

**ON GEOMETRIC AND STOCHASTIC MEAN VALUES
FOR SMALL GEODESIC SPHERES
IN RIEMANNIAN MANIFOLDS**

By

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1. Introduction.

The characterization of the harmonic, Einstein and super-Einstein spaces by means of the first or the second mean values (or the relations between them) for small geodesic spheres in a Riemannian manifold is recently studied by many authors ([3], [7], [10], [17], etc.). Among them, O. Kowalski [10] characterized the three spaces by the degree of concordance of the two mean values in some sense, proposing new classes of spaces which should be located between the harmonic and the super-Einstein spaces. On the other hand M. Pinsky [12] verified that the stochastic mean values are also useful for the characterization of the Einstein spaces.

In this paper, we study the above three mean values more in detail and fill the blanks in the previous works (Theorem 2 below). The main tool for our proof is Schauder's estimate, which enables us to treat C^∞ manifolds even more easily than Cauchy-Kowalewski's method for C^ω manifolds used in most of the previous papers. We also introduce some other new conditions which also characterize the above three spaces, that is, the conditions (M2), (M4) and (L2)–(L4) (see section 2 for the definitions). The condition (M2) is a variation of (M3) given in [3]. But it seems to be more natural than (M3) in connection with (M1). The conditions (M4) and (L4) are closely related to Helgason's expansion in [8: p. 435]. Indeed, the above three mean values coincide with Helgason's, if the manifold is an Euclidean space or a globally symmetric space of rank one, and we are interested in to what extent the Laplacian determines the mean values. Our results include the assertion; one of the operators induced by the three mean values is expanded by means of a sequence of polynomials of Laplacian if and only if the manifold is harmonic (Theorem 2 (1)). As a by-product, we also obtain some sufficient conditions for a C^∞ manifold to be analytic (Theorem 1). In the course of our proof, we partially answer for smooth manifolds to Kowalski's conjecture given in [10] (Theorem 3).

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In section 2, we state our results and list the related previous works in detail. Our main results are stated in Theorem 2. In section 3, we give the proof of Theorems 1 and 3. Section 4 is devoted to the proof of Parts (2) and (3) of Theorem 2. The proof of Part (1) of Theorem 2 will be given in section 5. The most pages there are exhausted for the proof of the necessity of the conditions (M4) and (L4) for the harmonicity of the manifold, where we will make use of Cauchy-Kowalewski's method.

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2. Statement of results.

Let (M, g) be an n -dimensional connected C^∞ Riemannian manifold with $n \geq 3$ and $B_m(\varepsilon)$ be the geodesic ball in M at center $m \in M$ with small radius $\varepsilon > 0$. The *first mean value* $M_m(\varepsilon, f)$ for a real valued continuous function f is defined by

$$M_m(\varepsilon, f) = (\text{vol}(\partial B_m(\varepsilon)))^{-1} \int_{\partial B_m(\varepsilon)} f(\omega) d\sigma(\omega),$$

where $d\sigma$ stands for the volume element on the geodesic sphere $\partial B_m(\varepsilon)$. Similarly, the *second mean value* $L_m(\varepsilon, f)$ for an f is defined by

$$L_m(\varepsilon, f) = (\text{vol}(S^{n-1}(1)))^{-1} \int_{S^{n-1}(1)} (f \circ \exp_m(\varepsilon u)) du,$$

where \exp_m is the exponential map at $m \in M$ and du is the usual volume element on the $(n-1)$ -dimensional unit sphere $S^{n-1}(1)$.

In his paper [10], O. Kowalski conjectured the next

KOWALSKI'S CONJECTURE. *For an analytic manifold (M, g) , the following conditions are mutually equivalent:*

(i) *for each $m \in M$, the mean value formula*

$$M_m(\varepsilon, f) = f(m) + O(\varepsilon^{2k+2}) \quad (\varepsilon \rightarrow 0)$$

holds for all harmonic functions f near m ;

(ii) *for each $m \in M$, the mean value formula*

$$L_m(\varepsilon, f) = f(m) + O(\varepsilon^{2k+2}) \quad (\varepsilon \rightarrow 0)$$

holds for all harmonic functions f near m ;

(iii) *for each $m \in M$, the estimate*

$$M_m(\varepsilon, f) = L_m(\varepsilon, f) + O(\varepsilon^{2k+2}) \quad (\varepsilon \rightarrow 0)$$

holds for all harmonic functions f near m ;

(iv) for each $m \in M$, the estimate

$$M_m(\varepsilon, f) = L_m(\varepsilon, f) + O(\varepsilon^{2k+2}) \quad (\varepsilon \rightarrow 0)$$

holds for all functions f of class C^{2k+2} near m .

In the above, k is a natural number or ∞ and, in the case of $k = \infty$, the formulae are understood to hold without remainder terms.

Let $X = (X(t), P_m)$ ($m \in M$) be a Brownian motion on (M, g) , i. e. the diffusion process on (M, g) whose infinitesimal operator is the Laplacian Δ on (M, g) (see [9] or [14] for the precise definitions). Let also T_ε be the first exit time from the geodesic ball $B_m(\varepsilon)$, i. e. $T_\varepsilon = \inf\{t > 0 : X(t) \notin B_m(\varepsilon)\}$. The *stochastic mean value* for an f and the *mean exit time* from $B_m(\varepsilon)$ are defined by $E_m f(X(T_\varepsilon))$ and $E_m T_\varepsilon$ respectively, where E_m denotes the expectation with respect to the probability measure P_m .

Also we set $r(p) = d(m, p)$ the distance between m and p , $A_m(\varepsilon) = \text{vol}(\partial B_m(\varepsilon))$ the volume of the geodesic sphere $\partial B_m(\varepsilon)$ and

$$\Phi_m(\varepsilon) = \int_0^\varepsilon A_m^{-1}(s) ds \int_0^s A_m(t) dt.$$

Finally a function f is called *bi-harmonic* near m if it is defined and smooth in a neighbourhood of m and Δf is harmonic there.

In this paper we are also concerned with the following conditions:

(M1) for each $m \in M$, the estimate

$$M_m(\varepsilon, f) = E_m f(X(T_\varepsilon)) + O(\varepsilon^{2k+2}) \quad (\varepsilon \rightarrow 0)$$

holds for all functions f of class C^{2k+2} near m ;

(M2) for each $m \in M$, the mean value formula

$$M_m(\varepsilon, f) = f(m) + (E_m T_\varepsilon) \Delta f(m) + O(\varepsilon^{2k+2}) \quad (\varepsilon \rightarrow 0)$$

holds for all bi-harmonic functions f near m ;

(M3) for each $m \in M$, the mean value formula

$$M_m(\varepsilon, f) = f(m) + \Phi_m(\varepsilon) \Delta f(m) + O(\varepsilon^{2k+2}) \quad (\varepsilon \rightarrow 0)$$

holds for all bi-harmonic functions f near m ;

(M4) there exists a sequence of polynomials p_j , $j = 1, 2, \dots, k$ without constant terms such that, for each $m \in M$, the expansion

$$M_m(\varepsilon, f) = f(m) + \sum_{j=1}^k p_j(\Delta) f(m) \varepsilon^{2j} + O(\varepsilon^{2k+2}) \quad (\varepsilon \rightarrow 0)$$

holds for all functions f of class C^{2k+2} near m .

The conditions (L1)-(L4) are defined in the same way as (M1)-(M4) are done respectively with the first mean value $M_m(\epsilon, f)$ replaced by the second one $L_m(\epsilon, f)$.

In the case of $k=\infty$, the conditions (M4) and (L4) are understood to hold for all analytic functions f at m .

First we note the following

THEOREM 1. *Let (M, g) be an n -dimensional connected C^6 Riemannian manifold with $n \geq 3$. Suppose further that one of the conditions (i)-(iv), (M1)-(M4) and (L1)-(L4) holds with $k=2$. Then (M, g) is an Einstein space. Especially it is an analytic space.*

We note that the conclusions of Theorem 1 are also valid for $C^{4,\alpha}$ manifolds and for the conditions (i)-(iv), (M1)-(M4), (L1)-(L4) with the remainder terms replaced by $O(\epsilon^{4+\alpha})$ for some $0 < \alpha \leq 1$ (and in the conditions (iv), (M1) and (L1) C^6 class replaced by $C^{4,\alpha}$ class) by almost the same proof as in the sequel.

For an $m \in M$, let $(U; x^1, x^2, \dots, x^n)$ be a normal coordinate system around m , and denote by (g_{ij}) and (R_{ijkl}) the metric tensor and the curvature tensor with respect to the normal frame $(\partial/\partial x^1, \partial/\partial x^2, \dots, \partial/\partial x^n)$. The Ricci tensor and the scalar curvature are denoted by (ρ_{ij}) and τ respectively; $\rho_{ij} = R^u{}_{iuj}$, $\tau = \rho^u{}_u$. We also denote the length of a tensor $T = (T_{i_1 i_2 \dots i_p})$ by $|T|$, i. e. $|T|^2 = T_{i_1 i_2 \dots i_p} T^{i_1 i_2 \dots i_p}$. Some other symbols such as $\langle \nabla f, \nabla \tau \rangle$, $\langle \nabla^2 f, \rho \rangle$, etc. are the same as in [7].

We call an Einstein space *super-Einstein* if $|R|$ is constant and $R_{ipqr} R_j{}^{pqr} = |R|^2 g_{ij}/n$. Similarly, we call the space (M, g) *harmonic* if, for each $m \in M$, there exist an $\epsilon > 0$ and a function $F: (0, \epsilon) \rightarrow R$ such that the function $f(n) = F(d(m, n))$ is harmonic in $B_m(\epsilon) \setminus \{m\}$.

The main objective of this paper is the following

THEOREM 2. *Let (M, g) be an n -dimensional connected C^ω Riemannian manifold with $n \geq 3$. Then the following assertions hold.*

(1) *Each of the conditions (i)-(iv), (M1)-(M4) and (L1)-(L4) with $k=\infty$ is necessary and sufficient for that (M, g) be a harmonic space.*

(2) *Each of the conditions (i)-(iv), (M1)-(M4) and (L1)-(L4) with $k=2$ is necessary and sufficient for that (M, g) be an Einstein space.*

(3) *Each of the conditions (i)-(iv), (M1)-(M4) and (L1)-(L4) with $k=3$ is necessary and sufficient for that (M, g) be a super-Einstein space.*

Combining Theorems 1, 2 and [2: Theorem 5.1], we can easily prove the following

COROLLARY 1. *The assertions (1)–(3) in Theorem 2 are also valid for an n -dimensional connected C^∞ Riemannian manifold with $n \geq 3$, except for the sufficiency of (M4) and (L4) in the assertion (1).*

For the proof of Theorems 1 and 2, we need the next theorem.

THEOREM 3. *Let (M, g) be an n -dimensional connected C^∞ Riemannian manifold with $n \geq 3$ and fix a $k \in \{1, 2, \dots, \infty\}$. Then the following assertions hold.*

- (1) *The condition (i) is necessary and sufficient for (M1).*
- (2) *The condition (ii) is necessary and sufficient for (L1).*
- (3) *The condition (iii) is necessary and sufficient for (iv).*

As is noted in sect. 1, several parts of the above Theorems are not new. We list the related previous works as far as we know. (1) The equivalence of (i) with $k = \infty$ and the harmonicity of the manifold was obtained by T. J. Willmore [17]. (2) The equivalence of (M3) with $k = \infty$ and the harmonicity of the manifold was obtained by A. Friedman [3]. He also proved the equivalence of (M3) with $k = 2$ and that (M, g) is an Einstein space. (3) The equivalence of (i) with $k = 2$ and that (M, g) is Einsteinian was obtained by A. Gray and T. J. Willmore [7] for C^ω manifolds. They also proved the equivalence of (i) with $k = 3$ and that (M, g) is super-Einsteinian. (4) All the assertions in Theorem 2 concerning the conditions (ii)–(iv) were proved by O. Kowalski [10] for C^ω manifolds. (5) O. Kowalski [11] proved the assertion (3) of Theorem 3 for C^ω manifolds. (6) M. Pinsky [12] showed that each of the conditions (M1) and (L1) with $k = 2$ is equivalent to that (M, g) is an Einstein space for C^ω manifolds. (7) We were inspired the condition (L4) by T. Sunada (private communication), which is easily verified to hold for an Euclidean space and a globally symmetric space of rank one from [8: Chap. X, Proposition 2.10] and [16: Lemma 5.4].

3. Proof of Theorems 1 and 3.

We will prove Theorem 3 first.

PROOF OF THEOREM 3. First we will show the assertion (3). Since the necessity is obvious, we will only prove the sufficiency of the condition (iii). Fix a natural number k and an $m \in M$, and let $(U; x^1, x^2, \dots, x^n)$ be the normal

coordinate system around m . For a function h of class C^k in U and an open set V in U , we set

$$|h|_{C^k(V)} = \sum_{j=0}^k \sum_{i_1, i_2, \dots, i_j} \sup_{p \in V} |\partial_{i_1} \partial_{i_2} \cdots \partial_{i_j} h(p)|$$

where $\partial_i = \partial / \partial x^i$. It is well known that there exist two sequences $\{M_{m,j}\}_{j=1,2,\dots,k}$ and $\{L_{m,j}\}_{j=1,2,\dots,k}$ of linear differential operators satisfying the following condition; for each $\varepsilon_1 > 0$ with $\bar{B}_m(\varepsilon_1) \subset U$, one can choose a positive constant K_1 such that

$$(3.1) \quad \begin{aligned} M_m(\varepsilon, h) &= h(m) + \sum_{j=1}^k M_{m,j} h(m) \varepsilon^{2j} + P_{m,k}(h) \varepsilon^{2k+2}, \\ \text{where } |P_{m,k}(h)| &\leq K_1 |h|_{C^{2k+2}(B_m(\varepsilon))}, \\ L_m(\varepsilon, h) &= h(m) + \sum_{j=1}^k L_{m,j} h(m) \varepsilon^{2j} + Q_{m,k}(h) \varepsilon^{2k+2}, \\ \text{where } |Q_{m,k}(h)| &\leq K_1 |h|_{C^{2k+2}(B_m(\varepsilon))}, \end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_1)$ and all functions h in $C^{2k+2}(B_m(\varepsilon_1))$ (see [7] and [11]).

Now assume that the condition (iii) holds and choose a function f of class C^{2k+2} near m . We may assume that f is defined in U and belongs to $C^{2k+2}(U)$. It then follows that

$$(3.2) \quad K_2 \equiv |f|_{C^{2k+2}(U)} < +\infty.$$

Further, for each $\varepsilon \in (0, \varepsilon_1)$, the function u_ε defined by

$$u_\varepsilon(p) = E_p f(X(T_\varepsilon)), \quad p \in B_m(\varepsilon),$$

is harmonic in $B_m(\varepsilon)$, continuous in $\bar{B}_m(\varepsilon)$ and satisfies

$$u_\varepsilon(\xi) = f(\xi), \quad \xi \in \partial B_m(\varepsilon).$$

Hence we have from the condition (iii) and the relations in (3.1) that

$$|M_m(r, u_\varepsilon) - L_m(r, u_\varepsilon)| \leq 2K_1 r^{2k+2} |u_\varepsilon|_{C^{2k+2}(B_m(\varepsilon))}$$

for all $r \in (0, \varepsilon)$ first and then, by letting $r \uparrow \varepsilon$, that

$$(3.3) \quad |M_m(\varepsilon, f) - L_m(\varepsilon, f)| \leq 2K_1 \varepsilon^{2k+2} |u_\varepsilon|_{C^{2k+2}(B_m(\varepsilon))}.$$

But due to Schauder's estimate (see [4: (6.80) and Problem 6.2], e. g.), it follows that

$$(3.4) \quad |u_\varepsilon|_{C^{2k+2}(B_m(\varepsilon))} \leq K_3 |f|_{C^{2k+2}(B_m(\varepsilon))}$$

for some positive constant K_3 . Now the formulae (3.2)-(3.4) imply

$$|M_m(\varepsilon, f) - L_m(\varepsilon, f)| \leq 2K_1 K_2 K_3 \varepsilon^{2k+2}, \quad \varepsilon \in (0, \varepsilon_1),$$

which proves the assertion (iv) for this case. In the case of $k = \infty$, we have

$$M_m(r, u_\varepsilon) = L_m(r, u_\varepsilon), \quad r \in (0, \varepsilon),$$

first and then $M_m(\varepsilon, f) = L_m(\varepsilon, f)$ in place of (3.3), verifying (iv). The assertion (3) is proved.

The proof of the assertions (1) and (2) is almost the same. Indeed, it holds by Dynkin's formula that $u(m) = E_m u(X(T_\varepsilon))$ for all harmonic functions u in U . Hence the necessity follows. Further, assuming (i) (resp. (ii)), we have

$$\begin{aligned} |M_m(\varepsilon, f) - E_m f(X(T_\varepsilon))| &\leq K_4 \varepsilon^{2k+2} |u_\varepsilon|_{C^{2k+2}(B_m(\varepsilon))} \\ (\text{resp. } |L_m(\varepsilon, f) - E_m f(X(T_\varepsilon))| &\leq K_4 \varepsilon^{2k+2} |u_\varepsilon|_{C^{2k+2}(B_m(\varepsilon))}) \end{aligned}$$

in place of (3.3), and we obtain the conclusion by the same arguments.

PROOF OF THEOREM 1. We first note the following asymptotic formulae given in [7], [10] and [12] respectively (their formulae are stated for C^ω manifolds, but their proofs are also valid for C^6 manifolds without any change);

$$\begin{aligned} (3.5) \quad M_m(\varepsilon, f) &= f(m) + \frac{\varepsilon^2}{2n} \Delta f(m) \\ &+ \frac{\varepsilon^4}{4! n(n+2)} \left(3\Delta^2 f - 2\langle \nabla^2 f, \rho \rangle - 3\langle \nabla f, \nabla \tau \rangle + \frac{4}{n} \tau \Delta f \right) (m) + O(\varepsilon^6). \end{aligned}$$

$$\begin{aligned} (3.6) \quad L_m(\varepsilon, f) &= f(m) + \frac{\varepsilon^2}{2n} \Delta f(m) \\ &+ \frac{\varepsilon^4}{4! n(n+2)} \left(3\Delta^2 f + 2\langle \nabla^2 f, \rho \rangle + \langle \nabla f, \nabla \tau \rangle \right) (m) + O(\varepsilon^6). \end{aligned}$$

$$(3.7) \quad E_m f(X(T_\varepsilon)) = f(m) + \frac{\varepsilon^2}{2n} \Delta f(m) + \frac{\varepsilon^4}{4! n(n+2)} \left(3\Delta^2 f + \frac{2}{n} \tau \Delta f \right) (m) + O(\varepsilon^6).$$

Now each of the conditions (M2)-(M4) (resp. (L2)-(L4)) trivially implies (i) (resp. (ii)). Hence due to Theorem 3 it is enough to show that each of the conditions (M1), (L1) and (iv) with $k=2$ deduces the conclusions of Theorem 1. But, by (3.5)-(3.7), each of them implies

$$(3.8) \quad \left(\langle \nabla^2 f, \rho \rangle + a \langle \nabla f, \nabla \tau \rangle - \frac{1}{n} \tau \Delta f \right) (m) = 0$$

for some positive constant a . Now let (x^1, x^2, \dots, x^n) be a harmonic coordinate around m . Then, by [2: Theorem 2.1], the coordinate functions $x^i, i=1, 2, \dots, n$ are of class C^6 . Hence, substituting $f=x^i$ in (3.8), we have $\nabla_i \tau(m)=0, i=1, 2, \dots, n$. This implies that the scalar curvature τ is constant. Substitute next $f=x^i x^j$ in (3.8). Then we have

$$\left(\sum_{p,q=1}^n (\partial_p \partial_q x^i x^j) \rho_{pq}\right)(m) = \left(\frac{\tau}{n} \sum_{p=1}^n \partial_p^2 x^i x^j\right)(m), \quad i, j=1, 2, \dots, n.$$

This implies $\rho_{ij}(m) = \tau(m) \delta_{ij}/n$, whence (M, g) is an Einstein space. The analyticity of (M, g) is now a direct consequence of [2: Theorem 5.2].

4. Proof of Parts (2) and (3) of Theorem 2.

First we prepare the next theorem.

THEOREM 4.1. *Let (M, g) be the same as in Theorem 2. Then it follows that*

$$(4.1) \quad E_m T_\varepsilon = \Phi_m(\varepsilon) + O(\varepsilon^6) \quad (\varepsilon \rightarrow 0)$$

for all $m \in M$. Further it holds that

$$(4.2) \quad E_m T_\varepsilon = \Phi_m(\varepsilon) + O(\varepsilon^8) \quad (\varepsilon \rightarrow 0)$$

for all $m \in M$ if and only if (M, g) is an Einstein space.

PROOF. Due to [6], the volume $A_m(\varepsilon)$ of the geodesic sphere $\partial B_m(\varepsilon)$ satisfies

$$A_m(\varepsilon) = \frac{\pi^{n/2} \varepsilon^{n-1}}{\Gamma(n/2+1)} (n + (n+2)A\varepsilon^2 + (n+4)B\varepsilon^4 + O(\varepsilon^6))$$

as $\varepsilon \rightarrow 0$, where

$$A = -\tau(m)/6(n+2),$$

$$B = \frac{1}{3 \cdot 5! (n+2)(n+4)} (-18\Delta\tau + 5\tau^2 + 8|\rho|^2 - 3|R|^2)(m).$$

Hence we have

$$(4.3) \quad \begin{aligned} \Phi_m(\varepsilon) &= \frac{1}{2n} \varepsilon^2 - \frac{A}{2n^2} \varepsilon^4 + \frac{1}{6n} \left(\frac{2(n+2)}{n^2} A^2 - \frac{4}{n} B \right) \varepsilon^6 + O(\varepsilon^8) \\ &= \frac{1}{2n} \varepsilon^2 + \frac{2\tau(m)}{4! n^2 (n+2)} \varepsilon^4 \\ &\quad + \frac{4}{6! n^2 (n+2)(n+4)} \left(6\Delta\tau + \frac{20}{3n} \tau^2 - \frac{8}{3} |\rho|^2 + |R|^2 \right) (m) \varepsilon^6 + O(\varepsilon^8). \end{aligned}$$

On the other hand, it is shown in [5] that

$$(4.4) \quad \begin{aligned} E_m T_\varepsilon &= \frac{1}{2n} \varepsilon^2 + \frac{2\tau(m)}{4! n^2 (n+2)} \varepsilon^4 \\ &\quad + \frac{4}{6! n^2 (n+2)(n+4)} \left(6\Delta\tau + \frac{5}{n} \tau^2 - |\rho|^2 + |R|^2 \right) (m) \varepsilon^6 + O(\varepsilon^8). \end{aligned}$$

Comparing (4.3) and (4.4), we obtain (4.1) in general, and (4.2) if (M, g) is an Einstein space.

Suppose next that (4.2) holds. Then comparing the coefficients of ε^6 in (4.3) and (4.4), we have

$$(4.5) \quad \left(6\Delta\tau + \frac{5}{n}\tau^2 - |\rho|^2 + |R|^2\right)(m) = \left(6\Delta\tau + \frac{20}{3n}\tau^2 - \frac{8}{3}|\rho|^2 + |R|^2\right)(m).$$

Hence it follows that $|\rho|^2(m) = \tau^2(m)/n$ first, and then $\rho_{ij} = \tau g_{ij}/n$. Further, in view of $n \geq 3$, we see that τ is constant. Thus (M, g) is an Einstein space.

REMARK 4.1. According to our computation, if (M, g) is an Einstein space, then it even follows that

$$(4.6) \quad E_m T_\varepsilon = \Phi_m(\varepsilon) + O(\varepsilon^{10}) \quad (\varepsilon \rightarrow 0).$$

But we do not show it, since it will not be used in the following proof of Theorem 2.

PROOF OF THEOREM 2 (2). Since the sufficiency of each of the conditions for that (M, g) be an Einstein space is shown in the proof of Theorem 1, we have only to show the necessity. In the following proof, we assume that (M, g) is an Einstein space and that $k=2$.

The conditions (i)-(iv), (M1) and (L1) are already verified in [10] and [12]. These with (3.7) imply

$$(4.7) \quad \begin{aligned} M_m(\varepsilon, f) &= L_m(\varepsilon, f) + O(\varepsilon^6) = E_m f(X(T_\varepsilon)) + O(\varepsilon^6) \\ &= f(m) + \frac{\varepsilon^2}{2n} \Delta f(m) + \frac{\varepsilon^4}{4! n(n+2)} \left(3\Delta^2 f + \frac{2}{n} \tau \Delta f\right)(m) + O(\varepsilon^6). \end{aligned}$$

Hence we have (M4) and (L4) and, by (4.4), (M2) and (L2). The conditions (M3) and (L3) now follow from (M2), (L2) and Theorem 4.1.

PROOF OF THEOREM 2 (3). In the following proof we assume that $k=3$.

SUFFICIENCY. The sufficiency of each of the conditions (i)-(iv) for that (M, g) be a super-Einstein space is proved in [7] and [10]. Since each of the conditions (M2)-(M4) (resp. (L2)-(L4)) implies (i) (resp. (ii)), the sufficiency of each of them is clear. Finally the sufficiency of (M1) and (L1) follows from Theorem 3 and the above facts.

NECESSITY. Suppose that (M, g) is a super-Einstein space. The conditions (i)-(iv) are shown by [7] and [10]. This with Theorem 3 implies (M1) and (L1). Hence it follows from [7: Theorem 4.5] that

$$\begin{aligned}
(4.8) \quad M_m(\varepsilon, f) &= L_m(\varepsilon, f) + O(\varepsilon^8) = E_m f(X(T_\varepsilon)) + O(\varepsilon^8) \\
&= f(m) + \frac{\varepsilon^2}{2n} \Delta f(m) + \frac{\varepsilon^4}{4! n(n+2)} \left(3\Delta^2 f + \frac{2}{n} \tau \Delta f \right)(m) \\
&\quad + \frac{\varepsilon^6}{6! n(n+2)(n+4)} \left\{ 15\Delta^3 f + \frac{30\tau}{n} \Delta^2 f + \left(\frac{16\tau^2}{n^2} + \frac{4}{n} |R|^2 \right) \Delta f \right\}(m) + O(\varepsilon^8).
\end{aligned}$$

Now we can prove all the rest in the same way as in the proof of the assertion (2).

5. Proof of Part (1) of Theorem 2.

For the proof of Theorem 2 (1), we prepare the following

THEOREM 5.1. *The manifold (M, g) is harmonic if and only if, for each $m \in M$,*

$$(5.1) \quad E_p T_\varepsilon = \Phi_m(\varepsilon) - \Phi_m(r(p)), \quad p \in B_m(\varepsilon)$$

for all sufficiently small $\varepsilon > 0$.

PROOF. Suppose that (M, g) is a harmonic space. It then follows from [1: p. 160] that $\Delta r = A'_m(r)/A_m(r)$. Hence, by [1: (6.22)], the function $u(p) = \Phi_m(\varepsilon) - \Phi_m(r(p))$ in $\bar{B}_m(\varepsilon)$ satisfies

$$\Delta u = -(\Phi_m'' + A'_m(r)\Phi_m'(r)/A_m(r)) = -1, \quad p \in B_m(\varepsilon)$$

and $u(\xi) = 0$, $\xi \in \partial B_m(\varepsilon)$. Hence, by Dynkin's formula, we have

$$u(X(T_\varepsilon)) - u(p) = E_p \int_0^{T_\varepsilon} \Delta u(X(s)) ds = -E_p T_\varepsilon.$$

Since $u(X(T_\varepsilon)) = 0$, (5.1) follows.

Conversely, if (5.1) holds, then by the well known formula $\Delta(E_p T_\varepsilon) = -1$, we have

$$\Phi_m''(r) + \Delta r \Phi_m'(r) = 1, \quad 0 < r < \varepsilon.$$

From the definition of $\Phi_m(r)$, this implies $\Delta r = A'_m(r)/A_m(r)$. Hence, due to [1: p. 160] again, the space (M, g) is harmonic.

We call a function f λ -harmonic near m , if it is defined and smooth in a neighbourhood of m and satisfies $\Delta f = \lambda f$ there.

LEMMA 5.1. *Let (M, g) be an n -dimensional C^ω manifold and \mathcal{D}_m be a linear differential operator at m . If $\mathcal{D}_m f(m) = 0$ for all λ -harmonic functions f near m and all real λ , then $\mathcal{D}_m = 0$.*

PROOF. We will make use of Cauchy-Kowalewski's method. In the following proof, $|x|$ and $|\lambda|$ are assumed to be sufficiently small so that all the power series in the sequel converge absolutely (it is easy to check that this can be done).

Let Z_+ be the set of all nonnegative integers and (x^1, x^2, \dots, x^n) be a normal coordinate around m . For each $\alpha = (\alpha^1, \alpha^2, \dots, \alpha^n) \in Z_+^n$ and $x = (x^1, x^2, \dots, x^n) \in R^n$, we denote $|\alpha| = \alpha^1 + \alpha^2 + \dots + \alpha^n$ and

$$\partial^\alpha = (\partial_1)^{\alpha^1} (\partial_2)^{\alpha^2} \dots (\partial_n)^{\alpha^n}, \quad x^\alpha = (x^1)^{\alpha^1} (x^2)^{\alpha^2} \dots (x^n)^{\alpha^n}.$$

Sometimes we denote $\tilde{\alpha} = (\alpha^1, \alpha^2, \dots, \alpha^{n-1})$ and $\tilde{x} = (x^1, x^2, \dots, x^{n-1})$. Thus we have $\alpha = (\tilde{\alpha}, \alpha^n)$, $x = (\tilde{x}, x^n)$, $|\tilde{\alpha}| = \alpha^1 + \alpha^2 + \dots + \alpha^{n-1}$ and

$$\partial^{\tilde{\alpha}} = (\partial_1)^{\alpha^1} (\partial_2)^{\alpha^2} \dots (\partial_{n-1})^{\alpha^{n-1}}, \quad \tilde{x}^{\tilde{\alpha}} = (x^1)^{\alpha^1} (x^2)^{\alpha^2} \dots (x^{n-1})^{\alpha^{n-1}}.$$

Now the operator \mathcal{D}_m is represented as

$$\mathcal{D}_m f(m) = \sum_{|\alpha| \leq 2l+1} a_\alpha \partial^\alpha f(0), \quad \text{for smooth } f,$$

with some $l \in Z_+$ and constants a_α , $|\alpha| \leq 2l+1$ (by adding dummy terms if necessary). Further, denoting $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$, we have

$$\Delta = g^{ij} \partial_i \partial_j + g^j \partial_j,$$

where $g^j = \partial_i g^{ij} + (\partial_i g) g^{ij} / 2g$. But g^{ij} are expressed as

$$g^{ij}(x) = \delta_{ij} + \sum_{p=0}^{\infty} \sum_{|\tilde{\alpha}|+p \geq 2} g_{\tilde{\alpha}, p}^{ij} \tilde{x}^{\tilde{\alpha}} (x^n)^p$$

for some constants $g_{\tilde{\alpha}, p}^{ij}$ (see [15] e. g.). Hence we have

$$g^j(x) = \sum_{p=0}^{\infty} \sum_{|\tilde{\alpha}|+p \geq 1} g_{\tilde{\alpha}, p}^j \tilde{x}^{\tilde{\alpha}} (x^n)^p$$

for some constants $g_{\tilde{\alpha}, p}^j$. Let φ_0 and φ_1 be two analytic functions at $\tilde{O} = (0, 0, \dots, 0) \in R^{n-1}$. Then, by Cauchy-Kowalewski's theorem we can find a λ -harmonic function f_λ near m with

$$f_\lambda(\tilde{x}, 0) = \varphi_0(\tilde{x}), \quad \partial_n f_\lambda(\tilde{x}, 0) = \varphi_1(\tilde{x}).$$

Indeed, expressing f_λ as

$$(5.2) \quad f_\lambda(x) = \sum_{p=0}^{\infty} f_{\lambda, p}(\tilde{x}) (x^n)^p,$$

we have

$$(5.3) \quad (p+2)(p+1) \left(1 + \sum_{|\tilde{\alpha}| \geq 2} g_{\tilde{\alpha}, 0}^{n\tilde{\alpha}} \tilde{x}^{\tilde{\alpha}}\right) f_{\lambda, p+2}(\tilde{x}) = \lambda f_{\lambda, p}(\tilde{x}) + U_{\lambda, p}^{(1)}(\tilde{x}),$$

$$p = 0, 1, 2, \dots,$$

where

$$\begin{aligned}
U_{\lambda, p}^{(1)}(\tilde{x}) = & - \sum_{i=1}^{n-1} \partial_i^2 f_{\lambda, p}(\tilde{x}) - \sum_{q=0}^{p-1} \sum_{|\tilde{\alpha}|+p-q \geq 2} g_{\tilde{\alpha}, p-q}^{n,n} \tilde{x}^{\tilde{\alpha}} f_{\lambda, q+2}(\tilde{x})(q+2)(q+1) \\
& - \sum_{q=0}^p \sum_{|\tilde{\alpha}|+p-q \geq 2} \left\{ \sum_{i,j=1}^{n-1} g_{\tilde{\alpha}, p-q}^{i,j} \tilde{x}^{\tilde{\alpha}} \partial_i \partial_j f_{\lambda, q}(\tilde{x}) \right. \\
& \qquad \qquad \qquad \left. + 2 \sum_{i=1}^{n-1} g_{\tilde{\alpha}, p-q}^{i,n} \tilde{x}^{\tilde{\alpha}} \partial_i f_{\lambda, q+1}(\tilde{x})(q+1) \right\} \\
& - \sum_{q=0}^p \sum_{|\tilde{\alpha}|+p-q \geq 1} \left\{ \sum_{i=1}^{n-1} g_{\tilde{\alpha}, p-q}^i \tilde{x}^{\tilde{\alpha}} \partial_i f_{\lambda, q}(\tilde{x}) + g_{\tilde{\alpha}, p-q}^n \tilde{x}^{\tilde{\alpha}} f_{\lambda, q+1}(\tilde{x})(q+1) \right\}.
\end{aligned}$$

This implies

$$f_{\lambda, p+2}(\tilde{x}) = \frac{\lambda}{(p+2)(p+1)} f_{\lambda, p}(\tilde{x})(1+V(\tilde{x})) + U_{\lambda, p}^{(2)}(\tilde{x}), \quad p=0, 1, 2, \dots,$$

where V is an analytic function at $\tilde{O} \in R^{n-1}$ with $V(\tilde{O}) = \partial_i V(\tilde{O}) = 0$, $i=1, 2, \dots, n-1$, and $U_{\lambda, p}^{(2)}$ is a linear functional of $\{f_{\lambda, q}, \partial_i f_{\lambda, q}, \partial_i \partial_j f_{\lambda, q} : q=0, 1, \dots, p+1, i, j=1, 2, \dots, n-1\}$ with analytic coefficients independent of λ . Hence by induction we obtain

$$\begin{aligned}
(5.4) \quad f_{\lambda, 2p}(\tilde{x}) &= \frac{\lambda^p}{2p!} \left\{ \varphi_0(\tilde{x})(1+V_{2p}(\tilde{x})) + \sum_{q=0}^{p-1} \lambda^q U_{2p, q}(\tilde{x}) \right\}, \\
f_{\lambda, 2p+1}(\tilde{x}) &= \frac{\lambda^p}{(2p+1)!} \left\{ \varphi_1(\tilde{x})(1+V_{2p+1}(\tilde{x})) + \sum_{q=0}^{p-1} \lambda^q U_{2p+1, q}(\tilde{x}) \right\},
\end{aligned}$$

for $p=0, 1, 2, \dots$, where $U_{p, q}$ and V_p are analytic functions at $\tilde{O} \in R^{n-1}$ with $V_p(\tilde{O}) = \partial_i V_p(\tilde{O}) = 0$, $i=1, 2, \dots, n-1$. Hence we can determine the functions $f_{\lambda, p}$ inductively, and obtain the function f_λ by (5.2). Now substituting the formulae in (5.4) into (5.2), we obtain from the assumption $\mathcal{D}_m f_\lambda(m) = 0$ that

$$\begin{aligned}
(5.5) \quad & \sum_{p=0}^l \sum_{|\tilde{\alpha}| \leq 2l+1-2p} a_{\tilde{\alpha}, 2p} \partial^{\tilde{\alpha}} \left\{ \lambda^p \varphi_0(1+V_{2p}) + \sum_{q=0}^{p-1} \lambda^q U_{2p, q} \right\}(\tilde{O}) \\
& + \sum_{p=0}^l \sum_{|\tilde{\alpha}| \leq 2l-2p} a_{\tilde{\alpha}, 2p+1} \partial^{\tilde{\alpha}} \left\{ \lambda^p \varphi_1(1+V_{2p+1}) + \sum_{q=0}^{p-1} \lambda^q U_{2p+1, q} \right\}(\tilde{O}) = 0.
\end{aligned}$$

We will show from (5.5) by induction that

$$\begin{aligned}
(5.6) \quad a_{\tilde{\alpha}, 2p} &= 0, \quad |\tilde{\alpha}| \leq 2l+1-2p, \\
a_{\tilde{\alpha}, 2p+1} &= 0, \quad |\tilde{\alpha}| \leq 2l-2p,
\end{aligned}$$

for all $p=0, 1, 2, \dots, l$. First we note that the coefficient of λ^l in (5.5) is equal to 0;

$$(5.7) \quad \sum_{|\tilde{\alpha}| \leq 1} a_{\tilde{\alpha}, 2l} \partial^{\tilde{\alpha}} \{ \varphi_0(1+V_{2l}) \}(\tilde{O}) + a_{\tilde{\alpha}, 2l+1} \{ \varphi_1(1+V_{2l+1}) \}(\tilde{O}) = 0$$

for any analytic functions φ_0 and φ_1 at $\tilde{O} \in R^{n-1}$. Taking various functions for φ_0 and φ_1 in (5.7), we can easily see that (5.6) is valid for $p=l$. We next assume that (5.6) is valid for all $p=p_0+1, p_0+2, \dots, l$. Then the formula (5.5) is reduced to

$$(5.8) \quad \sum_{p=0}^{p_0} \sum_{|\tilde{\alpha}| \leq 2l+1-2p} a_{\tilde{\alpha}, 2p} \partial^{\tilde{\alpha}} \left\{ \lambda^p \varphi_0(1+V_{2p}) + \sum_{q=0}^{p-1} \lambda^q U_{2p,q} \right\}(\tilde{O}) \\ + \sum_{p=0}^{p_0} \sum_{|\tilde{\alpha}| \leq 2l-2p} a_{\tilde{\alpha}, 2p+1} \partial^{\tilde{\alpha}} \left\{ \lambda^p \varphi_1(1+V_{2p+1}) + \sum_{q=0}^{p-1} \lambda^q U_{2p+1,q} \right\}(\tilde{O}) = 0,$$

and taking the coefficient of λ^{p_0} we have

$$(5.9) \quad \sum_{|\tilde{\alpha}| \leq 2l+1-2p_0} a_{\tilde{\alpha}, 2p_0} \partial^{\tilde{\alpha}} \{ \varphi_0(1+V_{2p_0}) \}(\tilde{O}) \\ + \sum_{|\tilde{\alpha}| \leq 2l-2p_0} a_{\tilde{\alpha}, 2p_0+1} \partial^{\tilde{\alpha}} \{ \varphi_1(1+V_{2p_0+1}) \}(\tilde{O}) = 0$$

for any analytic functions φ_0 and φ_1 at $\tilde{O} \in R^{n-1}$. It is easy to see that (5.9) implies (5.6) with $p=p_0$ as in the above. The proof is completed.

LEMMA 5.2. *Let (M, g) be an n -dimensional C^ω harmonic space. Then there exists a sequence of polynomials $p_j, j=1, 2, \dots$, without constant terms such that, for all $m \in M$ and all λ -harmonic functions f near m ,*

$$(5.10) \quad M_m(\varepsilon, f) = f(m) + \sum_{j=1}^{\infty} p_j(\Delta) f(m) \varepsilon^{2j}$$

for all sufficiently small $\varepsilon > 0$.

PROOF. Let f be a λ -harmonic function near m . Then due to [3: (6.1) and (6.2)] the first mean value $M_m(\varepsilon, f)$ satisfies

$$(5.11) \quad M_m(\varepsilon, f) = \delta_{m, \lambda}(\varepsilon) f(m)$$

for all sufficiently small $\varepsilon > 0$, where $\delta_{m, \lambda}$ is defined by the solution of the equation

$$(5.12) \quad \delta'_{m, \lambda}(\varepsilon) + \frac{\lambda}{A_m(\varepsilon)} \int_0^\varepsilon A_m(r) \delta_{m, \lambda}(r) dr = 0, \quad \delta_{m, \lambda}(0) = 1.$$

Further, by [7: Corollary 4.3], the volume $A_m(\varepsilon)$ of the geodesic sphere $\partial B_m(\varepsilon)$ is represented as $A_m(r) = r^{n-1} h_m(r^2)$ for an analytic function h_m at $0 \in R$. On the other hand, it is clear that

$$A_m(r) = r^{n-1} \Theta(r) \text{vol}(S^{n-1}(1)),$$

for the function $\Theta(r)$ given in [1: Proposition 6.16]. Hence the function h_m is independent of the choice of $m \in M$. Now solving (5.12) by Cauchy-Kowalewski's

method, we obtain a sequence of polynomials p_j , $j=1, 2, \dots$, without constant terms such that

$$(5.13) \quad M_m(\varepsilon, f) = f(m) + \sum_{j=1}^{\infty} p_j(\lambda) f(m) \varepsilon^{2j}$$

for all sufficiently small $\varepsilon > 0$. Substitution of the relations $\lambda^p f(m) = \Delta^p f(m)$, $p=1, 2, \dots$, into (5.13) leads us to (5.10).

PROOF OF THEOREM 2 (1). In the following proof we assume $k = \infty$.

SUFFICIENCY. The proof of the sufficiency of each of the conditions (i)-(iv), (M1)-(M4) and (L1)-(L4) is similar to that of the sufficiency in Theorem 2 (3). The details will be omitted.

NECESSITY. Suppose that (M, g) is a harmonic space. The conditions (i)-(iv) are shown in [10] and [17]. These with Theorem 3 imply (M1) and (L1). Further the condition (M3) is verified by [3]. Hence, by Theorem 5.1 and (iv), we have (M2), (L2) and (L3). On the other hand it follows from Lemma 5.2 that the coefficients in the expansion (3.1) satisfy

$$(5.14) \quad P_{m,j} f(m) = p_j(\Delta) f(m), \quad j=1, 2, \dots$$

for all λ -harmonic functions f near m . But Lemma 5.1 means that a linear differential operator is uniquely determined by its values operated to λ -harmonic functions near m . Hence we have (M4). The condition (L4) now follows from (iv) and (M4).

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