TSUKUBA J. MATH.
Vol. 13 No. 1 (1989), 1—5

NORMAL HIGHER WEIERSTRASS POINTS

By

R. F. LAX

Let X denote a compact Riemann surface of genus $g>1$ and let Ω denote the sheaf of sections of the canonical bundle on X. Put $D_n=\dim_C H^0(X, \Omega^{\otimes n})=(2n-1)(g-1)+\delta_{1n}$. An integer γ is called an n-gap at P if there exists $\sigma\in H^0(X, \Omega^{\otimes n})$ such that $\mathrm{ord}_P\sigma=\gamma-1$. At each point $P\in X$ there is a sequence of D_n n-gaps

$$1=\gamma_1(P)<\gamma_2(P)<\cdots<\gamma_{D_n}(P)\leqq 2n(g-1)+1$$

and P is called a Weierstrass point of order n if $\gamma_{D_n}>D_n$ (cf. [1, 2]). The n-weight of P, denoted $W_n(P)$, is defined by $W_n(P)=\sum_{i=1}^{D_n}(\gamma_i(P)-i)$ and one has

$$\sum_{P\in X} W_1(P)=g^3-g \quad \text{and} \quad \sum_{P\in X} W_n(P)=gD_n^2 \qquad \text{for } n\geqq 2\,.$$

A Weierstrass point P of order n is called *normal* if the sequence of n-gaps at P is $1, 2, \cdots, D_n-1, D_n+1$ or, equivalently, if the n-weight of P is 1.

The "classical" Weierstrass points are the Weierstrass points of order 1 and it is well-known that a generic Riemann surface has only normal classical Weierstrass points. The analogous statement for higher order Weierstrass points fails for genus two by virtue of the following result.

PROPOSITION 1. *Let X be a hyperelliptic Riemann surface of genus $g\geqq 2$ and let P denote one of the $2g+2$ (hyperelliptic) classical Weierstrass points on X. Then $W_n(P)=g(g+1)/2$ for all $n\geqq 2$.*

PROOF. [1].

It follows from the Proposition that if X is any compact Riemann surface of genus two, then X must have weight three Weierstrass points of order n for all $n\geqq 2$. Our goal in this note is to show how one may use results of H. B. Laufer [3] to show that if X is a generic compact Riemann surface of genus $g>2$, then X has only normal Weierstrass points of order n for all n.

Following Laufer, for $P\in X$, let $d_n(P)$ denote the maximal order of a zero at P of sections in $H^0(X, \Omega^{\otimes n})$ and let $w_n(P)$ denote the smallest nonnegative

Received September 3, 1987.

integer such that there does not exist $\sigma \in H^0(X, \Omega^{\otimes n})$ with a zero at P of order $w_n(P)$. Thus, $d_n(P)+1$ is the largest n-gap at P and $w_n(P)+1$ is the smallest positive non-n-gap at P. Then Laufer [3] proved the following important result.

THEOREM 1. *Let X be a compact Riemann surface of genus $g \geqq 2$. Let $\pi: \mathfrak{X} \to T$ be the complete effectively parametrized deformation of X. Let $P \in X$. Put*

$$G_1 = \{Q \in \mathfrak{X},\ Q \text{ near } P: d_1(Q) = d_1(P)\}.$$

Fix $n \geqq 1$.

(1) *Put $G = \{Q \in \mathfrak{X},\ Q \text{ near } P: d_n(Q) = d_n(P)\}$. If $n \geqq 2$, $d_n(P) = 2n(g-1)$ and $d_1(P) = 2g-2$, then G coincides with G_1. In all other cases, G is a submanifold of $\mathfrak{X}$ of dimension $3g-3+D_n-d_n(P)$.*

(2) *Put $W = \{Q \in \mathfrak{X},\ Q \text{ near } P: w_n(Q) = w_n(P)\}$. Suppose $w_n(P) \leqq D_n - 1$. If $n \geqq 2$, $w_n(P) = (2n-2)(g-1)-1$ and $d_1(P) = 2g-2$, then W coincides with G_1. In all other cases, W is a submanifold of $\mathfrak{X}$ of dimension $3g-2-D_n+w_n(P)$.*

Let $V \to T$ denote the "universal" curve over the Teichmüller space parametrizing Teichmüller surfaces of genus g. In [4], we defined complex subspaces $W_k^r(K^n)$ of V. Although we only considered $k \leqq D_n$ in [4], the definition makes sense for any value of k and we make no restriction here. (Also, we take this opportunity to point out that the two "+3"s on page 3 of [4] should be "−3"s.) It is easy to see, as in [5], that the underlying sets of the $W_k^r(K^n)$ satisfy the following:

(1) If $k \leqq D_n$, then

$$|W_k^r(K^n)| = \{(t, P): \text{at } P \in V_t, \text{ there are at least } r \text{ non-}n\text{-gaps} \leqq k\}.$$

(2) If $k \geqq D_n$, then

$$|W_k^r(K^n)| = \{(t, P): \text{at } P \in V_t, \text{ there are at least } r\ n\text{-gaps} > k\}.$$

We will write $W_k(K^n)$ for $W_k^1(K^n)$. As a consequence of Laufer's Theorem, we have

THEOREM 2.

(1) *Suppose $k \leqq D_n$ and $k \neq (2n-2)(g-1)$. Then $W_k(K^n) \backslash W_{k-1}(K^n)$, if nonempty, is smooth of pure dimension $3g-3-D_n+k$. If $k = 2(n-2)(g-1)$, then $W_k(K^n) \backslash (W_{k-1}(K^n) \cup W_{2g-2}(K))$, if nonempty, is smooth of pure dimension $3g-3-D_n+k$, while if a component of $W_k(K^n) \backslash W_{k-1}(K^n)$ is contained in $W_{2g-2}(K)$, then that component has dimension $2g-1$.*

(2) *Suppose $k \geqq D_n$ and $k \neq 2n(g-1)$. Then $W_k(K^n) \backslash W_{k+1}(K^n)$, if nonempty,*

is smooth of pure dimension $3g-3+D_n-k$. *If* $k=2n(g-1)$, *then* $W_k(K^n)\setminus(W_{k+1}(K^n)\cup W_{2g-2}(K))$, *if nonempty, is smooth of pure dimension* $3g-3+D_n-k$, *while if a component of* $W_k(K^n)\setminus W_{k+1}(K^n)$ *is contained in* $W_{2g-2}(K)$, *then that component has dimension* $2g-1$.

PROOF. The only point which may not be obvious is the observation that $W_{2g-2}(K)$ has pure dimension $2g-1$ by [5, Thm. 2].

COROLLARY 1. *Suppose* $g\geqq 3$. *If* $k\neq D_n$, *then dim* $W_k(K^n)<3g-3$.

PROOF. Suppose $k=(2n-2)(g-1)<D_n$ and a component of $W_k(K^n)\setminus W_{k-1}(K^n)$ is contained in $W_{2g-2}(K)$. By Theorem 2, this component has dimension $2g-1$ and this is less than $3g-3$ since g is greater than two. Now, every component of $W_2(K^n)$ has dimension less than $3g-3$. It follows from this and Theorem 2 that every component of $W_3(K^n)$ has dimension less than $3g-3$ and, continuing this reasoning, it is then easy to see that every component of $W_k(K^n)$ for $k<D_n$ has dimension less than $3g-3$.

Similarly, if $k=2n(g-1)$ and a component of $W_k(K^n)\setminus W_{k+1}(K^n)$ is contained in $W_{2g-2}(K)$, then that component has dimension $2g-1<3g-3$. Now, every component of $W_{2n(g-1)}(K^n)$ has dimension less than $3g-3$. It follows from this and Theorem 2 that every component of $W_{2n(g-1)-1}(K^n)$ has dimension less than $3g-3$ and, continuing in this manner, every component of $W_k(K^n)$ for $k>D_n$ has dimension less than $3g-3$. ■

Our main result is

THEOREM 3. *Suppose* $g\geqq 3$. *Then a generic Riemann surface of genus* g *has only normal Weierstrass points of order* n *for all* $n\geqq 1$. *In particular, for all* $n\geqq 1$ *and all* $g\geqq 3$, *there exist compact Riemann surfaces of genus* g *with normal Weierstrass points of order* n.

PROOF. By Corollary 1, no component of $W_k(K^n)\setminus W_{k-1}(K^n)$, for $k<D_n$ can dominate the Teichmüller space. Hence on a generic Riemann surface of genus g, the first non-n-gap at each Weierstrass point of order n must be D_n. Also, for $k>D_n$, no component of $W_k(K^n)\setminus W_{k+1}(K^n)$ can dominate the Teichmüller space. Hence on a generic Riemann surface of genus g, the last n-gap at each Weierstrass point of order n must be D_n+1. ■

We now consider the case $g=2$. In this case, as we observed in [4], $|W_3(K^2)|=|W_2(K^2)|=|W_2(K)|$ (and we showed that $W_3(K^2)$ is not reduced).

The key observation, in terms of Laufer's Theorem, when $g=2$ is that now $2g-1=3g-3$. Now, if $n\geqq 2$, then each of the six classical Weierstrass points on a curve of genus 2 has weight 3 as a Weierstrass point of order n and the total weight of Weierstrass points of order n on such a curve is $2(2n-1)^2$. Note that if $n\geqq 3$, then the total weight of Weierstrass points of order n is greater than 18 and so there must be Weierstrass points of order n which do not belong to $W_2(K)$. By applying Laufer's Theorem as in the proof of Corollary 1, one may see that $\dim W_k(K^n)\backslash W_2(K)<3g-3=3$ if $k\neq D_n$, and so by reasoning as in the proof of Theorem 3 we obtain

THEOREM 4. *For $n\geqq 3$, a generic Riemann surface of genus 2 has six Weierstrass points of order n of weight 3 each and $2(2n-1)^2-18$ normal Weierstrass points of order n.*

To conclude, we will indicate how the theory of Weierstrass points on singular curves may also be used to demonstrate the existence of normal higher order Weierstrass points. Suppose X_0 is an irreducible projective curve of arithmetic genus g with only nodes and cusps as singularities. Let ω denote the sheaf of dualizing differentials on X_0. By [9] there exists a flat proper map $\pi: S\to C$ from a smooth surface S to a smooth curve C and a point $c\in C$ such that X_0 is the fiber of π over c. Restricting to a neighborhood of c we may, and will, assume that all other fibers of π are smooth curves. Then S is a family of Gorenstein curves over C and there exists a bundle $\omega_{S/C}$ of relative dualizing differentials whose restriction to X_0 is ω.

Now suppose there exists a point $P\in X_0$ which is a normal Weierstrass point of ω^n in the sense of [6]. Then by [7] for all points $c'\in C$ sufficiently close to c, there will be a (normal) Weierstrass point of weight one of the nth tensor power of the canonical bundle on the smooth curve which is the fiber of π over c'.

The idea here is that if X_0 is a simple curve, for example a rational curve, then the higher order Weierstrass points on X_0 will be computable, at least in theory. To illustrate this, let X_0 denote a rational curve with three simple cusps. Then it was shown in [8] that each of the cusps is a Weierstrass point of order 2 of weight 35 and that there are three normal Weierstrass points of order 2. By the reasoning described above, this gives an alternate proof of the existence of normal Weierstrass points of order 2 on a smooth curve of genus 3.

References

[1] Duma, A., Holomorphe Differentiale höherer Ordnung auf kompakten Riemannschen Flächen, Schriftenreihe der Univ. Münster, Serie 2, Heft **14**, 1978.

[2] Farkas, H. and Kra, I., Riemann surfaces, Springer, New York-Heidelberg-Berlin, 1890.

[3] Laufer, H. B., On generalized Weierstrass points and rings with no prime elements, in Riemann surfaces and related topics, Proceedings of the 1978 Stony Conference, Princeton, 1980.

[4] Lax, R. F., On higher order Weierstrass points of the universal curve, Math. Z. **162** (1978), 1-6.

[5] Lax, R. F., Gap sequences and moduli in genus 4, Math. Z. **175** (1980), 67-75.

[6] Lax, R. F., On the distributon of Weierstrass points on singular curves, Israel Math. J. **57** (1987), 107-115.

[7] Lax, R. F., Weierstrass weight and degenerations, Proc. Amer. Math. Soc., **101** (1987), 8-10.

[8] Lax, R. F. and Widland, C., Weierstrass points on rational cuspidal curves, Boll. U. M. I., (7) **2**-A (1988), 65-71.

[9] Winters, G. B., On the existence of certain families of curves, Amer. J. Math. **96** (1974), 215-228.

Institut Mittag-Leffler, Djursholm, Sweden
Department of Mathematics,
LSU, Baton Rouge, LA 70803,
USA (current address)