ISOTROPIC MINIMAL SUBMANIFOLDS IN A SPACE FORM

By

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Let $\tilde{M}^m(\tilde{c})$ be an *m-dimensional space form* of constant curvature $\tilde{c}$, that is, an m-dimensional Riemannian manifold of constant curvature $\tilde{c}$. By the Theorem in [5], the author determined n-dimensional minimal submanifolds in $\tilde{M}^m(\tilde{c})$ with the sectional curvature not less than $n\tilde{c}/2(n+1)$. We should pay attention to the value next to $n\tilde{c}/2(n+1)$, so that we could classify minimal submanifolds in $\tilde{M}^m(\tilde{c})$ with the sectional curvature not less than it.

In the present paper, we will classify n-dimensional isotropic minimal submanifolds in $\tilde{M}^m(\tilde{c})$ with the sectional curvature not less than some value. Indeed, we will prove the following

THEOREM A. *Let M be a connected compact n-dimensional $(n \geq 3)$ orientable submanifold isotropically and minimally immersed in an $(n+\nu)$-dimensional space form $\tilde{M}$ of constant curvature $\tilde{c}$. If the sectional curvature of M is not less than $n\tilde{c}/3(n+2)$, then M is of constant curvature $\tilde{c}$ or $n\tilde{c}/3(n+2)$, or the second fundamental form is parallel.*

We may assume that $0 < \tilde{c}$ by (2.17) and Remark in §2, that is, $\tilde{M}$ is a sphere $S^m(\tilde{c})$ of constant curvature $\tilde{c}$. When M is of constant curvature, by the results in [2], according as the sectional curvature is $\tilde{c}$ or $n\tilde{c}/3(n+2)$, M is a great sphere of $S^m(\tilde{c})$ or the immersion is *the standard minimal one of degree* 3 *from a shere into a sphere* as stated in [2], which we will call *the generalized Veronese submaniolfd* in the present paper. When the second fundamental form is parallel, the above immersion is the planar geodesic one, which is determined in [8]. As a Corollary to Theorem A, using the results in [2], [4] and [8], we have the following

THEOREM B. *Let M be a connected compact n-dimensional $(n \geq 3)$ orientable submanifold minimally and isotropically immersed in a sphere $S(\tilde{c})$ of constant curvature $\tilde{c}$. If the immersion is full and the sectional curvature K_σ satisfies the inequality: $n\tilde{c}/3(n+2) \leq K_\sigma \leq \tilde{c}$, then M is a great sphere of $S(\tilde{c})$, a Veronese submanifold, or a generalized Veronese submanifold.*

§ 1. Preliminaries.

Let M be an n-dimensional submanifold immersed in an $(n+p)$-dimensional space form $\tilde{M}=\tilde{M}^{n+p}(\tilde{c})$ of constant curvature $\tilde{c}$, (i. e., Riemannian submanifold with induced Riemannian metric). We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation on M (resp. $\tilde{M}$). Then the second fundamental form (the shape operator) σ of the immersion is given by

$$\sigma(X,\ Y)=\tilde{\nabla}_X Y-\nabla_X Y, \quad \text{where } X \text{ and } Y \text{ are tangent vectors,}$$

and it satisfies $\sigma(X,\ Y)=\sigma(Y,\ X)$. We choose a local field of orthonormal frames $e_1, e_2, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}$ in $\tilde{M}$ in such a way that restricted to M, $e_1, e_2, \cdots, e_n$ are tangent to M (and, consequently the remaining vectors are normal to M). Let B be the set of all such frames in $\tilde{M}$. With respect to the frame field of $\tilde{M}$ chosen above, let $\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_{n+p}$ be the field of dual frames. Then the structure equations of $\tilde{M}$ are given by (*)

$$(1.1) \quad d\tilde{\omega}_A=\sum\tilde{\omega}_{AB}\wedge\tilde{\omega}_B, \qquad \tilde{\omega}_{AB}+\tilde{\omega}_{BA}=0,$$

$$d\tilde{\omega}_{AB}=\sum\tilde{\omega}_{AC}\wedge\tilde{\omega}_{CB}-\tilde{c}\tilde{\omega}_A\wedge\tilde{\omega}_B.$$

Restricting these forms to M, we have the structure equations of M:

$$\omega_\alpha=0, \qquad \omega_{i\alpha}=\sum h_{ij}^\alpha\omega_j, \qquad h_{ij}^\alpha=h_{ji}^\alpha,$$

$$d\omega_i=\sum\omega_{ij}\wedge\omega_j, \qquad \omega_{ij}+\omega_{ji}=0,$$

$$(1.2) \quad d\omega_{ij}=\sum\omega_{ik}\wedge\omega_{kj}-\Omega_{ij}, \qquad \Omega_{ij}=\frac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l,$$

$$R_{ijkl}=\tilde{c}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum(h_{ik}^\alpha h_{jl}^\alpha-h_{il}^\alpha h_{jk}^\alpha).$$

$$(1.2) \quad d\omega_{\alpha\beta}=\sum\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}-\Omega_{\alpha\beta}, \qquad \Omega_{\alpha\beta}=\frac{1}{2}\sum R_{\alpha\beta ij}\omega_i\wedge\omega_j,$$

$$R_{\alpha\beta ij}=\sum(h_{ik}^\alpha h_{kj}^\beta-h_{jk}^\alpha h_{ki}^\beta).$$

Then, the second fundamental form σ can be written as

$$\sigma(X,\ Y)=\sum h_{ij}^\alpha\omega_i(X)\omega_j(Y)e_\alpha.$$

If we define $h_{i_1 i_2 \cdots i_{k+1} m}^\alpha$ $(1\le k)$ by

$$(1.4) \quad \sum h_{i_1 \cdots i_{k+1} m}^\alpha\omega_m := d h_{i_1 \cdots i_{k+1}}^\alpha+\sum h_{i_1 \cdots i_{j-1} m i_{j+1} \cdots i_{k+1}}^\alpha\omega_{m i_j}+\sum h_{i_1 \cdots i_{k+1}}^\beta\omega_{\beta\alpha},$$

then we have

(*) We use the following convention on the range of indices unless otherwise stated; $A, B, C, \cdots =1, 2, \cdots . n+p$; $i, j, k, \cdots =1, 2, \cdots, n$; $\alpha, \beta, \gamma, \cdots =n+1, n+2, \cdots, n+p$. We agree that repeated indices under a summation sign without indication are summed over the respective ranges.

(1.5) $\qquad h^\alpha_{i_1\cdots i_k lm} - h^\alpha_{i_1\cdots i_k ml} = \sum h^\alpha_{i_1\cdots i_{j-1} r i_{j+1}\cdots i_k} R_{r i_j lm} + \sum h^\beta_{i_1\cdots i_k} R_{\beta\alpha lm}.$

If M is of constant curvature c, then we have

(1.6) $\qquad R_{ijkm} = c(\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}).$

The vector $\sigma(X, X)$ is called the *normal curvature vector* in the direction of a unit vector X. If the normal curvature vector has the same length λ for any unit tangent vector to M, then the immersion is said to be λ-*isotropic*. The immersion is called *minimal* if $\sum h^\alpha_{mm} = 0$ for all α. We easily see that the immersion is λ-isotropic if and only if the components of the second fundamental form satisfy the following relations ([7]);

(1.7) $\qquad \sum h^\alpha_{ij}h^\alpha_{km} + \sum h^\alpha_{im}h^\alpha_{jk} + \sum h^\alpha_{ik}h^\alpha_{jm} = \lambda^2(\delta_{ij}\delta_{km} + \delta_{im}\delta_{jk} + \delta_{ik}\delta_{jm}).$

Now, we consider the functions on M defined by

$$S := \|\sigma\|^2 = \sum h^\alpha_{ij}h^\alpha_{ij}, \qquad L_N := \sum h^\alpha_{ij}h^\alpha_{km}h^\beta_{ij}h^\beta_{km},$$

$$K_N := \sum R_{\alpha\beta ij}R_{\alpha\beta ij} = \sum(h^\alpha_{ik}h^\beta_{kj} - h^\beta_{ik}h^\alpha_{kj})(h^\alpha_{il}h^\beta_{lj} - h^\beta_{il}h^\alpha_{lj}).$$

Then we know the following differential equation ([1]);

(1.8) $\qquad \dfrac{1}{2}\Delta S = \|\sigma_3\|^2 + n\tilde{c}S - L_N - K_N, \qquad$ where $\quad \|\sigma_3\|^2 := \sum h^\alpha_{ijk}h^\alpha_{ijk}.$

§ 2. The proof of Theorems.

We suppose that M is a λ-*isotropic* and *minimal* submanifold in a space form form $\tilde{M}$ of constant curvature $\tilde{c}$. Then, from (1.7), we have

(2.1) $\qquad \sum\limits_m h^\alpha_{im}h^\alpha_{mj} = \dfrac{(n+2)}{2}\lambda^2\delta_{ij},$

that is, M is an Einstein manifold. If $n \geq 3$, as is well known, λ is constant on M. From (1.2) and (1.7) we have

(2.2) $\qquad 3\sum h^\alpha_{ij}h^\alpha_{kl} = (\lambda^2 - 2\tilde{c})\delta_{ij}\delta_{kl} + (\lambda^2 + \tilde{c})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - R_{ilkj} - R_{iklj},$

which implies

(2.3) $\qquad \sum h^\alpha_{km}R_{ikmj} = (\lambda^2 + \tilde{c})h^\alpha_{ij} - \dfrac{3}{2}\sum h^\alpha_{km}h^\beta_{km}h^\beta_{ij}.$

Since λ is constant, from (1.7), for all i, j, k and l, we have

$$\sum h^\alpha_{ijm}h^\alpha_{kl} + \sum h^\alpha_{ikm}h^\alpha_{jl} + \sum h^\alpha_{jkm}h^\alpha_{il} = -\{\sum h^\alpha_{ij}h^\alpha_{klm} + \sum h^\alpha_{ik}h^\alpha_{jlm} + \sum h^\alpha_{jk}h^\alpha_{ilm}\},$$

which implies

$(2.4)_{ijmkl} \quad \sum h^\alpha_{ijm}h^\alpha_{kl} - \sum h^\alpha_{ij}h^\alpha_{klm} = \sum h^\alpha_{ijl}h^\alpha_{km} + \sum h^\alpha_{ijk}h^\alpha_{lm} + 2(\sum h^\alpha_{im}h^\alpha_{jkl} + \sum h^\alpha_{jm}h^\alpha_{ikl}).$

Making $(2.4)_{ijmkl}+(2.4)_{klmij}$, we have

(2.5) $\sum h_{im}^{\alpha}h_{jkl}^{\alpha}+\sum h_{jm}^{\alpha}h_{kli}^{\alpha}+\sum h_{km}^{\alpha}h_{lij}^{\alpha}+\sum h_{lm}^{\alpha}h_{ijk}^{\alpha}=0.$

It follows from $(2.4)_{ijkml}$ and (2.5) that we have

(2.6) $\sum h_{ij}^{\alpha}h_{klm}^{\alpha}-\sum h_{ijm}^{\alpha}h_{kl}^{\alpha}=\sum h_{km}^{\alpha}h_{ijl}^{\alpha}+\sum h_{lm}^{\alpha}h_{ijk}^{\alpha}.$

Since M is minimal, from (2.6) we get

(2.7) $\sum h_{im}^{\alpha}h_{jmk}^{\alpha}=0.$

Furthermore, from (2.6) we get

(2.8)
$$\sum h_{ij}^{\alpha}h_{klm}^{\alpha}h_{kl}^{\beta}h_{ijm}^{\beta}=\sum h_{ijm}^{\alpha}h_{kl}^{\alpha}h_{kl}^{\beta}h_{ijm}^{\beta}+2\sum h_{km}^{\alpha}h_{ijl}^{\alpha}h_{kl}^{\beta}h_{ijm}^{\beta},$$
$$\sum h_{ij}^{\alpha}h_{klm}^{\alpha}h_{kl}^{\beta}h_{ijm}^{\beta}=-\sum h_{km}^{\alpha}h_{ijl}^{\alpha}h_{lm}^{\beta}h_{ijk}^{\beta}=-\sum h_{mk}^{\alpha}h_{ijl}^{\alpha}h_{kl}^{\beta}h_{ijm}^{\beta},$$

whcih imply

(2.9) $3\sum h_{ij}^{\alpha}h_{klm}^{\alpha}h_{kl}^{\beta}h_{ijm}^{\beta}=\sum h_{ij}^{\alpha}h_{klm}^{\alpha}h_{ij}^{\beta}h_{klm}^{\beta}.$

Letting $\Delta h_{ij}^{\beta}=\sum h_{ijmm}^{\beta}$, from (2.6) we have

$$\sum h_{ij}^{\beta}\Delta h_{kl}^{\beta}-\sum(\Delta h_{ij}^{\beta})h_{kl}^{\beta}=\sum h_{km}^{\beta}h_{ijlm}^{\beta}+\sum h_{lm}^{\beta}h_{ijkm}^{\beta},$$

which implies that

(2.10) $\sum h_{ij}^{\alpha}h_{kl}^{\alpha}h_{km}^{\beta}h_{ijlm}^{\beta}=0.$

It follows from (1.5) and (2.7) that we have

$$\sum h_{ij}^{\alpha}h_{kl}^{\alpha}h_{klm}^{\beta}h_{ijm}^{\beta}=-\sum h_{ij}^{\alpha}h_{kl}^{\alpha}h_{km}^{\beta}h_{ijml}^{\beta}=-\sum h_{ij}^{\alpha}h_{kl}^{\alpha}h_{km}^{\beta}h_{ijlm}^{\beta}$$
$$+\sum h_{ij}^{\alpha}h_{kl}^{\alpha}h_{km}^{\beta}\{h_{pj}^{\beta}R_{pilm}+h_{pi}^{\beta}R_{pjlm}+h_{ij}^{\gamma}R_{\gamma\beta lm}\},$$

which, together with (1.2), (1.3), (1.7), (2.3) and (2.9), implies

(2.11) $\sum h_{ij}^{\alpha}h_{kl}^{\alpha}h_{klm}^{\beta}h_{ijm}^{\beta}=\left\{\dfrac{(n+4)\lambda^{2}}{2}-\tilde{c}\right\}L_{N}-\lambda^{2}(n\tilde{c}+2\lambda^{2})S$

$$+2\,\mathrm{Trace}\,(H^{\alpha}H^{\beta}H^{\gamma}H^{\alpha}H^{\beta}H^{\gamma})$$

$$=\left\{\dfrac{n+10}{2}\lambda^{2}-\tilde{c}\right\}L_{N}-n\lambda^{2}(\tilde{c}-\lambda^{2})S-\dfrac{1}{2}\,\mathrm{Trace}\,A^{3}$$

$$-2\sum(\mathrm{Trace}\,H^{\alpha}H^{\beta}H^{\gamma})(\mathrm{Trace}\,H^{\alpha}H^{\beta}H^{\gamma}),$$

where $A=(\sum h_{ij}^{\alpha}h_{ij}^{\beta})$ and $H^{\alpha}=(h_{ij}^{\alpha})$. By means of (1.2), (1.5), (2.1) and (2.3), we have

(2.12) $\Delta h_{ij}^{\alpha}=\sum h_{ijmm}^{\alpha}=n(\tilde{c}-\lambda^{2})h_{ij}^{\alpha}-2\sum h_{km}^{\alpha}h_{km}^{\beta}h_{ij}^{\beta}.$

Since $S=\sum h_{ij}^{\alpha}h_{ij}^{\alpha}=n(n+2)\lambda^{2}/2$ is constant on M, using (1.7), we rewrite (1.8) as

(2.13) $\|\sigma_{3}\|^{2}=2L_{N}-n(\tilde{c}-\lambda^{2})S.$

Making use of (1.5), (2.7) and (2.12), we have

$$\sum h^{\alpha}_{ijmlm} - \sum h^{\alpha}_{ijmml} = \sum h^{\alpha}_{pjm} R_{pilm} + \sum h^{\alpha}_{ipm} R_{pjlm} + \sum h^{\alpha}_{ijp} R_{pmlm} + \sum h^{\beta}_{ijm} R_{\beta\alpha lm}$$

$$\sum h^{\alpha}_{ijlmm} - \sum h^{\alpha}_{ijmlm} = \sum h^{\alpha}_{pjm} R_{pilm} + \sum h^{\alpha}_{pim} R_{pjlm} + \sum h^{\beta}_{ijm} R_{\beta\alpha lm}$$

$$+ \sum h^{\beta}_{ij} h^{\beta}_{klm} h^{\alpha}_{km} - \sum h^{\beta}_{ij} h^{\beta}_{km} h^{\alpha}_{klm},$$

$$\sum h^{\alpha}_{ijmml} = n(\tilde{c} - \lambda^2) h^{\alpha}_{ijl} - 2\sum h^{\alpha}_{klm} h^{\beta}_{km} h^{\beta}_{ij} - 2\sum h^{\alpha}_{km} h^{\beta}_{klm} h^{\beta}_{ij} - 2\sum h^{\alpha}_{km} h^{\beta}_{km} h^{\beta}_{ijl}.$$

Summing up these equations and making use of (2.2)∼(2.11), we have

$$\Delta\|\sigma_3\|^2 = 2\sum h^{\alpha}_{ijkm} h^{\alpha}_{ijkm} + 2\sum h^{\alpha}_{ijk} \Delta h^{\alpha}_{ijk} = \{2(2n+3)\tilde{c} - 3(n-2)\lambda^2\}\|\sigma_3\|^2$$

$$+ 2\|\sigma_4\|^2 - 6\sum h^{\alpha}_{ij} h^{\alpha}_{klm} h^{\beta}_{ij} h^{\beta}_{klm} - 18\sum h^{\alpha}_{ij} h^{\alpha}_{kl} h^{\beta}_{ijm} h^{\beta}_{klm},$$

that is,

(2.14) $$\Delta\|\sigma_3\|^2 = 2\|\sigma_4\|^2 + \{2(2n+3)\tilde{c} - 3(n-2)\lambda^2\}\|\sigma_3\|^2 - 6\sum h^{\alpha}_{ij} h^{\alpha}_{klm} h^{\beta}_{ij} h^{\beta}_{klm}$$

$$- 18\left\{\frac{n+10}{2}\lambda^2 - \tilde{c}\right\} L_N + 18n\lambda^2(\tilde{c} - \lambda^2)S$$

$$+ 9\,\text{Trace}\,A^3 + 36\sum(\text{Trace}\,H^{\alpha}H^{\beta}H^{\gamma})(\text{Trace}\,H^{\alpha}H^{\beta}H^{\gamma}),$$

where $\|\sigma_4\|^2 = \sum h^{\alpha}_{ijkm} h^{\alpha}_{ijkm}$. On the other hand, using (2.9) and (2.12), we have

(2.15) $$\frac{1}{3}\Delta L_N = \frac{3}{4}\sum h^{\alpha}_{ij} h^{\alpha}_{klm} h^{\beta}_{ij} h^{\beta}_{klm} + \sum h^{\alpha}_{ij} h^{\alpha}_{kl} h^{\beta}_{ijm} h^{\beta}_{klm} + n(\tilde{c} - \lambda^2) L_N - 2\,\text{Trace}\,A^3.$$

Now, we consider the function $f = (2/9)\|\sigma_3\|^2 + (1/4)L_N$. Making use of (2.14) and (2.15), we have

(2.16) $$\Delta f = \frac{4}{9}\left(\|\sigma_4\|^2 + \left\{(2n+3)\tilde{c} - \frac{3(n-2)}{2}\lambda^2\right\}\|\sigma_3\|^2\right) + 4n\lambda^2(\tilde{c} - \lambda^2)S$$

$$+ \sum h^{\alpha}_{ij} h^{\alpha}_{kl} h^{\beta}_{ijm} h^{\beta}_{klm} - \left(4\left\{\frac{n+10}{2}\lambda^2 - \tilde{c}\right\} - n(\tilde{c} - \lambda^2)\right) L_N$$

$$+ 8\sum(\text{Trace}\,H^{\alpha}H^{\beta}H^{\gamma})(\text{Trace}\,H^{\alpha}H^{\beta}H^{\gamma}).$$

On the other hand, from (1.2) and (2.3) we have

(2.17) $$n\left(\tilde{c} - \frac{\lambda^2}{2}\right)S - \frac{3}{2}L_N = \sum h^{\alpha}_{ij} h^{\alpha}_{km} R_{kijm} + \sum h^{\alpha}_{ij} h^{\alpha}_{ki} R_{kmjm}.$$

For each α. let $h^{\alpha}_1, h^{\alpha}_2, \cdots, h^{\alpha}_n$ be the eigenvalues of H^{α}. Then we have

$$\sum_{i,j,k,m}(h^{\alpha}_{ij} h^{\alpha}_{km} R_{kijm} + h^{\alpha}_{ij} h^{\alpha}_{ki} R_{kmjm}) = \frac{1}{2}\sum_{i,k}(h^{\alpha}_i - h^{\alpha}_k)^2 R_{ikik}$$

$$\geq \frac{1}{2}\sum_{i,k}(h^{\alpha}_i - h^{\alpha}_k)^2 c = nc\sum_i(h^{\alpha}_i)^2 = nc\,\text{Trace}\,(H^{\alpha})^2,$$

where c is the minimum of the sectional curvature of M and the equality holds only when the sectional curvature is constant. This, together with (2.17), implies

$$n\Big(\tilde{c}-\frac{\lambda^2}{2}\Big)S-\frac{3}{2}L_N \geqq ncS,$$

that is,

(2.18) $L_N \leqq \dfrac{n}{3}\{2(\tilde{c}-c)-\lambda^2\}S,$

where the equality holds only when the sectional curvature is constant.

Pemark. *It follows from our assumption* $n\tilde{c}/3(n+2)\leqq c$ *that*

(2.19) $0\leqq \tilde{c}.$

When $\tilde{M}$ *is simply connected and complete, since* M *is a compact minimal snbmanifold of* $\tilde{M}$, $\tilde{c}$ *must be positive.*

It follows from (1.7) and $nL_N \geqq K_N$ (see [4]) that we have

(2.20) $\dfrac{n\lambda^2}{n-1}S \leqq L_N,$

where the equality holds only when the sectional curvature is constant. From (2.18) and (2.20) we have

(2.21) $0\leqq L_N-\dfrac{n\lambda^2}{n-1}S \leqq \dfrac{2n}{9(n+2)}\Big\{2(n+3)\tilde{c}-\dfrac{3(n+2)^2}{2(n-1)}\lambda^2\Big\}S.$

Since S *is constant on* M, *if* $S=\sum h_{ij}^{\alpha}h_{ij}^{\alpha}=0$, *then* M *is totally geodesic.* Therefore, from now on we may assume that $S\neq 0$. Let us now find the lower bounds for $\sum h_{ij}^{\alpha}h_{kl}^{\alpha}h_{ijm}^{\beta}h_{klm}^{\beta}$, $\sum(\text{Trace }H^{\alpha}H^{\beta}H^{\gamma})(\text{Trace }H^{\alpha}H^{\beta}H^{\gamma})$ and $\|\sigma_4\|^2$ when $S\neq 0$, that is, we will prove the following.

Lemma. *If* $S\neq 0$, *then we have the following inequalities;*

(2.22) $\|\sigma_4\|^2 \geqq \dfrac{3(n+2)}{(n+4)}\Big\{\dfrac{n+1}{n-1}\lambda^2-\tilde{c}\Big\}\|\sigma_3\|^2+\dfrac{6(n+2)}{n(n+4)S}\|\sigma_3\|^2\Big\{L_N-\dfrac{n\lambda^2}{n-1}S\Big\},$

(2.23) $\sum h_{ij}^{\alpha}h_{kl}^{\alpha}h_{ijm}^{\beta}h_{klm}^{\beta} \geqq \dfrac{n\lambda^2}{n-1}\|\sigma_3\|^2+\dfrac{1}{nS}\|\sigma_3\|^2\Big\{L_N-\dfrac{n\lambda^2}{n-1}S\Big\}.$

(2.24) $2\sum(\text{Trace }H^{\alpha}H^{\beta}H^{\gamma})(\text{Trace }H^{\alpha}H^{\beta}H^{\gamma}) \geqq \dfrac{(n-2)(n+4)}{2(n-1)}\lambda^2 L_N$

$$-\dfrac{(n+4)}{2nS}L_N\Big\{L_N-\dfrac{n\lambda^2}{n-1}S\Big\}.$$

Proof of Lemma. Taking the length of the tensors

and

$K_{ijkm}^{\alpha}=h_{ijkm}^{\alpha}-h_{ijmk}^{\alpha}+z(\delta_{im}h_{jk}^{\alpha}+\delta_{jm}h_{ik}^{\alpha}-\delta_{jk}h_{im}^{\alpha}-\delta_{ik}h_{jm}^{\alpha})$

$L_{ijkm}^{\alpha}=h_{ijkm}^{\alpha}-x(\delta_{ij}h_{km}^{\alpha}+\delta_{ik}h_{jm}^{\alpha}+\delta_{jk}h_{im}^{\alpha})+y(\delta_{km}h_{ij}^{\alpha}+\delta_{jm}h_{ik}^{\alpha}+\delta_{im}h_{jk}^{\alpha}),$

where $z=\|\sigma_3\|^2/nS$, $x=2y/(n+2)$ and $y=(n+2)\|\sigma_3\|^2/n(n+4)S$, we have (2.22) and (2.23), because we have $\sum(h^\alpha_{ijkm}-h^\alpha_{ijmk})(h^\alpha_{ijkm}-h^\alpha_{ijmk})=8\sum h^\alpha_{ij}h^\alpha_{kl}h^\beta_{ijm}h^\beta_{klm}-4(\lambda+\tilde{c})\|\sigma_3\|^2$ from (1.2), (1.3), (1.5), (2.3) and (2.11). Next, taking the length of the tensor

$$P^\alpha_{ijkm}=\sum h^\beta_{ij}h^\beta_{kr}h^\alpha_{rm}+\sum h^\beta_{ir}h^\beta_{km}h^\alpha_{rj}-a(\delta_{jk}h^\alpha_{im}+2\delta_{ik}h^\alpha_{jm}+\delta_{im}h^\alpha_{jk})$$
$$+b(\delta_{ij}h^\alpha_{km}+\delta_{km}h^\alpha_{ij}).$$

we have

(2.25) $\quad 2\sum(\text{Trace } H^\alpha H^\beta H^r)(\text{Trace } H^\alpha H^\beta H^r)\geqq -2(3n+4)a^2S+2a\{(3n+8)\lambda^2S-L_N\}$
$$+16abS-2b^2nS-4bL_N-(n+2)\lambda^2L_N,$$

because $\sum h^\alpha_{im}h^\alpha_{mj}=\dfrac{(n+2)\lambda^2}{2}\delta_{ij}$ and $\sum h^\alpha_{ij}h^\alpha_{kl}h^\beta_{il}h^\beta_{kj}=\sum(\sum h^\alpha_{ij}h^\alpha_{kl})\{\sum h^\beta_{ij}h^\beta_{kl}+\tilde{c}(\delta_{ij}\delta_{kl}$ $-\delta_{il}\delta_{kj})+R_{iklj}\}=\lambda^2S-(1/2)L_N$ from (2.2). For some positive constant x and any positive constant ε such that

(2.26) $\qquad \dfrac{2x+1}{2(x+1)}\varepsilon+L_N\leqq 2aS<nbS\leqq L_N+\varepsilon,$

we set

$$2(3n+4)aS=(3n+8)\lambda^2S-L_N+4b^2nSx/a+8(1-x)bS.$$

Then, from (2.25) we have

(2.27) $\qquad 2\sum(\text{Trace } H^\alpha H^\beta H^r)(\text{Trace } H^\alpha H^\beta H^r)\geqq(3n+8)\lambda^2Sa-aL_N-4bL_N$
$$-(n+2)\lambda^2L_N+2\{-(2x+1)nbS+4(x+1)aS\}b.$$

Since $L_N=2(x+1)L_N+(2x+1)\varepsilon-(2x+1)(L_N+\varepsilon)\leqq-(2x+1)nbS+4(x+1)aS$ by (2.26), using (2.26), (2.27) implies

(2.28) $\quad 2\sum(\text{Trace } H^\alpha H^\beta H^r)(\text{Trace } H^\alpha H^\beta H^r)>\dfrac{(n-2)(n+4)}{2(n-1)}\lambda^2L_N$

$$-\dfrac{(n+4)}{2nS}L_N\left\{L_N-\dfrac{n\lambda^2}{n-1}S\right\}+\left(\dfrac{(3n+8)(2x+1)}{4(x+1)}\lambda^2-\dfrac{n+4}{2nS}L_N\right)\varepsilon.$$

Since ε is any positive constant, we have (2.24).

Now, we will prove Theorem A. Making use of (2.22), (2.23) and (2.24), from (2.16) we have

(2.29) $\qquad \Delta f\geqq\left(\dfrac{4(n+1)}{9(n+4)}\left\{2(n+3)\tilde{c}-\dfrac{3(n+2)^2}{2(n-1)}\lambda^2\right\}\right.$

$$\left.-\dfrac{3n^2+13n+20}{3n(n+4)S}\left\{L_N-\dfrac{n\lambda^2}{n-1}S\right\}\right)\|\sigma_3\|^2,$$

which, together with (2.21), implies

(2.30) $$\Delta f \geqq \frac{2(3n^2+5n-8)}{27(n+4)(n+2)}\left\{2(n+3)\tilde{c}-\frac{3(n+2)^2}{2(n-1)}\lambda^2\right\}\|\sigma_3\|^2\geqq 0,$$

Therefore, if M is compact and orientable, we have $\int_M \Delta f d\,\mathrm{Vol}\,M=0$, and so we get $\Delta f=0$ on M, that is,

f is constant on M and $\left\{2(n+3)\tilde{c}-\dfrac{3(n+2)^2}{2(n-1)}\lambda^2\right\}\|\sigma_3\|=0$,

which implies that if M is not totally geodesic, then

$$\|\sigma_3\|=0 \quad\text{or}\quad 2(n+3)\tilde{c}=\frac{3(n+2)^2}{2(n-1)}\lambda^2.$$

If $S\neq 0$ and $\|\sigma_3\|\neq 0$, then it follows from (2.20) and (2.21) that M is of constant curvature $\dfrac{n\tilde{c}}{3(n+2)}$. In this case, as stated in [2] or [6], the immersion may be considered as a standard minimal one of degree 3 from a sphere into a sphere. If $S\neq 0$ and $\|\sigma_3\|=0$, then the second fundamental form is parallel, and so the immersion is the planar geodesic one which is determined by K. Sakamoto in [8]. Thus we have proved Theorem A.

Next, we will prove Theorem B. If the sectional curvature K_σ of M satisfies the inequality $\dfrac{n\tilde{c}}{3(n+2)}\leqq K_\sigma\leqq\tilde{c}$, then, by Theorem A, we see that M is of constant curvature $\tilde{c}$ or $n\tilde{c}/3(n+2)$, or the second fundamental form of M is parallel. Looking over the curvatures of planar geodesic submanifolds in [8], we easily see that M must be of constant curvature $\tilde{c}$ or $n\tilde{c}/2(n+1)$ when $\|\sigma_3\|$ $=0$. By the results in [5] and [6], according as K_σ is $\tilde{c}$, $n\tilde{c}/2(n+1)$, or $n\tilde{c}/$ $3(n+2)$, M is a great sphere of $S(\tilde{c})$, a Veronese submanifold, or a generalized Veronese submanifold.

References

[1] Chern, S.S., doCarmo, M. and Kobayashi, S.,   Minimal submanifolds of a sphere with second fundamental form of constant length, Functional Analysis and Related Fields, Springer, 1970, 59-75.

[2] doCarmo, M.P. and Wallach, N.,   Minimal immersions of spheres into spheres, Ann. Math., **93** (1971), 43-62.

[3] Itoh, T. and Ogiue, K.,   Isotropic immersions and Veronese manifolds, Trans. Amer. Math. Soc., **209** (1975), 109-117.

[4] Itoh, T.,   On Veronese manifolds, J. Math. Soc. Japan, **27** (1975), 497-506.

[5] ——————,   Addendum to my paper "On Veronese manifolds", J. Math. Soc. Japan, **30** (1978), 71-72.

[6] Nakagawa, H.,   On a certain minimal immersion of a Riemannian manifold into a sphere, Kodai Math. J., **3** (1980), 321-340.

[7] O'Neill, B.,   Isotropic and Kaehler immersions, Canad. J. Math., **17** (1965), 909-915.

[8] Sakamoto, K., Planar geodesic immersions, Tohoku Math. J., **29** (1977),

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