsilon}, \tag{5.6}$$

since $MN<x^{2/3-7\varepsilon}$.

We turn to R_2. We shall show $\sup|R_2|\ll x^{1-3\varepsilon/2}$, from which Lemma 4 follows. Actually, by (5.1) (5.2) (5.6), we then get

$$R\ll(\log D)^8\sum_{0<\delta\kappa\leq x/q}\sum\tau(\delta)^2\sup(|R_2|+|R_3|)+\frac{x}{q}x^{1-\varepsilon}\ll\frac{x}{q}x^{1-\varepsilon}.$$

Now,

$$R_2=\frac{1}{N_1N_2}\sum_{\substack{m_1\\(m_1,\,m_2)=1\\(m_1m_2,\,q)=1}}\sum_{m_2}\alpha_1(m_1)\alpha_2(m_2)\int_{x/\delta m_1m_2}^{2x-\delta kq/\delta m_1m_2}\sum_{0<|h|\leq H}\sum_{\substack{n_1\\(m_1n_1,\,m_2n_2)=1\\(n_1n_2,\,q)=1\\\mu^2(n_1n_2)=1}}\sum_{n_2}\beta_1(n_1)\beta_2(n_2)$$

$$\cdot\frac{N_1N_2}{n_1n_2}e\Big(\frac{ht}{n_1n_2}\Big)\cdot e\Big(hkq\frac{\overline{m_1n_1}}{m_2n_2}\Big)dt$$

$$\ll\frac{1}{N_1N_2}\sum_{\substack{m_1\\(m_1,m_2)=1\\(m_2,q)=1}}\sum_{m_2}\int_{x/4\delta M_1M_2}^{2x/\delta M_1M_2}\Big|\sum_{\substack{0<h\le H\\(m_1n_1,m_2n_2)=1\\(n_1n_2,q)=1\\\mu^2(n_1n_2)=1}}\sum_{n_1}\sum_{n_2}\beta_1(n_1)\beta_2(n_2)\frac{N_1N_2}{n_1n_2}e\Big(\frac{ht}{n_1n_2}\Big)$$

$$\cdot e\Big(hkq\frac{\overline{m_1n_1}}{m_2n_2}\Big)\Big|dt$$

$$=\frac{1}{N_1N_2}\int_{x/4\delta M_1M_2}^{2x/\delta M_1M_2}\sum_{m_1}\sum_{m_2}\Big|\sum_{h}\sum_{n_1}\sum_{n_2}c_{hn_1n_2}(t)\cdot e\Big(hkq\frac{\overline{m_1n_1}}{m_2n_2}\Big)\Big|dt \tag{5.7}$$

where $|c_{hn_1n_2}(t)|\leqq 1$.

We proceed to consider the sum

$$S=S(t;q,k;M_1,M_2,N_1,N_2)$$

$$=\sum_{m_1}\sum_{m_2}\Big|\sum_{h}\sum_{n_1}\sum_{n_2}c_{hn_1n_2}(t)\cdot e\Big(hkq\frac{\overline{m_1n_2}}{m_2n_2}\Big)\Big|^2$$

$$=\sum_{\substack{0<h_1,h_2\le H\\(n_1,n_2)=(n_3,n_4)=1\\(n_1n_2n_3n_4,q)=1\\\mu^2(n_1n_2)=\mu^2(n_3n_4)=1}}\sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{n_4}c_{h_1n_1n_2}(t)\bar{c}_{h_2n_3n_4}(t)\sum_{\substack{m_1\\(m_1n_1,m_2n_2)=1\\(m_1n_3,m_2n_4)=1\\(m_2,q)=1}}\sum_{m_2}e\Big(kq\Big(h_1\frac{\overline{m_1n_1}}{m_2n_2}-h_2\frac{\overline{m_1n_3}}{m_2n_4}\Big)\Big).$$

Put

$$b=h_1(m_1n_1)^*\frac{1}{(n_2,n_3)}\frac{n_4}{(n_1,n_4)}-h_2(m_1n_3)^{**}\frac{1}{(n_1,n_4)}\frac{n_2}{(n_2,n_3)}$$

where

$$(m_1n_1)^*m_1n_1\equiv 1 \pmod{m_2n_2},$$

$$(m_1n_3)^{**}m_1n_3\equiv 1 \pmod{m_2n_4},$$

then

$$m_1n_1n_3b=h_1(m_1n_1)^*m_1n_1\frac{n_3}{(n_2,n_3)}\frac{n_4}{(n_1,n_4)}-h_2(m_1n_3)^{**}m_1n_3\frac{n_1}{(n_1,n_4)}\frac{n_2}{(n_2,n_3)}$$

$$\equiv h_1\frac{n_3}{(n_2,n_3)}\frac{n_4}{(n_1,n_4)}-h_2\frac{n_1}{(n_1,n_4)}\frac{n_2}{(n_2,n_3)}\quad\Big(\operatorname{mod} m_2\frac{n_2}{(n_2,n_3)}\frac{n_4}{(n_1,n_4)}\Big)$$

$$=l,\ \text{say}.$$

Thus, we see

$$h_1\frac{m_1n_1}{m_2n_2}-h_2\frac{m_1n_3}{m_2n_4}\equiv l\frac{\overline{m_1n_1n_3}}{m_2\frac{n_2}{(n_2,n_3)}\frac{n_4}{(n_1,n_4)}}\quad(\operatorname{mod} 1),$$

Since $\mu^2(n_1n_2)=\mu^2(n_3n_4)=1$, $(m_1n_1, m_2n_2)=(m_1n_3, m_2n_4)=1$. Therefore we may write

$$S=\sum_{\substack{0<h_1,h_2\leqq H}}\ \sum_{\substack{n_j(j=1,2,3,4)\\ h_1n_3n_4-h_2n_1n_2=l(n_1,n_4)(n_2,n_3)\\ m_2n_2n_4=d(n_1,n_4)(n_2,n_3)}} c_{h_1n_1n_2}(t)\bar{c}_{h_2n_3n_4}(t)\sum_{m_1}\sum_{m_2} e\Big(kql\frac{\overline{m_1n_1n_3}}{d}\Big)$$

$$\leqq \sum_{h_1}\sum_{h_2}\sum_{\substack{n_j(j=1,2,3,4)\\ (n_1,n_2)=(n_3,n_4)=1\\ h_1n_3n_4-h_2n_1n_2=l(n_1,n_4)(n_2,n_3)\\ m_2n_2n_4=d(n_1,n_4)(n_2,n_3)\\ (d,q)=1}}\sum_{m_2}\Bigg|\sum_{\substack{n_1\sim M_1\\ (m_1,d(n_1,n_4)(n_2,n_3))=1}} e\Big(kql\frac{\overline{m_1n_1n_3}}{d}\Big)\Bigg|$$

For $l\neq 0$, we apply Lemma 3.

$$S\ll \sum_{\substack{h_1\ \ h_2\ \ n_j\\ h_1n_3m_4=h_2n_1n_2}}(M_1M_2)+x^{\varepsilon/2}\sum_{h_1}\sum_{h_2}\sum_{n_j}\sum_{\substack{m_2\\ h_1n_3n_4-h_2n_1n_2=l(n_1,n_4)(n_2,n_3)\neq 0\\ m_2n_2n_4=d(n_1,n_4)(n_2,n_3)\\ (d,q)=1}}(kql,d)^{1/2}d^{1/2}\Big(1+\frac{M_1}{d}\Big)$$

$$\ll M_1M_2\sum_{r\sim HN_1N_2}\tau_3(r)^2+x^{\varepsilon/2}\sum_{h_1}\sum_{h_2}\Big\{\sum_{n_j}\sum_{m_2}(kl,d)^{1/2}d^{1/2}+M_1\sum_{n_j}\sum_{m_2}\frac{(kl,d)^{1/2}}{d^{1/2}}\Big\}$$

$$\ll x^{\varepsilon}HM_1M_2N_1N_2+x^{\varepsilon/2}\sum_{h_1}\sum_{h_2}\Big(\sum_{h_j}\sum_{m_2}\frac{(kl,d)}{d}\Big)^{1/2}\Big\{\Big(\sum_{n_j}\sum_{m_2}d^2\Big)^{1/2}+M_1\Big(\sum_{n_j}\sum_{m_2}1\Big)^{1/2}\Big\}.$$

Here, by an elementary argument, we easily see that

$$\sum_{n_j}\sum_{m_2}\frac{(kl,d)}{d}\ll x^{\varepsilon}N_1^2,$$

$$\sum_{n_j}\sum_{m_2}d^2\ll(M_2N_2^2)^3N_1^2,$$

and

$$\sum_{n_j}\sum_{m_2}1\ll M_2N_2^2N_1^2.$$

Thus,

$$S\ll x^{\varepsilon}HM_1M_2N_1N_2+x^{\varepsilon}H^2(N_1^2(M_2N_2^2)^{3/2}+M_1M_2^{1/2}N_1^2N_2)$$

$$\ll x^{\varepsilon}H^2(x^{1-4\varepsilon}+N_1^2(M_2N_2^2)^{3/2}+M_1M_2^{1/2}N_1^2N_2). \tag{5.9}$$

Hence, in conjunction with (5.7) (5.8) (5.9), we have

$$R_2\ll\frac{1}{N_1N_2}\int_{x/4\delta M_1M_2}^{2x/\delta M_1M_2}\Big(\sum_{m_1}\sum_{m_2}1\Big)^{1/2}(S)^{1/2}dt$$

$$\ll\frac{x}{\delta M_1M_2N_1N_2}\{M_1M_2\cdot x^{\varepsilon}H^2(x^{1-4\varepsilon}+N_1^2(M_2N_2^2)^{3/2}+M_1M_2^{1/2}N_1^2N_2)\}^{1/2}$$

$$\ll x^{9\varepsilon/2}\{M_1M_2x^{1-4\varepsilon}+M_1M_2(N_1^2(M_2N_2^2)^{3/2}+M_1M_2^{1/2}N_1^2N_2)\}^{1/2}.$$

Since $M\leqq x^{1/2-4\varepsilon}$, $N\leqq x^{1/20}$, we get

$$\sup|R_2|\ll x^{9\varepsilon/2}(M^2x^{1-4\varepsilon}+M^{7/2}N^5)^{1/2}\ll x^{1-3\varepsilon/2},$$

as required.

This completes our proof of Theorem 1.

6. Proof of Theorem 2.

In this section we give a brief proof of Theorem 2. We may easily modify the proof of Theorem 1. Put

$$\Lambda'(n)=\sum_{d\mid n}\lambda_d,\quad \Lambda'=\sum_d\frac{\lambda_d}{d}$$

where λ_d's satisfy the conditions in Lemma 4. Moreover, for $x\leqq y\leqq 2x$, $\Delta=\Delta(x)>2$, write

$$E(y,\Delta)=\sum_{y-\Delta<n\leq y}\Lambda'(n)-\Delta\Lambda'.$$

Then we have

$$\int_x^{2x}|E(y,\Delta)|^2dy$$

$$=\sum_{x-\Delta<n\leq 2x}\Lambda'(n)^2\int_{\max(x,n)}^{\min(2x,n+\Delta)}dy+\sum_{\substack{x-\Delta<n_1,n_2\leq 2x\\ n_1\neq n_2\\ |n_1-n_2|\leq\Delta}}\Lambda'(n_1)\Lambda'(n_2)\int_{\max(x,n_1,n_2)}^{\min(2x,n_1+\Delta,n_2+\Delta)}dy$$

$$-2\Delta\Lambda'\sum_{x-\Delta<n\leq 2x}\Lambda'(n)\int_{\max(x,n)}^{\min(2x,n+\Delta)}dy+\Delta^2(\Lambda')^2x$$

$$=O\Big(\sum_{x-\Delta<n\leq 2x}\tau(n)^2\Delta\Big)+2\sum_{0<a\leq\Delta}\sum_{x-\Delta<n\leq 2x-a}\Lambda'(n)\Lambda'(n+a)\int_{\max(x,n+a)}^{\min(2x,n+\Delta)}dy$$

$$-2\Delta\Lambda'\Big\{\sum_{x<n\leq 2x-\Delta}\Lambda'(n)\cdot\Delta+O\Big(\sum_{\substack{x-\Delta<n\leq x\\ \text{or } 2x-\Delta<n\leq 2}}\tau(n)\cdot\Delta\Big)\Big\}+\Delta^2(\Lambda')^2((x-\Delta)+\Delta)$$

$$=O(\Delta x(\log x)^3)+2\sum_{0<a\leq\Delta}\Big\{\sum_{x<n\leq 2x-\Delta}\Lambda'(n)\Lambda'(n+a)(\Delta-a)$$

$$+O\Big(\sum_{\substack{x-\Delta<n\leq x\\ \text{or } 2x-\Delta<n\leq 2x-a}}\tau(n)\tau(n+a)\cdot\Delta\Big)\Big\}-2\Delta^2(\Lambda')\Big\{\sum_d\lambda_d\Big(\sum_{\substack{x<n\leq 2x-\Delta\\ d\mid n}}1\Big)\Big\}$$

$$+O(\Delta^3x^\epsilon)+\Delta^2(\Lambda')^2(x-\Delta)$$

$$=O(\Delta x(\log x)^3)+D'(x,\Delta)-\Delta^2(\Lambda')^2(x-\Delta)+O(\Delta^2D(\log D)+\Delta^3x^\epsilon),\text{ say.}$$

The above $D'(n,\Delta)$ is essentially equal to the sum $D(x,q)$ in (4.1). We therefore see

$$\int_x^{2x}|E(y,\Delta)|^2dy\ll\Delta x(\log x)^3+\Delta^2x^{1-\epsilon}+\Delta^3x^\epsilon.$$

The other modifications are immediate. Hence we get Theorem 2.

References

[1] Halberstam, H. and Richert, H. E., Sieve methods, Academic 1974.
[2] Harman, G., Almost-primes in short intervals, Math. Ann. **258** (1981), 107-112.
[3] Heath-Brown, D. R., Almost-primes in arithmetic progressions and short intervals, Math. Proc. Cambridge Phil. Soc. **83** (1978), 357-375.
[4] Hooley, C., Applications of sieve methods to the theory of numbers, Cambridge 1976.
[5] Iwaniec, H., A new form of the error term in the linear sieve, Acta Arith. **37** (1980), 307-320.
[6] Motohashi, Y., On almost-primes in arithmetic progressions, J. Math. Soc. Japan, **28** (1976), 363-383.
[7] Motohashi, Y., A note on almost-primes in short intervals, Proc. Japan Acad. Ser A Math. Sci. **55** (1979), 225-226.
[8] Selberg, A., On the normal density of primes in short intervals and the difference between consecutive primes, Arch. Math. Naturvid. **47** (1943), 87-105.
[9] Turán, P., Über die Primzahlen der arithmetischen Progression, Acta Sci. Math. Szeged. **8** (1936/37), 226-235.
[10] Wolke, D., Fast-Primzahlen in kurzen Intervallen, Math. Ann. **224** (1979), 233-242.

Institute of Mathematics
University of Tsukuba
Ibaraki Pref.
305 Japan