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SOME CRITERIA FOR REDUCIBLE ABELIAN VARIETIES

By

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Dedicated to Professor Yukihiro Kodama on his 60th birthday

Introduction.

A principally polarized abelian variety is called reducible if it is isomorphic to a product of two abelian varieties of positive dimensions. For a principally polarization L, it is well known that $L^{\otimes 2}$ determines a morphism. Its image is called a Wirtinger variety. If a principally polarized abelian variety is irreducible, then the Wirtinger variety coincides with the Kummer variety associated to this polarized abelian variety. Moreover if an abelian variety is sufficiently general, then the Wirtinger variety is not contained in any conics. On the other hand, if a principally polarized abelian variety is reducible, then the Wirtinger variety is contained in many conics. Our main purpose is to give conditions for reducibility of an abelian variety in terms of conics which contains the Wirtinger variety associated to the abelian variety.

Notations.

char(k): The characteristic of a field k

k^*: The group of all units of a field k

f^*: The pull back defined by a morphism f

$G^\wedge$: The character group of a finite group G

$\underline{L}$: The invertible sheaf associated to a line bundle L

$\mathcal{O}(D)$: The invertible sheaf associated to a divisor D

$K(\mathcal{L})$: The subgroup of an abelian variety defined as follows, $K(\mathcal{L})=\{x\in A;\ T_x^*(\mathcal{L})\cong\mathcal{L}\}$ where T_x is a translation morphism on A and $\mathcal{L}$ is an invertible sheaf on A

$NS(A)$: The Néron-Severi group on a variety A

S^nV: The n-th symmetric product of a vector space V

Map(A, B): The set of all maps from a set A to a set B

$\Gamma(A, \mathcal{L})$: The global sections of an invertible sheaf $\mathcal{L}$ on an abelian variety A

§1. Review.

Let k be a fixed algebraically closed field of char$(k)\neq 2$, and let A be a g-

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dimensional abelian variety defined over k. If L is an ample line bundle on A, then it is well known that $K(\underline{L})$ is a finite group and $K(\underline{L})\cong G\oplus G^{\wedge}$ where G is a finite abelian group isomorphic to $\boldsymbol{Z}/d_1\boldsymbol{Z}\oplus\cdots\oplus\boldsymbol{Z}/d_g\boldsymbol{Z}$ with $d_1|\cdots|d_g$. We take $d_i>0$ for $i=1,\cdots,g$. Put $\delta=(d_1,\cdots,d_g)$. Let $G(\underline{L})$ be the theta group of L defined by $\{(x,\phi);\ x\in K(\underline{L})$ and $\phi:\underline{L}\xrightarrow{\sim}T_x^*(\underline{L})\}$. In the following, we assume char$(k)\nmid d_g$.

THEOREM 1. *$G(\underline{L})$ has a unique irreducible representation $\Gamma(A,\underline{L})$ in which k^* acts by its natural character.*

PROOF. See Mumford [3].

Let $G(\delta)$ be the Heisenberg group, that is $G(\delta)=k^*\times G\times G^{\wedge}$ as sets with multiplication

$$(t,x,m)(t',x',m')=(tt'm'(x),x+x',m+m').$$

Put $V(\delta)=\mathrm{Map}(G,k)$. $V(\delta)$ is naturally a vector space over k and is a $G(\delta)$-module by

$$((t,x,m)f)(u)=tm(u)f(x+u)$$

where $(t,x,m)\in G(\delta)$ and $f\in V(\delta)$.

THEOREM 2. *$G(\delta)$ has a unique irreducible representation $V(\delta)$ in which k^* acts by its natural character.*

PROOF. See Mumford [3].

THEOREM 3. *$G(\underline{L})$ and $G(\delta)$ are isomorphic to each other as groups.*

PROOF. See Mumford [3].

Let δ be the delta function in $V(\delta)$ where x is in G defined by $\delta_x(y)=0$ if $y\neq x$ and $\delta_x(x)=1$. If α is an isomorphism from $G(\underline{L})$ to $G(\delta)$, then α induces the isomorphism $\beta:\Gamma(A,\underline{L})\to V(\delta)$. We put $q_L(x)$ by the "Nullwerte" (in the sense of Mumford) of $\beta^{-1}(\delta_x)$. Now we assume that L is totally symmetric and choose a symmetric theta structure on $(L,L^{\otimes 2})$ (see Mumford [3]). The symmetric theta structure induces $\beta_1:\Gamma(A,\underline{L})\xrightarrow{\sim}V(\delta)$ and $\beta_2:\Gamma(A,L^{\otimes 2})\xrightarrow{\sim}V(2\delta)$. Let s, s' be elements of $\Gamma(A,\underline{L})$. We put $f_1=\beta_1(s)$ and $f_2=\beta_2(s')$. Let $f_1*f_2=\beta_2(s\otimes s')$.

THEOREM 4. (*Multiplication formula*). *In above notations*

$$f_1*f_2(x)=\sum_{y\in x+G}f_1(x+y)f_2(x-y)q_{L^{\otimes 2}}(y).$$

PROOF. See Mumford [3].

§ 2. Examples.

Let $\delta=(d_1, \cdots, d_g)$ where $d_1, \cdots, d_g$ are positive integers with $d_1 | \cdots | d_g$. In this section we assume char$(k) \nmid d_g$. Let G_δ be the group $\boldsymbol{Z}/d_1\boldsymbol{Z}\oplus \cdots \oplus \boldsymbol{Z}/d_g\boldsymbol{Z}$.

DEFINITION. *We define $Sp(G_\delta)$ by*

$$\{\sigma\in \mathrm{Aut}(G_\delta\times G_\delta\hat{}\,);\ \text{For every } (x, m), (y, n)\in G_\delta\times G_\delta\hat{}\,,$$
$$((x, m), (y, n))_{Sp}=(\sigma(n, m), \sigma(y, n))_{Sp}\},$$

where $((x, m), (y, n))_{Sp}=n(-x)m(y)$ *and* $\sigma(x, m)=(\alpha x+\beta m, \gamma x+\delta m)$ *for* $\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$.

We put $\langle x, m\rangle=m(x)$ where $x\in G_\delta$ and $m\in G_\delta\hat{}$.

DEFINITION. *We define a group N_0 as follows,*

$$N_0=\left\{(\sigma, f);\ \sigma\in Sp(G_\delta) \text{ and } f: G_\delta\times G_\delta\hat{}\longrightarrow k^* \text{ with } f((x, m)+(x', m'))\right.$$
$$\left.=f((x, m))f((x', m'))\langle\alpha x+\beta m, \gamma x'+\delta m'\rangle/\langle x, m'\rangle \text{ where } \sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right\},$$

as sets. The multiplication of N_0 is defined by

$$(\sigma, f)(\sigma', f')=(\sigma\sigma', f'')$$

where $f''(w)=f'(\sigma w)f(w)$, $w\in G_\delta\times G_\delta\hat{}$.

As $\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in Sp(G(\delta))$, $\langle\alpha x+\beta m, \gamma x'+\delta m'\rangle/\langle x, m'\rangle=\langle\alpha x'+\beta m', \gamma x+\delta m\rangle/\langle x', m\rangle$. Therefore the multiplication of N_0 is well defined. Now we take an element (σ, f) in N_0. We define a map

$$n_{(\sigma, f)}((t, x, m))=(tf(x, m), \sigma(x, m)).$$

LEMMA. *Via $n_{(\sigma, f)}$, N_0 acts on $G(\delta)$ as a group of automorphisms over k^*.*

Let η be an automorphism of $G(\delta)$ over k^*. As $G(\delta)$ acts on $V(\delta)$, we can determine another $G(\delta)$-action on $V(\delta)$ via η. But in these two actions on $V(\delta)$, k^* acts by its natural character. Therefore these two actions are isomorphic to each other by theorem 2 in § 1. Therefore η determines a base change of $V(\delta)$.

EXAMPLE 1. $\delta=(2, \cdots, 2)$ and σ is a

$$\left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 1 & & & 0 & \\ & & \ddots & & & \ddots \\ & & 1 & & & 0 \\ \hline 1 & & & 0 & & \\ & 0 & & & 1 & \\ & & \ddots & & & \ddots \\ & & 0 & & & 1 \end{array}\right)$$

over char$(k)\neq 2$. In above notations, let $x={}^t(x_1, \cdots, x_g)$ and $m={}^t(m_1, \cdots, m_g)$ where x_i and m_i are elements of $\boldsymbol{Z}/2\boldsymbol{Z}$ $(i=1, \cdots, g)$. We define $\langle\ ,\ \rangle : G_\delta \times G_\delta\hat{}\ \rightarrow k^*$ by

$$\langle x, m\rangle=(-1)^{x_1m_1+\cdots+x_gm_g}.$$

In this situation, σ is an element of $Sp(G_\delta)$. Because

$$(\sigma(x, m), \sigma(x', m'))_{Sp}=(-1)^{x_1m_1'+x_2'm_2+\cdots+x_g'm_g}/(-1)^{x_1'm_1+x_2m_2'+\cdots+x_gm_g'}$$

where $x'={}^t(x_1', \cdots, x_g')$ and $m'={}^t(m_1', \cdots, m_g)$. We define a map $f: G_\delta \times G_\delta\hat{}\ \rightarrow k^*$ with

$$f(x, m)=(-1)^{x_1m_1}.$$

The pair (σ, f) is an element of N_0. In fact

$$\begin{aligned} f((x, m)+(x', m'))/f(x, m)f(x', m')&=(-1)^{(x_1+x_1')(m_1+m_1')}/(-1)^{x_1m_1}(-1)^{x_1'm_1'} \\ &=(-1)^{x_1'm_1+x_1m_1'}, \end{aligned}$$

on the other hand,

$$\begin{aligned} \langle\alpha x+\beta m, \gamma x'+\delta m'\rangle/\langle x, m'\rangle&=(-1)^{x_1'm_1+x_2m_2'+\cdots+x_gm_g'}/(-1)^{x_1m_1+\cdots+x_gm_g'} \\ &=(-1)^{x_1'm_1+x_1m_1'} \end{aligned}$$

where $\sigma=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$. Hence (σ, f) is an element of N_0.

Now we calculate the base change oi $V=V(\delta)$ defined by the above (σ, f). Let σ_0 be the base change of V defined by (σ, f). By definition

$$\sigma_0((t, x, m)\cdot v)=(tf(x, m), \sigma(x, m))\cdot v$$

for every element v of V. Let $t=1$, $x=0$ and $v=\delta_0$ where $0={}^t(0, \cdots, 0)$. As $(1, 0, m)\delta_0=\delta_0$, $\sigma_0((1, 0, m)\delta_0)=\sigma_0(\delta_0)$. Moreover $\sigma_0((1, 0, m)\delta_0)=(1, \sigma(0, m))\sigma_0(\delta_0)$ and $\sigma(0, m)=({}^t(m_1, 0, \cdots, 0), {}^t(0, m_2, \cdots, m_g))$. We put $\sigma_0(\delta_0)=\sum_{s\in(\boldsymbol{Z}/2\boldsymbol{Z})^g} h(s)\delta_s$. By above relations, we obtain

$$\sum_{s\in(\boldsymbol{Z}/2\boldsymbol{Z})^g} h(s)\delta_s=\sum_{s\in(\boldsymbol{Z}/2\boldsymbol{Z})^g}(-1)^{m_2s_2+\cdots+m_gs_g}f(s)\delta_{s+m_1e_1}$$

where $e_1={}^t(1, 0, \cdots, 0)$ and $s={}^t(s_1, \cdots, s_g)$. Therefore

$$\sigma_0(\delta_0)=c(\delta_0+\delta_{e_1})$$

where c is a constant. Similously we obtain

$$\sigma_0(\delta_{e_1})=c(\delta_0-\delta_{e_1}).$$

Moreover

$$\sigma_0(\delta_s)=c(\delta_{s-s_1e_1}+(-1)^{s_1}\delta_{s-(s_1-1)e_1})$$

where $s={}^t(s_1, \cdots, s_g)$.

EXAMPLE 2. $\delta=(4, \cdots, 4)$ and σ is

$$\left(\begin{array}{cccc|cccc} 0 & & & & -1 & & & \\ & 1 & & & & 0 & & \\ & & \ddots & & & & \ddots & \\ & & & 1 & & & & 0 \\ \hline 1 & & & & 0 & & & \\ & 0 & & & & 1 & & \\ & & \ddots & & & & \ddots & \\ & & & 0 & & & & 1 \end{array}\right)$$

over char$(k)\neq 2$. In above notations, let $x={}^t(x_1, \cdots, x_g)$ and $m={}^t(m_1, \cdots, m_g)$ where x_i and m_i are elements of $\boldsymbol{Z}/4\boldsymbol{Z}$ $(i=1, \cdots, g)$. We define $\langle\,,\rangle: G_\delta\times G_\delta\hat{}\to k^*$ by

$$\langle x, m\rangle=\sqrt{-1}^{x_1m_1+\cdots+x_gm_g}.$$

It is clear that σ is an element of $Sp)G_\delta)$. We define a map $f: G_\delta\times G_\delta\hat{}\to k^*$ with

$$f(x, m)=\sqrt{-1}^{x_1m_1}.$$

The pair (σ, f) is an element of N_0. In this situation, thebase change of $V(\delta)$ defined by (σ, f) is as follows,

$$\begin{aligned}\sigma(\delta_s)=c(&\delta_{s-s_1e_1}+\sqrt{-1}^{s_1}\delta_{s-(s_1-1)e_1}+(-1)^{s_1}\delta_{s-(s_1-2)e_1}\\ &+\sqrt{-1}^{3s_1}\delta_{s-(s_1+1)e_1})\end{aligned}$$

where $s={}^t(s_1, \cdots, s_g)$, $e_1={}^t(1, 0, \cdots, 0)$, c is constant and $\sqrt{-1}$ is an element of k with $\sqrt{-1}^2=-1$.

§ 3. Reducibility of abelian variety.

In this section we consider the canonical map $t: \Gamma(A, \mathcal{O}(2\theta))^{\otimes 2}\to\Gamma(A, \mathcal{O}(4\theta))$ where A is an abelian variety and θ is a theta divisor on A. We assume char$(k)\neq 2$ and fix a symmetric theta structure (α_1, α_2) on $(\mathcal{O}(2\theta), \mathcal{O}(4\theta))$ where α_1 is a group isomorphism from $G(\mathcal{O}(2\theta))$ to $G((2, \cdots, 2))$ over k^* and α_2 is a group isomorphism from $G(\mathcal{O}(4\theta))$ to $G((4, \cdots, 4))$ over k^*. Let δ be $(2, \cdots, 2)$. In this case we obtain a $G(\delta)$ module $V(\delta)$ and a $G(2\delta)$ module $V(2\delta)$ and t induces

$$\beta: V(\delta)^{\otimes 2}\longrightarrow V(2\delta).$$

We write $\beta(a\otimes b)=a*b$. Now the multiplication formula says that

$$\delta_s*\delta_t=\sum_{x\in(2\boldsymbol{Z}/4\boldsymbol{Z})^g} q_{L^{\otimes 4}}(\underline{s}-\underline{t}+x)\delta_{\underline{s}+\underline{t}+x}$$

where $\underline{L}=\mathcal{O}(\theta)$, s and t are elements of $(\boldsymbol{Z}/2\boldsymbol{Z})^g$ and $\underline{s}$, $\underline{t}$ are elements of $(\boldsymbol{Z}/4\boldsymbol{Z})^g$ with $\underline{s}$ mod $2=s$ and $\underline{t}$ mod $2=t$. Let $\underline{\beta}$ be a map

$$\underline{\beta}: S^2V(\delta) \longrightarrow V(2\delta)$$

induced by β. We put $\Delta_c=(\delta_x\odot\delta_{c-x})$ where c is an element of $(\boldsymbol{Z}/2\boldsymbol{Z})^g$ and x runs through a complete set of representative of $(\boldsymbol{Z}/2\boldsymbol{Z})^g/\{0, c\}$. Moreover we put $E_c=(\delta_{\underline{c}+x}+\delta_{-\underline{c}+x})$ where x runs through a complete set of representative of $(2\boldsymbol{Z}/4\boldsymbol{Z})^g/\{0, 2\underline{c}\}$. With the above notations, the map $\underline{\beta}$ is defined by

$$\underline{\beta}(\Delta_c)=E_cF_c$$

where F_c is an element of $M_{2g}(k)$ if $c=0$ and an element of $M_{2g-1}(k)$ if $c\neq 0$. As $(\Delta_c)_{c\in(\boldsymbol{Z}/2\boldsymbol{Z})^g}$ are basis of $S^2V(\delta)$, therefore $\underline{\beta}$ is represented by the following matrix

$$\begin{pmatrix} F_{x_1} & & \\ & \ddots & \\ & & F_{x_{2^g-1}} & \\ & & & F_0 \end{pmatrix}$$

where $(\boldsymbol{Z}/2\boldsymbol{Z})^g=\{0, x_1, \cdots, x_{2^g-1}\}$. Let G_c be a subgroup of $(\boldsymbol{Z}/2\boldsymbol{Z})^g$ satisfying $(\boldsymbol{Z}/2\boldsymbol{Z})^g=G_c\oplus(\boldsymbol{Z}/2\boldsymbol{Z})c$ in which c is a given non-zero element. Let $G_c^{(2)}$ be the subgroup of $(2\boldsymbol{Z}/4\boldsymbol{Z})^g\cong(\boldsymbol{Z}/2\boldsymbol{Z})^g$ corresponding to G_c. We fix such notations Now the multiplication formula says

$$\begin{aligned}\delta_x*\delta_{c-x}&=\sum_{\eta\in(2\boldsymbol{Z}/4\boldsymbol{Z})^g} q_{L^{\otimes 4}}(2\underline{x}-\underline{c}+\eta)\delta_{\underline{c}+\eta}\\ &=\sum_{\eta\in G_c^{(2)}} q_{L^{\otimes 4}}(2\underline{x}-\underline{c}+\eta)(\delta_{\underline{c}+\eta}+\delta_{-(\underline{c}+\eta)}).\end{aligned}$$

Therefore we obtain

$$\sum_{x\in G_c}\chi(x)\delta_x*\delta_{c-x}=\Big(\sum_{u\in G_c^{(2)}}\chi(u)q_{L^{\otimes 4}}(u-c)\Big)\Big(\sum_{v\in G_c^{(2)}}\chi(v)(\delta_{\underline{c}+v}+\delta_{-(c+v)})\Big)$$

where χ is a charactor of G_c. Let X_c be the set

$$\{\chi\in G_c\hat{}\,;\ \sum_{u\in G_c^{(2)}}\chi(u)q_{L^{\otimes 4}}(u-\underline{c})=0\}.$$

THEOREM. *In the above notations, if* $(\boldsymbol{Z}/2\boldsymbol{Z})^g=G_1\oplus G_2$, *and if for every* $x\in G_1$, $y\in G_2$ *with* $x\neq 0$ *and* $y\neq 0$ *the rank of* F_{x+y} *is at most* 2^{g-2} *and every* χ *contained in* $G_{x+y}\hat{}-X_{x+y}$ *have same value at* x, *then* $(A, \mathcal{O}(\theta))$ *is reducible.*

PROOF. By the assumption, the order of $G_{x+y}\hat{}-X_{x+y}$=the rank of F_{x+y}

$\leqq 2^{g-2}$. Therefore there exists a subgroup H of $G_{x+y}\hat{}$ which satisfies $\chi(x)=1$ for every $\chi\in H$, and some element $\rho\in G_{x+y}\hat{}$, we obtain $X_{x+y}\supset\rho+H$. Hence

$$\sum_{x\in H}\sum_{u\in G_{x+y}^{(2)}}\rho(u)\chi(u)q_{L^{\otimes 4}}(u-\underline{x}-\underline{y})=0$$

and

$$\sum_{\chi\in H}\chi(u)=2^{g-2}\quad\text{if } u=0 \text{ or } u=x$$
$$=0\quad\text{if } u\neq 0 \text{ and } u\neq x$$

by the definition of X_{x+y}. Therefore

$$q_{L^{\otimes 4}}(\underline{x}+\underline{y})+\rho(x)q_{L^{\otimes 4}}(\underline{x}-\underline{y}+u)=0.$$

Moreover

$$q_{L^{\otimes 4}}(\underline{x}+\underline{y}+u)+\rho(x)q_{L^{\otimes 4}}(\underline{x}-\underline{y}+u)=0$$

for every u contained in G_{x+y}. Hence

$$\begin{aligned}\delta_t*\delta_{x+y-t}&=\sum_{u\in G_{x+y}^{(2)}}q_{L^{\otimes 4}}(2\underline{t}-\underline{x}-\underline{y}+u)(\delta_{\underline{x}+\underline{y}+u}+\delta_{-(\underline{x}+\underline{y}+u)})\\&=-\rho(x)\sum_{u\in G_{x+y}^{(2)}}q_{L^{\otimes 4}}(2\underline{t}+\underline{x}-\underline{y}+u)(\delta_{\underline{x}+\underline{y}+u}+\delta_{-(\underline{x}+\underline{y}+u)})\\&=-\rho(x)\delta_{t+x}*\delta_{t-y},\end{aligned}$$

especially $\delta_0*\delta_{x+y}=-\rho(x)\delta_x*\delta_y$. Let $f(x+y)$ be $-\rho(x)$. This f is a function from $\{x+y;\ x\in G_1$ and $y\in G_2$ with $x\neq 0$ and $y\neq 0\}$ to $\{\pm 1\}$. We fix a symmetric theta structure (α_2, α_3) on $(\mathcal{O}(4\theta), \mathcal{O}(8\theta))$. We have already obtained

$$\delta_0*\delta_{x+y}=f(x+y)\delta_x*\delta_y$$
$$\delta_0*\delta_{x'+y'}=f(x'+y')\delta_{x'}*\delta_{y'}$$

for any non-zero $x, x'\in G_1$ with $x\neq x'$ and any non-zero $y, y'\in G_2$ with $y\neq y'$. Therefore

$$(\delta_0*\delta_{x+y})*(\delta_0*\delta_{x'+y'})=f(x+y)f(x'+y')(\delta_x*\delta_y)*(\delta_{x'}*\delta_{y'})$$

by the above symmetric theta structure. On the other hand

$$(\delta_0*\delta_0)*(\delta_{x+y}*\delta_{x'+y'})=f(x+x'+y+y')(\delta_0*\delta_0)*(\delta_{x+y'}*\delta_{x'+y}).$$

Hence we obtain the relation

$$f(x+x'+y+y')=f(x+y)f(x'+y')f(x+y')f(x'+y).$$

Let $\tilde{\delta}_{x+y}$ be $f(x+y)\delta_{x+y}$ if x is a non-zero element of G_1 and y is a non-zero element of G_2 and let $\tilde{\delta}_z$ be δ_z if z is an element of G_1 or G_2. The above relation says that

$$\tilde{\delta}_{x+y}*\tilde{\delta}_{x'+y'}=\tilde{\delta}_{x+y'}*\tilde{\delta}_{x'+y}$$

for $x, x' \in G_1 - \{0\}$ with $x \neq x'$ and $y, y' \in G_2 - \{0\}$ with $y \neq y'$. We denote $\phi: A \to \boldsymbol{P}^{2^g-1}$ by a morphism defined by $\underline{L}^{\otimes 2} \cong \mathcal{O}(2\theta)$. The above relations say that $\phi(A)$ is contained in some Segre variety embedded in $\boldsymbol{P}^{2^g-1}$ which is isomorphic to $\boldsymbol{P}^{2^{g_1}-1} \times \boldsymbol{P}^{2^{g_2}-1}$ where g_i is a dimension of G_i as a vector space over $\boldsymbol{Z}/2\boldsymbol{Z}$ $(i=1, 2)$. Let ϕ_i be a morphism from A to $\boldsymbol{P}^{2^{g_i}-1}$ $(i=1, 2)$ which is a composition of ϕ and the projection on $\boldsymbol{P}^{2^{g_1}-1} \times \boldsymbol{P}^{2^{g_2}-1}$ and let H_i be a hyperplane of $\boldsymbol{P}^{2^{g_i}-1}$ $(i=1, 2)$. We put B_i=the connected component of $K(\mathcal{O}(\phi_i{}^*H_i))$ containing 0 $(i=1, 2)$. It is clear that B_i is an abolian subvariety of A_i $(i=1, 2)$. Let A_i be the abelian variety A/B_i, let $p: A \to A_1 \times A_2$ be the canonical morphism and let η_i be the morphism from A_i to $\boldsymbol{P}^{2^{g_i}-1}$ defined by ϕ_i $(i=1, 2)$. As $\phi_i(-x) = \phi_i(x)$ for every $x \in A$ $(i=1, 2)$, hence $\eta_i(-x) = \eta_i(x)$ for every $x \in A_i$ $(i=1, 2)$. Therefore $\eta_i{}^*H_i$ is totally symmetric. This implies $\eta_i{}^*H_i$ is linearly equivalent to $2D_i$ for some divisor D_i on A_i $(i=1, 2)$. In this situations,

$$\dim \Gamma(A_i, \mathcal{O}(\eta_i{}^*H_i)) \geqq 2^{g_i}$$

$(i=1, 2)$. As $p^*(\eta_1{}^*H_1 \times A_2 + A_1 \times \eta_2{}^*H_2)$ is linearly equivalent to 2θ,

$$\begin{aligned} 2^g &= \dim \Gamma(A, \mathcal{O}(2\theta)) \\ &\geqq \dim \Gamma(A_1, \mathcal{O}(\eta_1{}^*H_1)) \dim \Gamma(A_2, \mathcal{O}(\eta_2{}^*H_2() \\ &\geqq 2^{g_1} 2^{g_2} = 2^g. \end{aligned}$$

Hence $\dim \Gamma(A_i, \mathcal{O}(\eta_i{}^*H_i)) = 2^{g_i}$ $(i=1, 2)$. Therefore $\eta_i{}^*H_i$ is linearly equivalent to $2\theta_i$ where θ_i is a principally polarization of A_i because $\eta_i{}^*H_i$ is linearly equivalent to $2D_i$ for some divisor D_i on A_i $(i=1, 2)$. So we obtain that dimension of A_i is g_i and p is a finite surjective morphism $(i=1, 2)$ (see Ohbuchi [4]). As 2θ is linearly equivalent to $p^*(\eta_1{}^*H_1 \times A_2 + A_1 \times \eta_2{}^*H_2)$, therefore 2θ is linearly equivalent to $2p^*(\theta_1 \times A_2 + A_1 \times \theta_2)$. Hence θ is algebraically equivalent to $p^*(\theta_1 \times A_2 + A_1 \times \theta_2)$ because $NS(A)$ is a torsion free module for any abelian variety A. Hence θ is linearly equivalent to $p^*(T_{z_1}{}^*\theta_1 \times A_2 + A_1 \times T_{z_2}{}^*\theta_2)$ for some $z_i \in A_i$ $(i=1, 2)$. Therefore p is an isomorphism and $(A, \mathcal{O}(\theta))$ is reducible polarized abelian variety (see Ohbuchi [4]). Thus we prove the theorem.

§4. Reducibility of 3-dimensional abelian variety.

In this section we prove the following theorem.

THEOREM. *Let A be a 3-dimensional abelian variety defined over algebraically closed field k of* char$(k) \neq 2$. *Let I be a kernel of $S^2\Gamma(A, \mathcal{O}(2\theta)) \to \Gamma(A, \mathcal{O}(4\theta))$. If dimension of I over $k \geqq 5$, then $(A, \mathcal{O}(\theta))$ is reducible.*

We put $\underline{L}=\mathcal{O}(\theta)$. We fix smmetric theta structures (α_1, α_2) and (α_2, α_3) on $(\underline{L}^{\otimes 2}, \underline{L}^{\otimes 4})$ and $(\underline{L}^{\otimes 4}, \underline{L}^{\otimes 8})$ respectively. Let $\delta_x^{(i)}$ be a delta function contained in $V((2^i, 2^i, 2^i))$ and let $G_{2^i}=G_{(2^i,2^i,2^i)}$ be a group $(\boldsymbol{Z}/2^i\boldsymbol{Z})^3$ $(i=1, 2, 3)$. Especially we dente δ_x by $\delta_x^{(1)}$. For every c contained in G_{2^i}, we take $\underline{c}$ which is an element of $G_{2^{i+1}}$ with $\underline{c} \bmod 2^i=c$ and take $\underline{\underline{c}}$ which is an element of $G_{2^{i+2}}$ with $\underline{\underline{c}} \bmod 2^{i+1}=\underline{c}$ $(i=1, 2)$. And for every c contained in $G_{2^{i+1}}$, we take c° which is an element of G_{2^i} with $2c=2c^\circ$. Let λ be an element of $G_2\hat{}$ and let a, b be elements of G_8 with $a \bmod 2=b \bmod 2$.

Definition. *In above notations, we define $T(\lambda; a, b)$ by*

$$T(\lambda;\, a,\, b)=\sum_{u\in G_2}\lambda(u)\delta^{(2)}_{c+b^\circ+2\underline{u}}*\delta^{(2)}_{c+2\underline{u}}$$

where c is an element of G_4 with $2\underline{c}=a-b$.

Definition. *For $\lambda\in G_2\hat{}$ abd $c\in G_8$, we define $q_1(\lambda, c)$ by*

$$q_1(\lambda,\, c)=\sum_{u\in G_2}\lambda(u)q_{L^{\otimes 8}}(c+4\underline{\underline{u}}).$$

To prove the theorem, we prepare the following lemmas.

Lemma 1. *For every λ contained in $G_2\hat{}$ and every a in G_8, there exists some $b\in G_2$ with $q_1(\lambda, a+4\underline{\underline{b}})\neq 0$.*

Proof. See Mumford [3].

Lemma 2. *The kernel of $S^2V(4, 4, 4)\rightarrow V(8, 8, 8)$ is generated by*

$$q_1(\lambda,\, c)T(\lambda;\, a,\, b)-q_1(\lambda,\, b)T(\lambda;\, a,\, c)$$

where a, b, c are elements of G_8 and $a \bmod 2=b \bmod 2=c \bmod 2$.

Proof. By Lemma 1 and Igusa [2] p. 167 theorem 5, this lemma is clear.

Proof of Theorem. By the notation of § 3, the homomorphism

$$\underline{\beta}: S^2V(2, 2, 2)\longrightarrow V(4, 4, 5)$$

is denoted by $\underline{\beta}(\Delta_c)=E_cF_c$ where c is an element of G_1. Therefore $\underline{\beta}$ is represented by the following 36×36 matrix,

$$F=\begin{pmatrix} F_{x_1} & & & \\ & \ddots & & \\ & & F_{x_7} & \\ & & & F_0 \end{pmatrix}$$

with respect to $(\Delta_c)_{c\in G_2}$, $(E_c)_{c\in G_2}$ where $G_2=\{x_1, \cdots, x_7, 0\}$, F_c is an element of

$M_4(k)$ if $c \in G_2$ is not 0 and F_0 is an element of $M_8(k)$. The assumption says that the rank of $F \leqq 31$. Therefore there exists at most 5 c's contained in G_2 with determinant of $F_c=0$. By the example in § 2, we may assume that at least 2 of these $c \neq 0$. We prove this theorem in two steps.

STEP 1; If there exists some c contained in G_2 with $c \neq 0$ and the rank of $F_c \leqq 2$, then the theorem is true.

We take a $c' \neq c$ which satisfies $\det F_{c'}=0$. Let χ_1 and χ_2 be element of $G_2\hat{}$ with $\chi_i(c)=1$ and

$$\sum_{u \in G_2/\{0,c\}} \chi_i(u) q_{L^{\otimes 4}}(2\underline{\underline{u}}-c)=0$$

where $i=1, 2$. We put

$$\{\chi \in G_2\hat{}\,;\ \chi(c)=1\}=\{\chi_1, \chi_2, \chi_3, \chi_4\}.$$

Let H_c be a kernel of $\chi_3\chi_4$. As H_c contains c and dimension of H_c as a vector space over $\boldsymbol{Z}/2\boldsymbol{Z}$ is 2, therefore there exists some t contained in H_c and $t \neq c$ with

$$(*)_c \quad \delta_0 * \delta_c = \pm \delta_t * \delta_{c+t}$$

and $H_c=\{0, c, t, c+t\}$. Now we take $a, b \in G_4$. By the definition,

$$T(\lambda; 2\underline{a}, 2\underline{b})=\sum_{u \in G_2} \lambda(u) \delta^{(2)}_{a+b+2\underline{u}} * \delta^{(2)}_{a-b+2\underline{u}}$$

for every λ contained in $G_2\hat{}$. Therefore the Nullwerte of $T(\lambda; 2\underline{a}, 2\underline{b})$ is

$$\sum_{u \in G_2} \lambda(u) q_{L^{\otimes 4}}(a+b+2\underline{u}) q_{L^{\otimes 4}}(a-b+2\underline{u}).$$

Especially the Nullwerte of $T(\lambda; 2\underline{\underline{d}}, 0)$ is

$$\sum_{u \in G_2} \lambda(u) q_{L^{\otimes 4}}(\underline{d}+2\underline{u})^2$$

for every d contained in G_2 and fixed $\underline{d}$. If χ is an element of H_c, then

$$\begin{aligned}\sum_{u \in G_2} \chi(u) q_{L^{\otimes 4}}(\underline{c}+2\underline{u})^2 &= 2 \sum_{u \in G_2/\{0,c\}} \chi(u) q_{L^{\otimes 4}}(\underline{c}+2\underline{u})^2 \\ &= 2 \sum_{u \in H_c} \chi(u)(1+\chi(t)) q_{L^{\otimes 4}}(\underline{c}+2\underline{u})^2,\end{aligned}$$

because $q_{L^{\otimes 4}}(\underline{c}+2\underline{u}) \pm q_{L^{\otimes 4}}(\underline{c}+2\underline{u}+2\underline{t})=0$ by the relation $(*)_c$. Hence for $\chi \in H_c$ with $\chi(t)=-1$, the Nullwerte of $T(\chi; 2\underline{c}, 0)=0$. Moreover we can take this χ with $\chi(c')=1$. Lemma 2 says that

$$q_1(\chi, b) T(\chi; 2\underline{e}, 0)=q_1(\chi, 0) T(\chi; 2\underline{e}, b)$$

where $e \in G_2$ and $b \in G_8$ with $b \bmod 2=0$. Now we prove that

$$(*)_{c'} \quad \delta_0 * \delta_{c'} = \pm \delta_t * \delta_{c'+t}.$$

The proof of $(*)_{c'}$ is done in two cases.

case 1). $q_1(\chi, 0)=0$.

As there exists $b\in G_8$ with $b \bmod 4=0$ by Lemma 1, $T(\chi; 2\underline{\underline{e}}, 0)=0$. In this case, we take $e=c'$. Then $T(\chi; 2\underline{\underline{c}}', 0)=0$. Therefore we obtain

$$\sum_{u\in G_2/H_c} \chi(u)(q_{L^{\otimes 4}}(\underline{c}'+2\underline{u})^2-q_{L^{\otimes 4}}(\underline{c}'+2\underline{u}+2\underline{t})^2)=0\,.$$

This equation and the relation given by $\det F_{c'}=0$ gives

$$q_{L^{\otimes 4}}(\underline{c}'+2\underline{u})\pm q_{L^{\otimes 4}}(\underline{c}'+2\underline{u}+2\underline{t})=0\,.$$

Hence we obtain $(*)_{c'}$.

case 2). $q_1(\chi, 0)\neq 0$.

By this condition,

$$T(\chi; 2\underline{\underline{e}}, b)=(q_1(\chi, b)/q_1(\chi, 0))T(\chi; 2\underline{\underline{e}}, 0)\,.$$

We put $e=c$. In this case, the Nullwerte of $T(\chi; 2\underline{\underline{c}}, 0)=0$. Hence the Nullwertw of $T(\chi; 2\underline{\underline{c}}, b)=0$ for any $b\in G_8$ with $b \bmod 2=0$. We take $b=2\underline{\underline{c}}'-2\underline{\underline{c}}$. This implies

$$\sum_{u\in G_2} \chi(u)q_{L^{\otimes 4}}(\underline{c}'+2\underline{u})q_{L^{\otimes 4}}(-\underline{c}'+2\underline{c}+2\underline{u})=0\,.$$

Hence

$$\sum_{u\in G_2} \chi(u)q_{L^{\otimes 4}}(\underline{c}'+2\underline{u})q_{L^{\otimes 4}}(\underline{c}'+2\underline{c}+2\underline{u})=0\,.$$

In this situation,

$$\begin{aligned}
&\sum_{u\in G_2} \chi(u)q_{L^{\otimes 4}}(\underline{c}'+2\underline{u})q_{L^{\otimes 4}}(\underline{c}'+2\underline{c}+2\underline{u})\\
&\quad =2 \sum_{u\in G_2/\{0,c\}} \chi(u)q_{L^{\otimes 4}}(\underline{c}'+2\underline{u})q_{L^{\otimes 4}}(\underline{c}'+2\underline{c}+2\underline{u})\\
&\quad =4 \sum_{u\in G_2/\{0,c,c',c+c'\}} \chi(u)q_{L^{\otimes 4}}(\underline{c}'+2\underline{u})q_{L^{\otimes 4}}(\underline{c}'+2\underline{c}+2\underline{u})\\
&\quad =4(q_{L^{\otimes 4}}(\underline{c}'+2\underline{u})q_{L^{\otimes 4}}(\underline{c}'+2\underline{c}+2\underline{u})-q_{L^{\otimes 4}}(c'+2\underline{u})q_{L^{\otimes 4}}(\underline{c}'+2\underline{c}+2\underline{u}))
\end{aligned}$$

Therefore this equation and relation given by $\det F_{c'}=0$ give

$$q_{L^{\otimes 4}}(\underline{c}'+2\underline{u})\pm q_{L^{\otimes 4}}(\underline{c}'+2\underline{t}+2\underline{u})=0\,.$$

Hence we obtain the relation $(*)_{c'}$.

Now it is clear that $c'\neq t$ by the definition of c', t and c. Let $c_0=c+t$ and $c_0'=c'+t$. The relations $(*)_c$ and $(*)_{c'}$ say that

$$\begin{aligned}
\delta_0*\delta_0*\delta_{c_0'}*\delta_{c_0'+c_0+t}&=\pm\delta_0*\delta_0*\delta_{t+c_0'}*\delta_{c_0+c_0'}\\
&=\pm\delta_0*\delta_{c_0'}*\delta_t*\delta_{c_0+c_0'}\,.
\end{aligned}$$

Hence $\delta_0*\delta_{c_0+c_0'+t}=\pm\delta_t*\delta_{c_0+c_0'}$. Therefore we obtain the relations

$$\delta_0{}^*\delta_{t+c_0}=\pm\delta_t{}^*\delta_{c_0}$$
$$\delta_0{}^*\delta_{t+c_0'}=\pm\delta_t{}^*\delta_{c_0'}$$
$$\delta_0{}^*\delta_{t+c_0+c_0'}=\pm\delta_t{}^*\delta_{c_0+c_0'}\,.$$

Hence we obtain this theorem by the theorem of § 3.

STEP 2; General case.

First, we show in the case of which $\det F_{c_i}=0$ $(i=1, \cdots, 5)$ and $c_i\neq 0$ for every i. In this case, there exists some χ_i contained in $G_2{}^\wedge$ with $\chi_i(c_i)=1$ and

$$\sum_{u\in G_2/\{0,\, c_i\}}\chi_i(u)\delta_u{}^*\delta_{c_i-u}=0$$

for $i=1, \cdots, 5$. We prepare the following Lemma.

LEMMA 3. *In above notations, there exists some i and j with $i\neq j$ and i, j $\{1, \cdots, 5\}$ which satisfies $\chi_i(c_j)=\chi_j(c_i)$. Moreover if $\chi_5=0$ and $c_5=0$, then there exists some i and j with $i\neq j$ and $i, j\in\{1, \cdots, 4\}$ which satisfies $\chi_i(c_j)=\chi_j(c_i)$.*

PROOF. If there exists some i and j with $i\neq j$ and $\chi_i=\chi_j$, then this lemma is clear. And if there exists some i with $\chi_i=0$, then this lemma is again clear. So we assume that $\chi_1, \cdots, \chi_5$ are all distinct and not equal to 0. We also assume that $c_1, \cdots, c_5$ are all distinct and not equal to 0. We put the set E_+ and E_- by

$$E_+=\{(i, j);\ i\neq j \text{ and } \chi_i(c_j)=1\}$$
$$E_-=\{(i, j);\ i\neq j \text{ and } \chi_i(\chi_j)=-1\}\,.$$

As $\chi_i(c_i)=1$ and $c_j\neq 0$ for every $i, j=1, \cdots, 5$, therefore the order of $E_+\leqq 7$ and the order of $E_-\geqq 13$. Hence the first part of this lemma is clear. Moreover in the case of $\chi_5=0$ and $c_5=0$ we put the set E_+', E_-' by

$$E_+'=\{(i, j);\ i\neq j,\ i, j=1, \cdots, 4 \text{ and } \chi_i(c_j)=1\}$$
$$E_-'=\{(i, j);\ i\neq j,\ i, j=1, \cdots, 4 \text{ and } \chi_i(c_j)=-1\}\,.$$

If the order of $E_+'\geqq 6$, then there exists some χ_i, χ_j $(i\neq j)$ with the order of $T_i=\{c_k;\chi_i(c_k)=1,\ k=1, \cdots, 5\}$ and $T_j=\{c_k;\chi_j(c_k)=1,\ k=1, \cdots, 5\}$ are both 4. As T_i and T_j are subgroup of G_2 and $T_i, T_j\subset\{c_1, \cdots, c_4, c_5=0\}$, hence $T_i=T_j$. This is a contradiction. Therefore the order of $E_+'\leqq 5$. Hence this lemma is clear.

Now we continue the proof of the theorem. By Lemma 3, we take i, j with $i\neq j$ and $\chi_i(c_j)=\chi_j(c_i)$. Therefore

$$\sum_{u\in G_2/\{0,c_i\}}\chi_i(u)\delta_u*\delta_{c_i-u}=\delta_0*\delta_{c_i}+\chi_i(c_j)\delta_{c_j}*\delta_{c_i+c_j}+\chi_i(v)\delta_v*\delta_{c_i+v}$$
$$+\chi_i(c_j+v)\delta_{v+c_j}*\delta_{v+c_i+c_j},$$
$$\sum_{u\in G_2/\{0,c_j\}}\chi_j(u)\delta_u*\delta_{c_j-u}=\delta_0*\delta_{c_j}+\chi_j(c_i)\delta_{c_i}*\delta_{c_j+c_i}+\chi_j(v)\delta_v*\delta_{c_i+v}$$
$$+\chi_j(c_i+v)\delta_{v+c_i}*\delta_{v+c_j+c_i}.$$

In this, this v is an element of G_2 with $c \bmod\{0, c_i, c_j, c_i+\delta_j\}\neq 0$. Therefore we obtain

$$(\delta_{c_i}+\delta_{c_j})*(\delta_0+\xi\delta_{c_i+c_j})+\eta(\delta_{c_i+v}+\delta_{c_j+v})*(\delta_v+\xi\delta_{v+c_i+c_j})=0$$

where $\xi, \eta\in\{\pm 1\}$. By the examples of § 2, we obtain the theorem.

Finally, we show this theorem in general case. If $\sum_{u\in G_2}\chi(u)\delta_u*\delta_u=0$ implies $\chi=0$, then these cases arereduced in Step 1 or the first case of this step, by the examples in § 2. Now we assume that $\sum_{u\in G_2}\delta_u*\delta_u=0$ and the rank of $F_0=7$. In this case, let $c_5=0$ and let $\chi_5=0$. By lemma 3, we also obtain this theorem. Therefore we prove this theorem.

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