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MULTI-TENSORS OF DIFFERENTIAL FORMS ON THE SIEGEL MODULAR VARIETY AND ON ITS SUBVARIETIES

By

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Let $A_n=H_n/\Gamma_n$, where H_n is the Siegel space and $\Gamma_n=Sp_{2n}(\boldsymbol{Z})$. A_n is called a *Siegel modular variety*, which is the coarse moduli variety of n-dimensional principally polarized abelian varieties over $\boldsymbol{C}$. Let $\tilde{A}_n$ be a projective non-singular model of A_n. $\tilde{A}_n$ is shown to be of general type for $n\geqq 9$ by Tai [10] ($n=8$ by Freitag [6], $n=7$ by Mumford [9]). Subvarieties of A_n are expected to have the same property if they are not too special.

Freitag [7] showed that $\tilde{A}_n$ carries many global sections of $\mathrm{Symm}^d(\Omega_{\tilde{A}_n}^{N-1})$, $N=n(n+1)/2$, if n is bounded from below by some n_0, and that any subvariety in A_n ($n\geqq n_0$) of codimension one is of type 'G' which is some weakened notion of "general type". He conjectured that n_0 would be taken to be ten, in connection with the argument of extensibility of holomorphic differentials to a non-singular model which is similar to Tai's one [9].

In this paper, we show that for $n\geqq 10$, $\tilde{A}_n$ carries many global sections of $(\Omega_{\tilde{A}_n}^{N-1})^{\otimes r}$ for some r (Theorem 1), and show the following;

THEOREM 2. *Let $n\geqq 10$. Then any subvariety in A_n of codimension one is of general type.*

We have the following corollary to this theorem (cf. Freitag [7]). We denote by $\Gamma_n(l)$, the principal congruence subgroup $\{M\in\Gamma_n \mid M\equiv 1_{2n} \bmod l\}$, 1_{2n} being the identity matrix of size $2n$, and by $A_{n,l}$, the quotient space $H_n/\Gamma_n(l)$.

COROLLARY. *If $K(\Gamma_n(l))$ denotes the function field of $A_{n,l}$ (namely, the Siegel modular function field for $\Gamma_n(l)$), then the automorphism group $\mathrm{Aut}_C(K(\Gamma_n(l)))$ over $\boldsymbol{C}$ is isomorphic to $\Gamma_n/\pm\Gamma_n(l)$ for $n\geqq 10$, i.e., the birational automorphism group of $A_{n,l}$ equals $\mathrm{Aut}_C(A_{n,l})\cong\Gamma_n/\pm\Gamma_n(l)$. In particular, A_n ($n\geqq 10$) has no non-trivial birational automorphism.*

This result is shown to be true under the condition that n is sufficiently large, in Freitag [7].

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Let ω be a matrix as in § 1, whose entries are differentials in $\Omega_{H_n}^{N-1}$. Then $\omega^{\otimes r}$ satisfies the formula $M\cdot\omega^{\otimes r}=|CZ+D|^{-r(n+1)}(CZ+D)^{\otimes r}\omega^{\otimes r}\,{}^t(CZ+D)^{\otimes r}$, $M=\begin{pmatrix}AB\\DC\end{pmatrix}\in Sp_{2n}(\boldsymbol{R})$ (Lemma 1). So if we construct a square matrix $\Lambda=\Lambda(Z)$ of size n^r whose entries are holomorphic functions on H_n such that $\Lambda(MZ)=|CZ+D|^{r(n+1)}\times({}^t(CZ+D)^{-1})^{\otimes r}\Lambda(Z)((CZ+D)^{-1})^{\otimes r}$, $M=\begin{pmatrix}AB\\CD\end{pmatrix}\in\Gamma_n$, then $tr(\Lambda\omega^{\otimes r})$ is Γ_n-invariant and is regarded as a section of $(\Omega_{A_n^\circ}^{N-1})^{\otimes r}$ $(n\geqq 3)$, A_n° denoting the smooth locus of A_n. The multi-tensor we consider is of this kind. Extensibility of $tr(\Lambda\omega^{\otimes r})$ to $\tilde{A}_n$ is proven under some conditions on Λ and on the degree n, which are similar to the case of pluri-canonical differential forms (Tai [10]). A restriction of such a multi-tensor of differentials to a subvariety D of codimension one gives a pluri-canonical differential form on D. Construction of a desired Λ is done by using transformation formulas for theta series of quadratic forms with spherical functions. Using this we show that for any D, there are such multi-tensors whose restriction to D gives enough non-trivial pluricanonical differentials.

1. $\boldsymbol{M}_{m,n}(*)$ denotes the set of $m\times n$ matrices with entries in $*$, and $\boldsymbol{M}_m=\boldsymbol{M}_{m,m}$. Let H_n be the Siegel space of degree n ; $H_n=\{Z\in\boldsymbol{M}_n(\boldsymbol{C})\,|\,{}^tZ=Z,\ \mathrm{Im}\,Z>0\}$. The symplectic group $Sp_{2n}(\boldsymbol{R})$ acts on H_n by the usual symplectic substitution

$$Z\rightarrow MZ=(AZ+B)(CZ+D)^{-1},\quad M=\begin{pmatrix}AB\\CD\end{pmatrix}.$$

Let $Z=(z_{ij})$, and let

$$\omega_{ij}=(-1)^{i+j}e_{ij}dz_{11}\wedge dz_{12}\wedge\cdots\wedge\check{dz_{ij}}\wedge\cdots\wedge dz_{nn}\quad(1\leqq i\leqq j\leqq n)$$

where the symbol e_{ij} denotes 1 if $i\neq j$, 2 if $i=j$, and $\check{dz_{ij}}$ means that dz_{ij} is omitted. ω_{ij} is a section of $\Omega_{H_n}^{N-1}$, $N=n(n+1)/2$, where $\Omega_{H_n}^{N-1}$ is a sheaf of holomorphic $(N-1)$-form. Let $\omega=(\omega_{ij})$. Then we have a transformation

$$M\cdot\omega=|CZ+D|^{-n-1}(CZ+D)\omega\,{}^t(CZ+D).$$

Let A, $B=(b_{ij})$ be square matrices of size n, m respectively. A tensor product $A\otimes B$ is defined to be

$$\begin{pmatrix}Ab_{11} & Ab_{12}\cdots & Ab_{1m}\\ Ab_{21} & & \vdots\\ \vdots & & \vdots\\ Ab_{m1} & \cdots\cdots & Ab_{mm}\end{pmatrix}\in M_{mn}.$$

Then we have (i) $(A\otimes B)(A'\otimes B')=AA'\otimes BB'$, A', B' being matrices of the same size as A, B respectively, (ii) ${}^t(A\otimes B)={}^tA\otimes{}^tB$, (iii) $c(A\otimes B)=(cA)\otimes B=A\otimes(cB)$

for a scalar c, (iv) $tr(A\otimes B)=tr(A)\cdot tr(B)$. The following is a direct consequence of the above:

LEMMA 1. *For a non-negative integer r, we have*

$$M\cdot\omega^{\otimes r}=|CZ+D|^{-r(n+1)}(CZ+D)^{\otimes r}\omega^{\otimes r}\,{}^t(CZ+D)^{\otimes r}.$$

Let $A=(a_{ij})\in \boldsymbol{M}_n$ and let us fix a positive integer r. Let I, J be ordered collections of r integers in $\{1, \cdots, n\}$ where a repeated choice is allowed. We define $A^{(I,J)}$ by

$$A^{(I,J)}=a_{i_1 j_1}\cdots a_{i_r j_r}$$

where $I=\{i_1, \cdots, i_r\}$, $J=\{j_1, \cdots, j_r\}$. Then a (k, l)-entry of a matrix $A^{\otimes r}$ is equal to $A^{(I,J)}$ if $k=1+\sum_{s=1}^{r}(i_s-1)n^{s-1}$, $l=1+\sum_{s=1}^{r}(j_s-1)n^{s-1}$ $(1\leqq k, l\leqq n^r)$. $\mathrm{sgn}(I)$ is defined by $\mathrm{sgn}(I)=\prod_{i\in I}(-1)^i$.

A holomorphic function f on H_n satisfying

$$f(MZ)=|CZ+D|^k f(Z) \qquad \text{for } M=\begin{pmatrix} AB \\ CD \end{pmatrix}\in\Gamma_n,$$

is called a (*Siegel*) *modular form of weight* k (when $n=1$, we need an additional condition that f is holomorphic also at the cusp). f admits the Fourier expansion

$$f(Z)=\sum_{S\geq 0} a(S)\boldsymbol{e}\left(tr\left(\frac{1}{2}ZS\right)\right)$$

where $\boldsymbol{e}(*)$ stands for $\exp(2\pi\sqrt{-1}\,*)$, and S runs over the set of semi-positive symmetric even matrices of size n. f is said to *vanish to order α at the cusp* if α is the minimum integer such that $a(S)=0$ for

$$S \quad \text{with} \quad \min_{g\in \boldsymbol{Z}^n,\neq 0}\left\{\frac{1}{2}S[g]\right\}<\alpha,$$

$S[g]$ denoting ${}^t gSg$. The minimum is equal to $\min\left\{\frac{1}{2}tr(ST)\right\}$, T running over the set of non-zero symmetric positive semi-definite integral matrices of size n (cf. Barnes and Cohn [3]). We denote by $\mathrm{ord}(f)$, the order α of vanishing at the cusp.

2. *Theta series and a matrix $\Psi_{F,r}[w]$*. Let m be an integer with $m\geqq 2(n-1)$, and let η be a complex $m\times(n-1)$ matrix satisfying both ${}^t\eta\eta=0$ rank $\eta=n-1$ (such exists). η $(1\leqq i\leqq n)$ denotes an $(n-1)\times n$ matrix given by

$$\eta_i=\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 10 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}.$$
$$\underbrace{\qquad}_{i}$$

We fix a positive symmetric matrix F of size m with rational coefficients. Let r be a positive integer, and let I, J be as in the preceding section. We define a theta series associated with F by setting

$$\theta_F^{(I,J)}\begin{bmatrix} u \\ v \end{bmatrix}(Z)=\mathrm{sgn}(I)\,\mathrm{sgn}(J)\sum_G \prod_{i\in I} |\eta_i{}^t(G+u)F^{1/2}\eta| \prod_{j\in J} |\eta_j{}^t(G+u)F^{1/2}\eta|$$
$$\times e\Big(tr\Big(\frac{1}{2}ZF[G+u]+{}^t(G+u)v\Big)\Big)$$

where G runs through all $m\times n$ integral matrices, and u, v are $m\times n$ matrices with rational coefficients. $\binom{u}{v}$ is called a theta characteristic. If $\{I', J'\}$ equals $\{I, J\}$ up to orders, then obviously $\theta_F^{(I',J')}\begin{bmatrix} u \\ v \end{bmatrix}(Z)=\theta_F^{(I,J)}\begin{bmatrix} u \\ v \end{bmatrix}(Z)$. Then we define $\Psi_{F,r}\begin{bmatrix} u \\ v \end{bmatrix}(Z)$ to be a square matrix of size n^r whose (k, l)-entry equals $\theta_F^{(I,J)}\begin{bmatrix} u \\ v \end{bmatrix}(Z)$ where $k=1+\sum_{s=1}^{r}(i_s-1)n^{s-1}$, $l=1+\sum_{s=1}^{r}(j_s-1)n^{s-1}$ with $I=\{i_1, \cdots i_r\}$, $J=\{j_1, \cdots, j_r\}$. $\Psi_{F,r}\begin{bmatrix} u \\ v \end{bmatrix}$ is a symmetric matrix. Our purpose of this section is to prove the following;

PROPOSITION 1. *Let* $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp_{2n}(\boldsymbol{Q})$ *such that* $A, D, F\otimes B, F^{-1}\otimes C$ *are integral. Put*

$$u_M=uA+F^{-1}vC+\frac{1}{2}{}^t(F^{-1})_\Delta({}^tAC)_\Delta, \quad v_M=FuB+vD+\frac{1}{2}{}^t(F_\Delta)({}^tBD)_\Delta,$$

$$E_F\Big(\binom{u}{v}, M\Big)=e\Big(\frac{1}{2}tr(-{}^t(uA+F^{-1}vC)(FuB+vD+{}^t(F_\Delta)({}^tBD)_\Delta)+{}^tuv)\Big),$$

where for a square matrix P, P_Δ *denotes the vector composed of diagonal elements of* P. *Then we have a transformation formula*

$$\Psi_{F,r}\begin{bmatrix} u \\ v \end{bmatrix}(MZ)=\chi_F(M)E_F\Big(\binom{u}{v}, M\Big)|CZ+D|^{(m/2)+2r}$$
$$\times({}^t(CZ+D)^{-1})^{\otimes r}\Psi_{F,r}\begin{bmatrix} u_M \\ v_M \end{bmatrix}(Z)((CZ+D)^{-1})^{\otimes r},$$

where $\chi_F(M)$ *is an eighth root of unity depending only on* F *and* M.

Since F, u, v are of rational coefficients, we have the following corollary.

COROLLARY. *There is an integer l such that*

$$\Psi_{F,r}\begin{bmatrix} u \\ v \end{bmatrix}(MZ)=\chi(M)|CZ+D|^{(m/2)+2r}$$

$$\times({}^t(CZ+D)^{-1})^{\otimes r}\Psi_{F,r}\begin{bmatrix} u \\ v \end{bmatrix}(Z)((CZ+D)^{-1})^{\otimes r},$$

for any $M=\begin{pmatrix} AB \\ CD \end{pmatrix}\in\Gamma_n(l)$ where χ is a map of $\Gamma_n(l)$ to the set of roots of unity. χ is killed by some power.

Let $X=(x_{ij})$ be an $m\times n$ variable matrix. We define another theta series associated with F by setting

$$\theta_F\begin{bmatrix} u \\ v \end{bmatrix}(Z, X)=\sum_G e\Big(tr\Big(\frac{1}{2}ZF[G+u]+{}^t(G+u)(X+v)\Big),$$

where G runs through all $m\times n$ integral matrices.

LEMMA 2. *Let the notations be as in Proposition* 1. *Then*

$$\theta_F\begin{bmatrix} u \\ v \end{bmatrix}(MZ, X)=\chi_F(M)E_F\Big(\begin{pmatrix} u \\ v \end{pmatrix}, M\Big)e\Big(\frac{1}{2}tr(C^t(CZ+D)^t XF^{-1}X)\Big)$$

$$\times|CZ+D|^{m/2}\theta_F\begin{bmatrix} u_M \\ v_M \end{bmatrix}(Z, X(CZ+D)).$$

The proof of the above lemma was given in Andrianov and Maloletkin [1] under some condition on $\begin{pmatrix} u \\ v \end{pmatrix}$, M, and in Tsuyumine [11], [12], the general case (not exactly this form).

Let $\partial=\Big(\frac{\partial}{\partial x_{ij}}\Big)$ be an $m\times n$ matrix of differential operators. Let r, and I, J be as above. We define a differential operator $L_{IJ\eta}$ by

$$L_{IJ\eta}=\frac{\operatorname{sgn}(I)\operatorname{sgn}(J)}{(2\pi\sqrt{-1})^{2r(n-1)}}\prod_{i\in I}\det({}^t\eta\partial{}^t\eta_i)\prod_{j\in J}({}^t\eta\partial{}^t\eta_j)$$

LEMMA 3. *Let P be a complex symmetric matrix of degree n, and Q, a complex $n\times m$ matrix and c, a constant. Then we have the following identity*:

$$L_{IJ\eta}\Big(e\Big(tr\Big(\frac{1}{2}P^tXX+QX\Big)+c\Big)\Big)$$

$$=\operatorname{sgn}(I)\operatorname{sgn}(J)\prod_{i\in I}|\eta_i(P^tX+Q)\eta|\prod_{j\in J}|\eta_j(P^tX+Q)\eta|$$

$$\times e\Big(tr\Big(\frac{1}{2}P^tXX+QX\Big)+c\Big).$$

The proof is given as a slight modification of that of Lemma 3 in Andrianov and Maloletkin [1]. So we skip the proof.

PROOF OF PROPOSITION 1. $\theta_F^{(I,J)}\begin{bmatrix} u \\ v \end{bmatrix}(Z)$ is equal to $L_{IJ\eta}\left(\theta_F\begin{bmatrix} u \\ v \end{bmatrix}(Z, F^{1/2}X)\right)\Big|_{X=0}$. Substituting X by $F^{1/2}X$ in the formula in Lemma 2 and applying $L_{IJ\eta}$ at $X=0$, we get, by Lemma 3,

$$\theta_F^{(I,J)}\begin{bmatrix} u \\ v \end{bmatrix}(MZ)$$

$$=\chi_F(M)E_F\left(\begin{pmatrix} u \\ v \end{pmatrix}, M\right)|CZ+D|^{m/2}\operatorname{sgn}(I)\operatorname{sgn}(J)$$

$$\times\sum_G\prod_{i\in I}|\eta_i(CZ+D)^t(G+u_M)F^{1/2}\eta|\prod_{j\in J}|\eta_j(CZ+D)^t(G+u_M)F^{1/2}\eta|$$

$$\times\mathbf{e}\left(tr\left(\frac{1}{2}ZF[G+u_M]+{}^t(G+u_M)v_M\right)\right).$$

Using the Laplace expansion

$$|\eta_i(CZ+D)^t(G+u_M)F^{1/2}\eta|=\sum_{s=1}^{n}|\eta_i(CZ+D)^t\eta_s|\,|\eta_s{}^t(G+u_M)F^{1/2}\eta|,$$

the equality becomes

$$\theta_F^{(I,J)}\begin{bmatrix} u \\ v \end{bmatrix}(MZ)$$

$$=\chi_F(M)E_F\left(\begin{pmatrix} u \\ v \end{pmatrix}, M\right)|CZ+D|^{m/2}\operatorname{sgn}(I)\operatorname{sgn}(J)\sum_G\left(\sum_S\sum_T\right)$$

$$\times\prod_{\substack{i\in I\\ s\in S}}|\eta_i(CZ+D)^t\eta_s|\prod_{\substack{j\in J\\ t\in T}}|\eta_j{}^t(CZ+D)^t\eta_t|\cdot|\eta_s{}^t(G+u_M)F^{1/2}\eta|$$

$$\times|\eta_t{}^t(G+u_M)F^{1/2}\eta|\,\mathbf{e}\left(tr\left(\frac{1}{2}ZF[G+u_M]+{}^t(G+u_M)v_M\right)\right),$$

S, T running through all the collections of r integers in $\{1, \cdots, n\}$ (admitting a repeated choice),

$$=\chi_F(M)E_F\left(\begin{pmatrix} u \\ v \end{pmatrix}, M\right)|CZ+D|^{m/2}\sum_{S,T}\prod_{\substack{i\in I\\ s\in S}}(-1)^{i+s}|\eta_i(CZ+D)^t\eta_s|$$

$$\times\prod_{\substack{j\in J\\ t\in T}}(-1)^{j+t}|\eta_j(CZ+D)^t\eta_t|\,\theta_F^{(S,T)}\begin{bmatrix} u_M \\ v_M \end{bmatrix}(Z).$$

$(-1)^{i+s}|\eta_i(CZ+D)^t\eta_s|$ is an (s, i)-entry of the cofactor matrix $(CZ+D)^*$ of $CZ+D$, and so

$$\theta_F^{(I,J)}\begin{bmatrix} u \\ v \end{bmatrix}(MZ)=\chi_F(M)E_F\left(\begin{pmatrix} u \\ v \end{pmatrix}, M\right)|CZ+D|^{m/2}\sum_{S,T}({}^t(CZ+D)^*)^{(I,S)}$$

$$\times\theta_F^{(S,T)}\begin{bmatrix} u_M \\ v_M \end{bmatrix}(Z)((CZ+D)^*)^{(T,J)}.$$

Thus we have

$$\Psi_{F,r}\begin{bmatrix} u \\ v \end{bmatrix}(MZ)$$

$$=\chi_F(M)E_F\left(\begin{pmatrix} u \\ v \end{pmatrix}, M\right)|CZ+D|^{m/2}({}^t(CZ+D)^*)^{\otimes r}\Psi_{F,r}\begin{bmatrix} u_M \\ v_M \end{bmatrix}(Z)((CZ+D)^*)^{\otimes r}$$

$$=\chi_F(M)E_F\left(\begin{pmatrix} u \\ v \end{pmatrix}, M\right)|CZ+D|^{(m/2)+2r}({}^t(CZ+D)^{-1})^{\otimes r}\Psi_{F,r}\begin{bmatrix} u_M \\ v_M \end{bmatrix}(Z)((CZ+D)^{-1})^{\otimes r}.$$

q. e. d.

3. *Further properties of* $\Psi_{F,r}\begin{bmatrix} u \\ v \end{bmatrix}$. The following formula is easy to see.

LEMMA 4. *Let* $\Psi_{F,r}\begin{bmatrix} u \\ v \end{bmatrix}(Z)$ *be as in the preceding section. Let* P *be any symmetric matrix of size* n *with rational coefficients, and let* $k\neq 0$ *be an integer such that* $k^2PF[G]$ *is even for any* $G\in \boldsymbol{M}_{m,n}(\boldsymbol{Z})$. *Then*

$$\Psi_{F,r}\begin{bmatrix} u \\ v \end{bmatrix}(Z+P)=k^{-2(n-1)r}\sum_w e\left(-\frac{1}{2}k^2PF[w+(u/k)]\right)\Psi_{k^2F,r}\begin{bmatrix} w+(u/k) \\ kv+kF(kw+u)P \end{bmatrix}(Z)$$

where w *runs over the representatives of* $\boldsymbol{M}_{m,n}\left(\frac{1}{k}\boldsymbol{Z}\right) \bmod \boldsymbol{Z}$.

Now we show that for any F and for any characteristic of the form $\begin{pmatrix} u \\ 0 \end{pmatrix}$, $\Psi_{F,r}\begin{bmatrix} u \\ 0 \end{bmatrix}$ is non-trivial for infinitely many r. Let $[\![i]\!]=(i, \cdots, i)\in \boldsymbol{M}_{1,2r}(\boldsymbol{Z})$ $(1\leqq i\leqq n)$. Then the Fourier coefficient $a(S)$ in the expansion of $\theta_F^{([\![i]\!])}\begin{bmatrix} u \\ 0 \end{bmatrix}$ is given by

$$a(S)=\sum_G |\eta_i{}^t(G+u)F^{1/2}\eta|^{2r}$$

where G runs over the finite set $\{G\in \boldsymbol{M}_{m,n}(\boldsymbol{Z})|F[G+u]=S\}$. There is an S such that $\{G|F[g+u]=S\}\neq\phi$, and $|\eta_i{}^t(G+u)F^{1/2}\eta|\neq 0$ for some G in the set. Then it is easy to see $a(S)\neq 0$ for infinitely many r (for instance, if it is zero for $r=1, 2, \cdots, \sharp\{G|F[G+u]=S\}$, then every $|\eta_i{}^t(G+u)F^{1/2}\eta|^2$ must be zero).

LEMMA 5. *Suppose that* $\Psi_{F,r}\begin{bmatrix} u \\ v \end{bmatrix}(Z)$ *is non-trivial. Let us take any complex symmetric matrix* W *of size* n. *Then* $tr\left(\Psi_{F,r}\begin{bmatrix} u \\ v \end{bmatrix}(Z)W^{\otimes r}\right)$ *is identically zero in* Z *if and only if* $W=0$.

PROOF. $\Psi_{F,r}\begin{bmatrix} u \\ v \end{bmatrix}(Z)$ satisfies a transformation formula in Corollary to Proposition 1. It follows that $\theta_F^{([\![1]\!])}\begin{bmatrix} u \\ v \end{bmatrix}(Z)$ is not identically zero. Let us suppose

$W \neq 0$. If we put $W=(w_{ij})$, then it is impossible that only w_{11} is non-zero. So some w_{ij} is not zero with $(i, j) \neq (1, 1)$. Again by the transformation formula, it follows from our assumption, that $tr\left(\Psi_{F,r}\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](Z)({}^tUWU)^{\otimes r}\right)=0$ for any $U \in GL_n(\boldsymbol{Z})(l)=\{U \in GL_n(\boldsymbol{Z}) \mid U \equiv 1_n \mod l\}$, l being as in Corollary to Proposition 1, and 1_n denoting the identity matrix of size n. Then we may assume $w_{ii} \neq 0$ for some i with $2 \leqq i \leqq n$. If we take as U,

$$U=\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ b & & & \ddots \\ & & & & 1 \end{pmatrix}$$

where b is an $(1, i)$-entry, then

$$tr\left(\Psi_{F,r}\left[\begin{matrix} u \\ v \end{matrix}\right](Z)({}^tUWU)^{\otimes r}\right)=\theta_F^{([1])}\left[\begin{matrix} u \\ v \end{matrix}\right]\cdot w_{ii}\cdot b^{2r}$$

$$+(\text{lower degree terms of } b),$$

which cannot be zero, a contradiction. q. e. d.

4. *Construction of* $\Psi(Z)$. Let $\Psi_{F,r}\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](Z)$, $\Gamma_n(l)$ be as in Corollary to Proposition 1. Then for some positive integer r', $\left(\Psi_{F,r}\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](Z)\right)^{\otimes r'}$ satisfies

$$\left(\Psi_{F,r}\left[\begin{matrix} u \\ v \end{matrix}\right](MZ)\right)^{\otimes r'}=|CZ+D|^{((m/2)+2r)r'}({}^t(CZ+D)^{-1})^{\otimes rr'}$$

$$\times\left(\Psi_{F,r}\left[\begin{matrix} u \\ v \end{matrix}\right](Z)\right)^{\otimes r'}((CZ+D)^{-1})^{\otimes rr'}$$

for $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n(l)$. Let $\{M_j\}$ be any system of representatives of Γ_n mod $\Gamma_n(l)$. Let us put

$$\Psi(Z)=\sum_j |C_jZ+D_j|^{-((m/2)+2r)r'}\,{}^t(C_jZ+D_j)^{\otimes rr'}\left(\Psi_{F,r}\left[\begin{matrix} u \\ v \end{matrix}\right](M_jZ)\right)^{\otimes r'}(C_jZ+D_j)^{\otimes rr'}$$

where $M_j=\begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}$. Then $\Psi(Z)$ satisfies

$$(*) \qquad \Psi(MZ)=|CZ+D|^{((m/2)+2r)r'}({}^t(CZ+D)^{-1})^{\otimes rr'}\Psi(Z)((CZ+D)^{-1})^{\otimes rr'}$$

for $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$.

PROPOSITION 2. *Let Z_0 be any point of H_n, and let W be any non-zero complex symmetric matrix of size n. Let m be an integer with $m \geqq 2(n-1)$. Then for infinitely many r and for infinitely many r', there is a symmetric matrix $\Psi(Z)$*

of size $n^{rr'}$ *satisfying the above transformation formula* (*) *for* Γ_n *such that* $tr(\Psi(Z_0)W^{\otimes rr'})\neq 0$.

PROOF. Let F' be a positive symmetric matrix of size m with rational coefficients. By Lemma 5, and by the argument just before it, $tr\left(\Psi_{F',r}\begin{bmatrix}u'\\v'\end{bmatrix}(Z)W^{\otimes r}\right)$ is not identically zero for infinitely many r if we take a suitable theta characteristic $\binom{u'}{v'}$. Since the analytic closure of $\{Z_0+P|$ rational symmetric matrices P of size $n\}\subset H_n$ equals H_n, there is P such that $tr\left(\Psi_{F',r}\begin{bmatrix}u'\\v'\end{bmatrix}(Z_0+P)W^{\otimes r}\right)\neq 0$. By Lemma 4, $\Psi_{F',r}\begin{bmatrix}u'\\v'\end{bmatrix}(Z+P)$ is written as a linear combination of the similar matrices as $\Psi_{F',r}\begin{bmatrix}u'\\v'\end{bmatrix}$ whose entries are theta series in Z. It follows that there is F of size m, and $\binom{u}{v}$ such that $tr\left(\Psi_{F,r}\begin{bmatrix}u\\v\end{bmatrix}(Z_0)W^{\otimes r}\right)\neq 0$. We make $\Psi(Z)$ from $\Psi_{F,r}\begin{bmatrix}u\\v\end{bmatrix}(Z)$ as in the above manner. Let $M_j=\begin{pmatrix}A_jB_j\\C_jD_j\end{pmatrix}$ be as above. We may assume that $M_1=1_{2n}$. Then

$$\begin{aligned}
&tr(\Psi(Z_0)W^{\otimes rr'})\\
&=tr\left(\left(\Psi_{F,r}\begin{bmatrix}u\\v\end{bmatrix}(Z_0)\right)^{\otimes r'}W^{\otimes rr'}\right)\\
&\quad+\sum_{j>1}tr\Big(|C_jZ_0+D_j|^{-((m/2)+2r)r'}\,{}^t(C_jZ_0+D_j)^{\otimes rr'}\left(\Psi_{F,r}\begin{bmatrix}u\\v\end{bmatrix}(M_jZ_0)\right)^{\otimes r'}\\
&\quad\times(C_jZ_0+D_j)^{\otimes rr'}W^{\otimes rr'}\Big)\\
&=\left(tr\left(\Psi_{F,r}\begin{bmatrix}u\\v\end{bmatrix}(Z_0)W^{\otimes r}\right)\right)^{r'}\\
&\quad+\sum_{j>1}tr\left(|C_jZ_0+D_j|^{-((m/2)+2r)}\,{}^t(C_jZ_0+D_j)^{\otimes r}\Psi_{F,r}\begin{bmatrix}u\\v\end{bmatrix}(M_jZ_0)(C_jZ_0+D_j)^{\otimes r}W^{\otimes r}\right)^{r'},
\end{aligned}$$

which is not zero for infinitely many r', since the first term is not zero.

q. e. d.

REMARK. Our argument actually shows that we can take infinitely many r, r' from the set of multiples of any fixed integer, which makes $tr(\Psi(Z_0)W^{\otimes rr'})$ non-zero in the proposition.

Let us put

$$\lambda_{m,r,r'}=tr(\Psi(Z)\omega^{\otimes rr'}).$$

By Lemma 1, we have the following;

PROPOSITION 3. *Suppose* $r(n-1)\geqq m/2$. *Then for any modular form* f *for* Γ_n *of weight* $(r(n-1)-m/2)r'$, $f\lambda_{m,r,r'}$ *is a* Γ_n-*invariant form in* $(\Omega_{H_n}^{N-1})^{\otimes rr'}$, *where* $(r(n-1)-m/2)r'$ *is supposed to be possible as weight.*

5. *Extensibility.* Let $\bar{A}_{n,l}$ be a toroidal compactification of $H_n/\Gamma_n(l)$, which is taken to be smooth and projective for $l\geqq 3$ ([2]). We may assume that $\bar{A}_n$ is a quotient of $\bar{A}_{n,l}$ by $\Gamma_n/\Gamma_n(l)$. In this section we assume $n\geqq 3$. Then the singularities of $\bar{A}_n$ are just the ramification locus of $\bar{A}_{n,l}\to\bar{A}_n$. Let $\bar{A}_n^{\circ}$ be the smooth locus of $\bar{A}_n$.

Let $f\lambda_{m,r,r'}$ be as in Proposition 3. $f\lambda_{m,r,r'}$ is holomorphic multi-tensor of differentials on the smooth locus of A_n. However we need to find a condition that it extends holomorphically to a projective non-singular model. By the similar argument as in the case of pluri-canonical differential forms ([2], Tai [10]), we have the following;

LEMMA 6. *If* f *vanishes at the cusp of order* $\geqq rr'$, *then* $f\lambda_{m,r,r'}$ *extends holomorphically to* $\bar{A}_n^{\circ}$.

Now we discuss the extensibility of $f\lambda_{m,r,r'}$ over the quotient singularities. As in the similar way as in Tai [10], Theorem 3.3, we have the following, whose proof is easy to recover, so that we skip it.

LEMMA 7. *Let* X *be a quotient* $\mathcal{D}/G$ *of an open domain* $\mathcal{D}$ *in* $\boldsymbol{C}^N$ *by a finite group* G *acting on* $\mathcal{D}$. *Let* $\tilde{X}\to X$ *be a desingularization. Suppose that* $g\in G$ *acts on the tangent space of a fixed point as a multiplication by* $\boldsymbol{e}(s_i)$, $i=1,\cdots,N$ *with* $s_i\in\boldsymbol{Q}$, $0\leqq s_i<1$. *Then if*

$$\sum_i s_i\geqq 1+\max\{s_i\}$$

is satisfied for each $g\neq 1$ *and for each of its fixed point, then a* G-*invariant form in* $(\Omega_{\mathcal{D}}^{N-1})^{\otimes r}$ *extends holomorphically to* $\tilde{X}$.

Tai [10] showed that every fixed point in $\bar{A}_n$ satisfies $\sum_i s_i\geqq 1$ for $n\geqq 5$. Following his method, we can find when the condition in the lemma is satisfied. Indeed it is sufficient for the condition, that $\sum_i s_i\geqq 2$. It is shown that every fixed point of order k satisfies it unless k equals 2, 3, 4, 5, 6, 8, 10, 12 or14. Then we check the condition in the lemma for the individual case of these k, and get our bound for it to hold;

$$n\geqq 7.$$

Now we have the following by Proposition 3, and by Lemmas 6 and 7.

PROPOSITION 4. *Let $n \geqq 7$. Let $\lambda_{m,r,r'}$ be as in the preceding section. Then if f is a modular form of weight $(r(n-1)-m/2)r'$ with* $\mathrm{ord}(f) \geqq rr'$, *then a multi-tensor $f\lambda_{m,r,r'}$ of differentials extends holomorphically to a projective non-singular model of A_n.*

6. Let $n \geqq 7$. Suppose that there is a non-trivial modular form f such that

$$\frac{(n-1)\,\mathrm{ord}(f)}{\mathrm{weight}(f)} > 1 .$$

Then for suitable integers α, β, every modular form g in $f^{\alpha k}A(\Gamma_n)_{\beta k}$, $k=1, 2, \cdots$, has enough vanishing order at the cusp, i.e., $(n-1)\,\mathrm{ord}(g)/\mathrm{weight}(g) \geqq 1/(1-(m/2r(n-1)))$ for a fixed m, $A(\Gamma_n)_{\beta k}$ denoting the vector space of modular form of weight βk. If $\mathrm{weight}(g)=(r(n-1)-m/2)r'$, then $g\lambda_{m,r,r'}$ extends to a section of $(\Omega^{N-1}_{\tilde{A}_n})^{\otimes rr'}$ (Proposition 4), and it is taken to be non-trivial (cf. the remark in § 4). So, in this case there are many global sections of $(\Omega^{N-1}_{\tilde{A}_n})^{\otimes r}$ for some r.

Freitag [6], [7] introduced the desired modular forms when $n \geqq 10$. Let $\vartheta[w](Z)$ be a theta constant;

$$\vartheta[w](Z) = \sum_{g \in \boldsymbol{Z}^n} \boldsymbol{e}\Big(\frac{1}{2}{}^t(g+u)Z(g+u) + {}^t(g+u)v\Big)$$

with a theta characteristic $w=\binom{u}{v} \in \frac{1}{2}\boldsymbol{Z}^{2n} \pmod{\boldsymbol{Z}}$, $\boldsymbol{e}(2\,{}^tuv)=1$. Then the functions

$$\prod_w \vartheta[w](Z)$$

and

$$f_l = \sum_{w_0} \sum_{w \neq w_0} \vartheta[w](Z)^{8l} \quad (l\text{: a natural number})$$

have the desired property when $n \geqq 10$. Moreover in Freitag [7], it is shown that on any subvariety in A_n of codimension one, all of f_l, $l=1, 2, \cdots$, do not vanish identically.

THEOREM 1. *Let $\tilde{A}_n$ be a non-singular model of the moduli A_n of principally polarized abelian varieties over $\boldsymbol{C}$ of dimension n. Suppose $n \geqq 10$. Then for any elements $\alpha_1, \cdots, \alpha_t$ of the modular function field, there exists r such that there are $t+1$ elements $\lambda_0, \lambda_1, \cdots, \lambda_t \in (\Omega^{N-1}_{\tilde{A}_n})^{\otimes r}$ with $\alpha_i = \lambda_i/\lambda_0$, N being $n(n+1)/2$.*

7. *Proof of Theorem* 2. Let D be any subvariety in A_n of codimension one. When $n \geqq 3$, there is a modular form $h(Z)$ of some weight p whose divisor equals D (Freitag [4], [5], or Tsuyumine [13]). Let us put

$$\psi_h=\left(e_{ij}\frac{\partial}{\partial z_{ij}}h\right),\quad e_{ij}=\begin{cases}2, & i=j,\\ 1, & i\neq j,\end{cases}$$

which is a symmetric matrix of size n. Let $\pi: H_n \rightarrow A_n$ be the canonical projection. Let $\pi^{-1}(D)^{\circ}$ be the smooth locus of $\pi^{-1}(D)$. Then by using $\left\{\sum\left(\frac{\partial}{\partial z_{ij}}h\right)dz_{ij}\right\}\Big|_{\pi^{-1}(D)^{\circ}}=0$, we get $\omega|_{\pi^{-1}(D)^{\circ}}=\psi_h\omega'$ where $\omega'\in\Omega^{N-1}_{\pi^{-1}(D)^{\circ}}$, $\neq 0$.

LEMMA 8. *Let $n\geqq 3$. Let D be any subvariety in A_n of codimension one. Then for infinitely many r and for infinitely many r' there are $\lambda_{m,r,r'}$ whose restrictions to $\pi^{-1}(D)$ do not vanish identically, where r, r' are taken from the set of multiples of any fixed integer.*

PROOF. Let h be as above. If Z_0 is a non-singular point of D, then $\psi_h(Z_0)$ is non-zero. Since $\lambda_{m,r,r'}|_{\pi^{-1}(D)^{\circ}}=tr(\Psi(Z)\psi_h(Z)^{\otimes rr'})\omega'^{\otimes rr'}$, our assertion is immediate from Proposition 2 (see also the remark just after it). q.e.d.

For D, we can take a modular form f with $(n-1)\operatorname{ord}(f)/\operatorname{weight}(f)>1$, not vanishing identically on D, provided that $n\geqq 10$. By Lemma 8, there are many multi-tensors of the form $g\lambda_{m,r,r'}$, g being a modular form, extending holomorphically to a non-singular model of A_n whose restrictions to D do not vanish identically. This proves Theorem 2.

PROOF OF COROLLARY. We denote by $A_{n,l}^*$, the Satake compactification of $A_{n,l}$. Let ϕ be any birational automorphism of $A_{n,l}^*$. We can eliminate points of indeterminancy for ϕ, by taking a commutative diagram

$$\begin{array}{ccc} & \tilde{A}_{n,l} & \\ {}^{\phi_1}\swarrow & & \searrow^{\phi_2} \\ A_{n,l}^* & \overset{}{\underset{\phi}{\dashrightarrow}} & A_{n,l}^* \end{array}$$

where ϕ_1, ϕ_2 are morphisms of blowing up (Hironaka [8], Chap. 0, §5). By our theorem, any subvariety in $A_{n,l}$ $(n\geqq 10)$ of codomension one is also of general type. Then the image of any exceptional divisor in $\tilde{A}_{n,l}$ via ϕ_2 cannot be a divisor in $A_{n,l}^*$. This implies that for elimination of points of indeterminancy, the diagram is not necessary, and that ϕ itself must be a morphism. Thus ϕ is an automorphism of $A_{n,l}^*$, which maps the cusps to the cusps. So ϕ

induces an automorphism of $H_n/\Gamma_n(l)$. Since Γ_n is the maximal discrete subgroup of $Sp_{2n}(\boldsymbol{R})$ having $\Gamma_n(l)$ as its normal subgroup, ϕ equals an automorphism induced by some element of $\Gamma_n/\pm\Gamma_n(l)$.

REMARK. By our argument, the assertion in the corollary holds for $n\geqq 2$ if l is sufficiently large.

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