TSUKUBA J. MATH.
Vol. 11 No. 1 (1987). 101—105

ANOTHER PROOF OF THE STRONG COMPLETENESS OF THE INTUITIONISTIC FUZZY LOGIC

By

Mitio TAKANO

Takeuti and Titani [3] introduced the system, which we shall call TT, for the intuitionistic fuzzy logic, and proved the following theorem:

STRONG COMPLETENESS THEOREM (Takeuti and Titani [3, Theorem 1.3]). *Suppose that the language of* TT *is countable. If a sequent* $\Sigma\Rightarrow\Delta$ *is valid then it is provable in* TT, *where* Σ *may be infinite.*

The purpose of this note is to give another proof of the above theorem.

The author expresses his thanks to Dr. Yuichi Komori, who gave him information on the logic studied in this note.

§1. Recall, first, that the axioms and inference rules of TT are those of the intuitionistic logic (Gentzen's LJ) together with the following ones:

EXTRA AXIOM SCHEMATA FOR TT.

1. $\Rightarrow(A\rightarrow B)\vee((A\rightarrow B)\rightarrow B)$;
2. $(A\rightarrow B)\rightarrow B\Rightarrow(B\rightarrow A)\vee B$;
3. $(A\wedge B)\rightarrow C\Rightarrow(A\rightarrow C)\vee(B\rightarrow C)$;
4. $A\rightarrow(B\vee C)\Rightarrow(A\rightarrow B)\vee(A\rightarrow C)$;
5. $\forall x(C\vee A(x))\Rightarrow C\vee\forall xA(x)$, where x does not occur in C;
6. $\forall xA(x)\rightarrow C\Rightarrow\exists x(A(x)\rightarrow D)\vee(D\rightarrow C)$, where x does not occur in D.

EXTRA INFERENCE RULE FOR TT.

$$\frac{\Gamma\Rightarrow A\vee(C\rightarrow p)\vee(p\rightarrow B)}{\Gamma\Rightarrow A\vee(C\rightarrow B)},$$

where p is any propositional variable not occurring in the lower sequent.

We call that system TT^- which is obtained from TT by deleting Extra Inference Rule for TT.

Received March 1, 1986.

Fifteen years before, Horn had introduced another system for the logic, which we shall call H, and had shown the weak completeness (Horn [**1**, Theorem 3.8]): *A formula is valid iff it is provable in* H. The system H has also been characterized by means of Kripke models in Ono [**2**, Theorem 3.3]. Recall that the axioms and inference rules of H are those of LJ together with the following axioms:

EA 1. $\forall x(C \vee A(x)) \Rightarrow C \vee \forall x A(x)$, where x does not occur in C;

EA 2. $\Rightarrow (A \rightarrow B) \vee (B \rightarrow A)$.

Then we claim the following theorem:

THEOREM. *Suppose that the language concerned is countable. The following properties* (a)-(d) *of a sequent* $\Sigma \Rightarrow \Delta$ *are equivalent, where* Σ *may be infinite*:

(a) $\Sigma \Rightarrow \Delta$ *is valid.*

(b) $\Sigma \Rightarrow \Delta$ *is provable in* H.

(c) $\Sigma \Rightarrow \Delta$ *is provable in* TT^-.

(d) $\Sigma \Rightarrow \Delta$ *is provable in* TT.

The proof of (a)$\Rightarrow$(b) is postponed until § 2. Since **EA 1** is identical with Extra Axiom Schema 5 for TT and **EA 2** follows from Extra Axiom Schemata 1 and 2 by the intuitionistic logic, (b) implies (c). Clearly (c) implies (d), while the proof of (d)$\Rightarrow$(a) is routine. Thus, the proofs of (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d) form another proof of Strong Completeness Theorem stated in the introduction.

The author confesses that he does not know any syntactical proof of (d)$\Rightarrow$(c).

§ 2. We shall prove that (a) implies (b). In this section, provability means that in H.

To show the contraposition we assume that $\Sigma \Rightarrow \Delta$ is unprovable, and we shall construct a model $\langle \mathcal{A}, [\![\ ]\!] \rangle$ in which $\Sigma \Rightarrow \Delta$ is not valid. As in [**3**] we further assume, for simplicity, that there exist infinitely many individual free variables which do not occur in $\Sigma \Rightarrow \Delta$, and that Δ consists of one formula A.

Let $\mathcal{A}$ and $\mathcal{F}$ be the sets of all terms and all formulas, respectively.

PROPOSITION 1. *There exists a set* $\mathcal{G}$ *of formulas which satisfies the following conditions* (1)-(3):

(1) $\Sigma \subseteq \mathcal{G}$ *and* $A \notin \mathcal{G}$.

(2) *If* $\vdash \mathcal{G} \Rightarrow B_1 \vee \cdots \vee B_\nu$, *then* $B_i \in \mathcal{G}$ *for some* i.

(3) *If* $B(t) \in \mathcal{G}$ *for every* t *in* $\mathcal{A}$, *then* $\forall x B(x) \in \mathcal{G}$.

PROOF. Let $\mathcal{F}=\{F_n \mid n=1, 2, \cdots\}$. We define a pair $\mathcal{G}_n$, $\mathcal{H}_n$ of subsets of $\mathcal{F}$ inductively as follows.

Let $\mathcal{G}_1=\Sigma$ and $\mathcal{H}_1=\Delta=\{A\}$.

Assume that $\mathcal{G}_n$ and $\mathcal{H}_n$ have been defined already. CASE 1: $\vdash \mathcal{G}_n \Rightarrow \bigvee \mathcal{H}_n \vee F_n$. Set $\mathcal{G}_{n+1}=\mathcal{G}_n \cup \{F_n\}$ and $\mathcal{H}_{n+1}=\mathcal{H}_n$. CASE 2: Otherwise. Set $\mathcal{G}_{n+1}=\mathcal{G}_n$, and $\mathcal{H}_{n+1}=\mathcal{H}_n \cup \{F_n, B(a)\}$ or $\mathcal{H}_{n+1}=\mathcal{H}_n \cup \{F_n\}$ according as F_n has the form $\forall x B(x)$ or not, where a is any individual free variable which does not occur in $\mathcal{G}_n \cup \mathcal{H}_n \cup \{F_n\}$.

In view of **EA 1** we see $\nvdash \mathcal{G}_n \Rightarrow \bigvee \mathcal{H}_n$ by induction on n, and $\bigcup_{n=1}^{\infty} \mathcal{H}_n = \mathcal{F} - \bigcup_{n=1}^{\infty} \mathcal{G}_n$. So $\mathcal{G}=\bigcup_{n=1}^{\infty} \mathcal{G}_n$ is the required set. □

We assume hereafter that a set $\mathcal{G}$ of formulas satisfying (1)-(3) is given.

Now we define the relation $\leqq^{\circ}$ and $\equiv$ on $\mathcal{F}$ by

$$B \leqq^{\circ} C \Leftrightarrow B \to C \in \mathcal{G} \quad \text{and} \quad B \equiv C \Leftrightarrow B \leqq^{\circ} C \ \& \ C \leqq^{\circ} B.$$

Then $\leqq^{\circ}$ is reflexive and transitive; since for every B, C and D, $\vdash \Rightarrow B \to B$ and $\vdash B \to C, C \to D \Rightarrow B \to D$, so $B \to B \in \mathcal{G}$ and if $B \to C \in \mathcal{G}$ and $C \to D \in \mathcal{G}$ then $B \to D \in \mathcal{G}$. Hence $\equiv$ is an equivalence relation on $\mathcal{F}$. For every B in $\mathcal{F}$ we let $|B|$ be the equivalence class under $\equiv$ to which B belongs, and $\mathcal{F}/\equiv$ the set of all equivalence classes. Next we define the relation $\leqq$ on $\mathcal{F}/\equiv$ by

$$|B| \leqq |C| \Leftrightarrow B \leqq^{\circ} C \Leftrightarrow B \to C \in \mathcal{G}.$$

This is an unambiguous definition, and $\langle \mathcal{F}/\equiv, \leqq \rangle$ forms an ordered structure.

PROPOSITION 2. *$\langle \mathcal{F}/\equiv, \leqq \rangle$ is a countable linearly ordered structure with the distinct maximal element $|A \to A|$ and the minimal element $|\neg(A \to A)|$.*

PROOF. Since $\mathcal{F}$ is countably infinite, $\mathcal{F}/\equiv$ is countable. For every B and C, $\vdash \Rightarrow (B \to C) \vee (C \to B)$ by **EA 2**, and so either $B \to C \in \mathcal{G}$ or $C \to B \in \mathcal{G}$; hence $\leqq$ is linear. For every B, $\vdash \Rightarrow B \to (A \to A)$ and $\vdash \Rightarrow \neg(A \to A) \to B$, and so $B \to (A \to A) \in \mathcal{G}$ and $\neg(A \to A) \to B \in \mathcal{G}$; hence $|A \to A|$ and $|\neg(A \to A)|$ are the maximal and the minimal elements, respectively. Since $\vdash (A \to A) \to \neg(A \to A) \Rightarrow A$ and since $A \notin \mathcal{G}$, $(A \to A) \to \neg(A \to A) \notin \mathcal{G}$; so $|A \to A| \neq |\neg(A \to A)|$. □

We abbreviate $|A \to A|$ and $|\neg(A \to A)|$ by **1** and **0**, respectively.

PROPOSITION 3. *The following properties hold in $\langle \mathcal{F}/\equiv, \leqq \rangle$:*

1°) $|B \wedge C| = \min(|B|, |C|)$.

2°) $|B \vee C| = \max(|B|, |C|)$.

3°) $|B \to C| = \mathbf{1}$ *if* $|B| \leqq |C|$; $|B \to C| = |C|$ *otherwise.*

4°) $|\neg B|=\mathbf{1}$ *if* $|B|=\mathbf{0}$; $|\neg B|=\mathbf{0}$ *otherwise.*

5°) $|\exists x B(x)|=\sup\{|B(t)| \mid t\in\mathcal{A}\}$.

6°) $|\forall x B(x)|=\inf\{|B(t)| \mid t\in\mathcal{A}\}$.

7°) $|B|=\mathbf{1}\Leftrightarrow B\in\mathcal{G}$.

PROOF. 1°) From $\vdash\Rightarrow B\wedge C\to B$, $\vdash\Rightarrow B\wedge C\to C$ and $\vdash D\to B,\ D\to C\Rightarrow D\to B\wedge C$ for every D, it follows $|B\wedge C|=\inf(|B|, |C|)$, from which 1°) follows since $\leqq$ is linear.

2°) is proved similarly to 1°).

3°) From $\vdash\Rightarrow(B\to C)\wedge B\to C$ and $\vdash D\wedge B\to C\Rightarrow D\to(B\to C)$ for every D, it follows $|B\to C|=\max\{|D| \mid |D\wedge B|\leqq|C|\}$. Hence in view of 1°), follows 3°) since $\leqq$ is linear.

4°) From $\vdash\Rightarrow\neg B\wedge B\to\neg(A\to A)$ and $\vdash D\wedge B\to\neg(A\to A)\Rightarrow D\to\neg B$ for every D, it follows $|\neg B|=\max\{|D| \mid |D\wedge B|=\mathbf{0}\}$. Hence in view of 1°), follows 4°) since $\leqq$ is linear.

5°) Since $\vdash\Rightarrow B(t)\to\exists x B(x)$, $|B(t)|\leqq|\exists x B(x)|$ for every t in $\mathcal{A}$. On the other hand, for every D,

$$\begin{aligned}
&|B(t)|\leqq|D| && \text{for every } t \text{ in } \mathcal{A}\\
&\Leftrightarrow B(t)\to D\in\mathcal{G} && \text{for every } t \text{ in } \mathcal{A}\\
&\Rightarrow\forall x(B(x)\to D)\in\mathcal{G} && \text{since (3)}\\
&\Rightarrow\exists x B(x)\to D\in\mathcal{G} && \text{since } \vdash\forall x(B(x)\to D)\Rightarrow\exists x B(x)\to D\\
&\Leftrightarrow|\exists x B(x)|\leqq|D|.
\end{aligned}$$

Hence 5°) follows.

6°) is proved similarly to 5°).

7°) Since $\vdash(A\to A)\to B\Rightarrow B$ and $\vdash B\Rightarrow(A\to A)\to B$,

$$|B|=\mathbf{1}\Leftrightarrow|A\to A|\leqq|B|\Leftrightarrow(A\to A)\to B\in\mathcal{G}\Leftrightarrow B\in\mathcal{G}. \qquad \square$$

PROPOSITION 4 (Horn [**1**, Lemma 3.7]). *If $\langle L, \leqq\rangle$ is a countable linearly ordered structure with the distinct maximal and minimal elements, then there exists a monomorphism on $\langle L, \leqq\rangle$ to $\langle[0, 1]\cap\boldsymbol{Q}, \leqq\rangle$ which preserves the maximal and the minimal elements as well as all existing supremums and infimums in $\langle L, \leqq\rangle$. Hence there exists such a monomorphism on $\langle L, \leqq\rangle$ to $\langle[0, 1], \leqq\rangle$.* $\square$

By Propositions 2 and 4, there exists a monomorphism h on $\langle\mathcal{F}/\equiv, \leqq\rangle$ into $\langle[0, 1], \leqq\rangle$ which preserves the maximal and the minimal elements as well as all existing supremums and infimums in $\langle\mathcal{F}/\equiv, \leqq\rangle$. Put $[\![B]\!]=h(|B|)$ for every B in $\mathcal{F}$, and we obtain a model $\langle\mathcal{A}, [\![\ ]\!]\rangle$ by Proposition 3. Note that for every B,

$$[\![B]\!]=1 \Leftrightarrow |B|=\mathbf{1} \Leftrightarrow B \in \mathcal{G} .$$

In this model,

$$B \in \Sigma \Rightarrow B \in \mathcal{G} \Leftrightarrow [\![B]\!]=1 ,$$

while $A \notin \mathcal{G}$ so $[\![A]\!] \neq 1$; so $\Sigma \Rightarrow \Delta$ is not valid.

Thus we have found, on the assumption that $\Sigma \Rightarrow \Delta$ is unprovable, a model $\langle \mathcal{A}, [\![\ ]\!] \rangle$ in which it is not valid. Q. E. D.

References

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Department of Mathematics
Faculty of Education
Niigata University
Niigata, 950-21 Japan