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APPROXIMATIVE SHAPE I

—BASIC NOTIONS—

By

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§ 0. Introduction.

Many mathematicians discussed the classical questions of the expansions of spaces and maps into polyhedral inverse systems. For expansions of spaces Freudenthal [**9**] showed that

(i) any compact metric space X admits a polyhedral inverse sequence $\mathcal{X}$ whose inverse limit is X.

(i) has very important meanings. Because it gives us a method to investigate X by means of a polyhedral inverse sequence $\mathcal{X}$. This idea goes back to Alexandroff and Lefschetz. It is a good and fruitfull idea in topology.

Naturally we have the question: *Can we use this idea for maps*? Essentially this is divided in two questions (ii) and (iii) stated below: Let X and Y be compact metric spaces. Let $\mathcal{X}=\{X_i, p_{i,j}, N\}$ and $\mathcal{Y}=\{Y_i, q_{i,j}, N\}$ be polyhedral inverse sequences such that $\lim\mathcal{X}=X$ and $\lim\mathcal{Y}=Y$. Here $\lim\mathcal{X}$ and N denote an inverse limit of $\mathcal{X}$ and the set of all positive integers, respectively.

(ii) For any map $f:X\to Y$, is there a system map $\boldsymbol{f}:\mathcal{X}\to\mathcal{Y}$ for some $\mathcal{X}$ and $\mathcal{Y}$ such that $f=\lim\boldsymbol{f}$?

(iii) For any $\mathcal{X}$, $\mathcal{Y}$ and any map $f:X\to Y$, is there a system map $\boldsymbol{f}:\mathcal{X}\to\mathcal{Y}$ such that $f=\lim\boldsymbol{f}$?

When we handle maps by this idea, we encounter some troubles. By examples we consider the above questions. Let C, I and R be the Cantor discontinuum, the unit interval and the real line, respectively. There is an onto map $f:C\to I$.

First we consider question (iii). Let $\mathcal{C}=\{C_i, p_{ij}, N\}$ and $\mathcal{J}=\{I_i, q_{ij}, N\}$ be inverse sequences such that $C=\lim\mathcal{C}$, $I=\lim\mathcal{J}$, all C_i are finite sets, all $I_i=I$ and all q_{ij} are the identity map $1_I:I\to I$. Let $\boldsymbol{p}=\{p_i:i\in N\}:C\to\mathcal{C}$ be an inverse limit. Let all $q_i:I\to I$ be 1_I. Then $\boldsymbol{q}=\{q_i:i\in N\}:I\to\mathcal{J}$ forms an inverse limit.

We assume that there is a system map $\boldsymbol{f}=\{f, f_i:i\in N\}:\mathcal{C}\to\mathcal{J}$ such that $\lim\boldsymbol{f}=f$. Then $q_if=f_ip_{f(i)}$ for each i. Since q_i and f are onto, f_i must be

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onto. Since $C_{f(i)}$ is finite, $I=I_i=f_i(C_{f(i)})$ is also finite. This is a contradiction Hence there is no such system map. Thus, in general, question (iii) is negative.

Next we consider question (ii). We may assume that C and I are closed subsets of R. Since R is an absolute retract, there exists a map $F: R\to R$ such that $F(x)=f(x)$ for $x\in C$. We can choose polyhedral neighborhood systems $\{U_i\}$ and $\{V_i\}$ of C and I in R, respectively, such that $U_{i+1}\subset U_i$, $V_{i+1}\subset V_i$, $F(U_i)\subset V_i$ for each i and $C=\cap\{U_i : i\in N\}$, $I=\cap\{V_i : i\in N\}$. We put $f_i=F|U_i : U_i\to V_i$ for each i and $p_{ij}: U_i\to U_j$, $q_{ij}: V_i\to V_j$ are inclusion maps for $i\geqq j$. Then $\boldsymbol{f}=\{1_N, f_i : i\in N\} : \mathcal{X}=\{U_i, p_{ij}, N\}\to\mathcal{Y}=\{V_i, q_{ij}, N\}$ forms a system map and $f=\lim\boldsymbol{f}: C=\lim\mathcal{X}\to I=\lim\mathcal{Y}$. Thus in this case question (ii) is positive.

By $\dim X$ we denote the (covering) dimension of a space X. Though $\dim C=0$, in the above construction $\dim U_i=1$ for each i. We can not choose 0-dimensional neighborhoods U_i of C in R. This is a disadvantage of this method.

The questions (ii) and (iii) are positively answered in the homotopy category. They gave the ANR-systems approach and Borsuk's original approach to shape theory (see Mardešic and Segal [**18**]).

Many mathematicians considered these phenomena. How to handle the maps? The most successfull treatment is given by Mioduszewski [**19**]. He showed the existence of approximative expansions of maps into polyhedral inverse sequences. However, his description is neither simple nor categorical.

In this paper we shall give a systematic approach to approximative expansions of maps into polyhedral inverse systems. Our method is natural and categorical. To do so we need some ideas and notions which are developed in shape theory.

In §**1** we give the terminology. In §**2** we introduce approximative procategories and discuss their basic properties. In §**3** we introduce approximative resolutions for spaces. This notion is related to inverse limits. In §**4** we introduce approximative resolutions of maps. This notion is the central notion of this paper. We show that any map has an approximative resolution with respect to any approximative polyhedral resolutions. This gives a positive answer to question (iii) by approximations. In §**5** we introduce the approximative shape category. This category is analogous to the shape category. In §**6** we show that the Tychonoff functor and the completion functor induce functors on the approximative shape category. In §**7** we introduce the realization functor. Finally we show that the approximative shape category is categorically isomorphic to the topological category of complete Tychonoff spaces. This gives us a method to investigate bad spaces and bad maps by means of polyhedra and maps between them.

The principle of shape theory is to investigate bad spaces and bad maps by means of the homotopy category of polyhedra. On the other hand, *our principle of approximative shape theory is to investigate bad shapes and bad maps by means of the category of polyhedra without any homotopies.* We say that the approximative shape theory is a shape theory without homotopies.

Our theory has many applications in topology. For example we will apply it to generalized absolute neighborhood retracts, fixed point theorems, shape fibrations, UV^n-maps, Steenrod homology (see [**28**]), Čech homology (see [**28**]) and so on. These applications shall be published in the sequels.

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§ 1. Preliminaries.

All spaces and maps are topological spaces and continuous functions, respectively. For a space X $1_X : X \to X$ denotes the identity map. For a subset $X_0 \subset X$ $\mathrm{Int}\, X_0$ and $\bar{X}_0$ denote the interior of X_0 and the closure of X_0 in X, respectively.

We assume that the reader is familiar with the theory of ANRs and with shape theory. Borsuk [**5**] and Hu [**11**] are standard textbooks for the theory of ANRs. Borsuk [**6**] and Mardešic and Segal [**18**], which is quoted by MS [**18**], are standard textbooks for shape theory. Without any specification we shall use the terminology and notions from the theory of ANRs and from shape theory. For undefined terminology and notions see Hu [**11**] and MS [**18**].

TOP denotes the category of all spaces and all maps. $\mathbf{TOP}_{3.5}$, **M**, **COM** and **CM** denote the full subcategories of **TOP** consisting of all Tychonoff spaces, all metric spaces, all compact (Hausdorff) spaces and all compact metric spaces, respectively. Polyhedra denote realizations of simplicial complexes with the *CW*-topology. AR and ANR denote an absolute retract and an absolute neighborhood retract for metric spaces, respectively. **POL**, $\mathbf{POL}_f$, **AR** and **ANR** denote the full subcategories of **TOP** consisting of all polyhedra, all finite polyhedra, all ARs and all ANRs, respectively.

Without any specification *coverings* mean always *normal open coverings* (see [1], [12] and [18]). Normal open coverings are equivalent to numerable open coverings or to open coverings with a partition of unity. $\mathcal{C}ov(X)$ denotes the set of all coverings of X. Let $\mathcal{V}, \mathcal{V}'\in \mathcal{C}ov(X)$. We say that $\mathcal{V}$ is a refiement of $\mathcal{V}'$, in notation $\mathcal{V}<\mathcal{V}'$, provided that for each $U\in\mathcal{V}$ there exists $U'\in\mathcal{V}'$ with $U\subset U'$. For a subset X_0 of X we define $st(X_0, \mathcal{V})=\cup\{U\in\mathcal{V} : U\cap X_0\neq\emptyset\}$ and $\mathcal{V}|X_0=\{U\cap X_0 : U\in\mathcal{V}\}\in\mathcal{C}ov(X_0)$. We define $st\mathcal{V}=\{st(U, \mathcal{V}) : U\in\mathcal{V}\}\in\mathcal{C}ov(X)$. For each integer $n\geqq 0$ we inductively define $st^0\mathcal{V}=\mathcal{V}$ and $st^{n+1}\mathcal{V}=st(st^n\mathcal{V})$. Note that for each $\mathcal{V}\in\mathcal{C}ov(X)$ and for each positive integer n there exists $\mathcal{V}'\in\mathcal{C}ov(X)$ such that $st^n\mathcal{V}'<\mathcal{V}$. Let $\mathcal{V}_i$, $i=1, 2, \cdots, n$, be coverings of X. $\mathcal{V}_1\wedge\mathcal{V}_2\wedge\cdots\wedge\mathcal{V}_n$ denotes the covering $\{U_1\cap U_2\cap\cdots\cap U_n : U_i\in\mathcal{V}_i$ and $i=1, 2, \cdots, n\}$ of X.

Let X_0 be a subspace of X. We say that X_0 is P-embedded in X provided that for each $\mathcal{V}_0\in\mathcal{C}ov(X_0)$ there exists $\mathcal{V}\in\mathcal{C}ov(X)$ such that $\mathcal{V}|X_0<\mathcal{V}_0$ (see [1]). In *MS* [18, p. 89] such an X_0 is said to be normally embedded in X. $\dim X$ denotes the covering dimension of a space X with respect to coverings (see [22]).

Let $f, g : X\to Y$ be maps and $\mathcal{W}\in\mathcal{C}ov(Y)$. $f^{-1}\mathcal{W}$ denotes the covering $\{f^{-1}(V) : V\in\mathcal{W}\}$ of X. We say that f and g are $\mathcal{W}$-near, in notation $(f, g)<\mathcal{W}$, provided that for each $x\in X$ there exists $V\in\mathcal{W}$ such that $f(x), g(x)\in V$. $f\simeq g$ denotes that f and g are homotopic. We say that f and g are $\mathcal{W}$-homotopic provided that there exists a homotopy $h : X\times I\to Y$ such that for each $x\in X$ $h(x, 0)=f(x)$, $h(x, 1)=g(x)$ and $h(x\times I)\subset V$ for some $V\in\mathcal{W}$. Here $I=[0, 1]$ is the unit interval. $H(f)$ denotes the homotopy class of f.

HTOP, **HPOL** and **HANR** denote the homotopy categories of **TOP**, **POL** and **ANR**, respectively. H: **TOP**$\to$**HTOP** denotes the homotopy functor. **Sh** and S : **HTOP**$\to$**Sh** denote the shape category and the shape functor. Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be categories. Ob $\boldsymbol{C}$ and Mor $\boldsymbol{C}$ denote the collections of all objects and all morphisms in $\boldsymbol{C}$, respectively. When $X, Y\in$ Ob $\boldsymbol{C}$, $\boldsymbol{C}(X, Y)$ denotes the set of all morphisms from X to Y in $\boldsymbol{C}$. Sometimes $X\in\boldsymbol{C}$ means $X\in$ Ob $\boldsymbol{C}$. When Ob $\boldsymbol{D}\subset$ Ob $\boldsymbol{C}$, $\boldsymbol{C}(\boldsymbol{D})$ denotes the full subcategory of $\boldsymbol{C}$ consisting of Ob $\boldsymbol{D}$. From our notations **Sh**(**CM**) is the shape category on compact metric spaces.

A preordering $>$ on a set A is a binary relation on A which is reflexive and transitive, i. e., (i) $a>a$ for each $a\in A$ and (ii) both $a>a'$ and $a'>a''$ imply that $a>a''$. We say that a preordered set $(A, >)$ is directed provided that for any $a, a'\in A$ there exists $a''\in A$ with $a''>a, a'$. We do not assume the antisymmetry condition: (iii) Both $a'>a$ and $a>a'$ imply $a'=a$. We say that a directed set $(A, >)$ is cofinite provided that for any $a\in A$ $P(a)=\{a'\in A : a>a'\}$ is a finite set. Let $(B, >)$ be a directed set. Let $s, t : A\to B$ be functions. $s>t$ means that $s(a)>t(a)$ for each $a\in A$. We say that s is an increasing function provided that $s(a')>s(a)$ for $a'>a$. We can easily show the following.

(1.1) LEMMA. *Let* $(A, >)$ *and* $(B, >)$ *be directed sets. Let* $s_i: A \to B$, $i=1, 2, \cdots, n$, *be functions. If* $(A, >)$ *is cofinite, then there exists an increasing function* $s: A \to B$ *such that* $s > s_i$ *for each* i. ■

The mark ■ denotes the end of a proof or of an example. When it appears just after a statement of a theorem, a proposition or a corollary, it means that the statement is obviously valid.

§ 2. The approximative pro-category.

In this section we introduce the notion of approximative pro-categories. This notion plays a fundamental role in our theory. It has a role similar to that of pro-categories in shape theory (see MS [18]).

Let $\boldsymbol{C}$ be a subcategory of **TOP**. We say that $(\mathcal{X}, \mathcal{U}) = \{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ is an approximative inverse system in $\boldsymbol{C}$ provided that it satisfies the following three conditions:

(AI1) $\mathcal{X} = \{X_a, p_{a',a}, A\}$ is an inverse system in $\boldsymbol{C}$, and A is cofinite and directed.

(AI2) For each $a \in A$ $\mathcal{U}_a$ is a covering of X_a satisfying that $p_{a',a}^{-1}\mathcal{U}_a > \mathcal{U}_{a'}$ for $a' > a$.

(AI3) For each $a \in A$ and for each $\mathcal{U} \in \mathcal{C}ov(X_a)$ there exists $a' > a$ such that $p_{a',a}^{-1}\mathcal{U} > \mathcal{U}_{a'}$.

Let $(\mathcal{Y}, \mathcal{V}) = \{(Y_b, \mathcal{V}_b), q_{b',b}, B\}$ be an approximative inverse system in $\boldsymbol{C}$. We say that $\boldsymbol{f} = \{f, f_b : b \in B\} : (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ is an approximative system map in $\boldsymbol{C}$ provided that $f: B \to A$ is a function and $f_b: X_{f(b)} \to Y_b$ is a map in $\boldsymbol{C}$ for each $b \in B$ satisfying the following two conditions:

(AM1) $f_b^{-1}\mathcal{V}_b > \mathcal{U}_{f(b)}$ for $b \in B$.

(AM2) For each $b' > b$ there exists $a > f(b), f(b')$ such that $(q_{b',b} f_{b'} p_{a,f(b')}, f_b p_{a,f(b)}) < \mathcal{V}_b$.

Sometimes we refer to approximative inverse systems in $\boldsymbol{C}$ and approximative system maps in $\boldsymbol{C}$ as to approximative $\boldsymbol{C}$-inverse systems and approximative $\boldsymbol{C}$-system maps, respectively.

Let $(\mathcal{Z}, \mathcal{W}) = \{(Z_c, \mathcal{W}_c), r_{c',c}, C\}$ be an approximative inverse system in $\boldsymbol{C}$ and $\boldsymbol{g} = \{g, g_c : c \in C\} : (\mathcal{Y}, \mathcal{V}) \to (\mathcal{Z}, \mathcal{W})$ an approximative system map in $\boldsymbol{C}$. We define $\boldsymbol{gf} = \{fg, g_c f_{g(c)} : c \in C\}$. In general, $\boldsymbol{gf}$ is not an approximative system map from $(\mathcal{X}, \mathcal{U})$ to $(\mathcal{Z}, \mathcal{W})$ in $\boldsymbol{C}$. Therefore we need some tricks.

(2.1) LEMMA. ***gf*** *forms an approximative system map from* $(\mathscr{X}, \mathcal{U})$ *to* $st(\mathscr{Z}, \mathcal{W})=\{(Z_c, st\mathcal{W}_c), r_{c',c}, C\}$ *in* ***C***.

To prove (2.1) we need the following:

(2.2) LEMMA. *Let* $f: X\to Y$ *be a map. Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be coverings of* X *and* Y, *respectively. If* $f^{-1}\mathcal{V}>\mathcal{U}$, *then* $f^{-1}st^n\mathcal{V}>st^n\mathcal{U}$ *for each integer* $n\geqq 0$. ■

Proof of (2.1). First we show (AI1)-(AI3) for $st(\mathscr{Z}, \mathcal{W})$. (AI1) is trivial and (AI2) follows from (2.2). We show (AI3). Take any $c\in C$ and any $\mathcal{W}\in \mathcal{C}ov(Z_c)$. There exists $\mathcal{W}'\in \mathcal{C}ov(Z_c)$ such that $st\mathcal{W}'<\mathcal{W}$. By (AI3) for $(\mathscr{Z}, \mathcal{W}')$ there exists $c'>c$ such that $r_{c',c}^{-1}\mathcal{W}'>\mathcal{W}_{c'}$. By (2.2) we have that $r_{c',c}^{-1}\mathcal{W}>r_{c',c}^{-1}st\mathcal{W}'>st\mathcal{W}_{c'}$. This means (AI3) for $st(\mathscr{Z}, \mathcal{W})$. Hence $st(\mathscr{Z}, \mathcal{W})$ forms an approximative inverse system.

Next we show that $\boldsymbol{gf}: (\mathscr{X}, \mathcal{U})\to st(\mathscr{Z}, \mathcal{W})$ is an approximative system map in ***C***. We show (AM1). Take any $c\in C$. By (AM1) for $\boldsymbol{f}$ and $\boldsymbol{g}$ $f_{g(c)}^{-1}g_c^{-1}\mathcal{W}_c> f_{g(c)}^{-1}\mathcal{V}_{g(c)}>\mathcal{U}_{fg(c)}$ and then by (2.2) $(g_cf_{g(c)})^{-1}st\mathcal{W}_c>st\mathcal{U}_{fg(c)}>\mathcal{U}_{fg(c)}$. This means (AM1) for $\boldsymbol{gf}$.

We show (AM2). Take any $c'>c$. By (AM2) for $\boldsymbol{g}$ there exists $b>g(c), g(c')$ such that

(1) $(g_cq_{b,g(c)}, r_{c',c}g_{c'}q_{b,g(c')})<\mathcal{W}_c$.

Since $b>g(c), g(c')$, by (AM2) for $\boldsymbol{f}$ there exists $a>fg(c), fg(c'), f(b)$ such that

(2) $(f_{g(c)}p_{a,fg(c)}, q_{b,g(c)}f_bp_{a,f(b)})<\mathcal{V}_{g(c)}$ and

(3) $(f_{g(c')}p_{a,fg(c')}, q_{b,g(c')}, f_bp_{a,f(b)})<\mathcal{V}_{g(c')}$.

By (2), (3) and (AM1) for $\boldsymbol{g}$

(4) $(g_cf_{g(c)}p_{a,fg(c)}, g_cq_{b,g(c)}f_bp_{a,f(b)})<\mathcal{W}_c$ and

(5) $(g_{c'}f_{g(c')}p_{a,fg(c')}, g_{c'}q_{b,g(c')}f_bp_{a,f(b)})<\mathcal{W}_{c'}$.

By (AI2) for $(\mathscr{Z}, \mathcal{W})$ and (5)

(6) $(r_{c',c}g_{c'}f_{g(c')}p_{a,fg(c')}, r_{c',c}g_{c'}q_{b,g(c')}f_bp_{a,f(b)})<\mathcal{W}_c$.

By (1)

(7) $(g_cq_{b,g(c)}f_bp_{a,f(b)}, r_{c',c}g_{c'}q_{b,g(c')}f_bp_{a,f(b)})<\mathcal{W}_c$.

By (4), (6) and (7)

(8) $(g_cf_{g(c)}p_{a,fg(c)}, r_{c',c}g_{c'}f_{g(c')}p_{a,fg(c')})<st\mathcal{W}_c$.

(8) means (AM2) for $\boldsymbol{gf}$. ■

Let $\boldsymbol{f}'=\{f', f'_b : b\in B\} : (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ and $\boldsymbol{g}'=\{g', g'_c : c\in C\} : (\mathcal{Y}, \mathcal{V})\to(\mathcal{Z}, \mathcal{W})$ be approximative system maps in $\boldsymbol{C}$. We say that $\boldsymbol{f}$ and $\boldsymbol{f}'$ are simply approximatively equivalent, in notation $\boldsymbol{r}=:\boldsymbol{f}'$, provided that for each $b\in B$ there exists $a>f(b)$, $f'(b)$ such that $(f_b p_{a,f(b)}, f'_b p_{a,f'(b)})<\mathcal{V}_b$. We say that $\boldsymbol{f}$ and $\boldsymbol{f}'$ are approximatively equivalent, in notation $\boldsymbol{f}\equiv:\boldsymbol{f}'$, provided that there exists a finite collection of approximative system maps $\boldsymbol{f}_i : (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ in $\boldsymbol{C}$, $i=1, 2, \cdots, n$, such that $\boldsymbol{f}=\boldsymbol{f}_1$, $\boldsymbol{f}'=\boldsymbol{f}_n$ and $\boldsymbol{f}_i=:\boldsymbol{f}_{i+1}$ for $i=1, 2, \cdots, n-1$. Obviously this relation $\equiv:$ forms an equivalence relation. $[\boldsymbol{f}]$ denotes the equivalence class of $\boldsymbol{f}$.

$1_{(\mathcal{X},\mathcal{U})}=\{1_A, 1_{X_a} : a\in A\} : (\mathcal{X}, \mathcal{U})\to(\mathcal{X}, \mathcal{U})$ is the identity approximative system map. Let $s : A\to A$ be an increasing function with $s>1_A$. We define $\boldsymbol{p}(s)=\{s, p_{s(a),a} : a\in A\}$.

(2.3) LEMMA. *$\boldsymbol{p}(s) : (\mathcal{X}, \mathcal{U})\to(\mathcal{X}, \mathcal{U})$ forms an approximative system map in $\boldsymbol{C}$ and $\boldsymbol{p}(s)=:1_{(\mathcal{X},\mathcal{U})}$.* ■

We say that s is an n-refinement function of $(\mathcal{X}, \mathcal{U})$ provided that $p^{-1}_{s(a),a}\mathcal{U}_a>st^n\mathcal{U}_{s(a)}$ for $a\in A$.

(2.4) LEMMA. *If s and s' are n-refinement and m-refinement functions of $(\mathcal{X}, \mathcal{U})$, respectively, then $s's$ is an $(n+m)$-refinement function of $(\mathcal{X}, \mathcal{U})$.* ■

(2.5) LEMMA. *Any approximative inverse system in $\boldsymbol{C}$ has an n-refinement function for each integer $n\geqq 0$.*

PROOF. We show that $(\mathcal{X}, \mathcal{U})$ has an n-refinement function. Since each $\mathcal{U}_a$ is a normal open covering, there exists $\mathcal{U}'_a\in \mathcal{C}ov(X_a)$ such that $st^n\mathcal{U}'_a<\mathcal{U}_a$. By (AI3) there exists a function $s' : A\to A$ such that $s'>1_A$ and $p^{-1}_{s'(a),a}\mathcal{U}'_a>\mathcal{U}_{s'(a)}$ for $a\in A$. By (1.1) there exists an increasing function $s : A\to A$ with $s>s'$. Thus by (AI2) and (2.2) $p^{-1}_{s(a),a}\mathcal{U}_a>p^{-1}_{s(a),a}st^n\mathcal{U}'_a=p^{-1}_{s(a),s'(a)}p^{-1}_{s'(a),a}st^n\mathcal{U}'_a>p^{-1}_{s(a),s'(a)}st^n\mathcal{U}_{s'(a)}>st^n\mathcal{U}_{s(a)}$ for each $a\in A$. Then s is the required n-refinement. ■

Let $t : B\to B$ be an increasing function with $t>1_B$. By (2.3) $\boldsymbol{q}(t) : (\mathcal{Y}, \mathcal{V})\to(\mathcal{Y}, \mathcal{V})$ is an approximative system map. From the definitions and (2.5) it is not difficult to show the following two lemmas:

(2.6) LEMMA. *$\boldsymbol{q}(t)\boldsymbol{f} : (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ forms an approximative system map in $\boldsymbol{C}$ and $\boldsymbol{q}(t)\boldsymbol{f}=:\boldsymbol{f}$.* ■

(2.7) LEMMA. *$\boldsymbol{f} \equiv : \boldsymbol{f}'$ iff there exists an increasing function $t: B \to B$ such that $t > 1_B$ and $\boldsymbol{q}(t)\boldsymbol{f} = : \boldsymbol{q}(t)\boldsymbol{f}'$.* ■

(2.8) LEMMA. *Let $u, u': C \to C$ be 1-refinement functions of $(\mathcal{Z}, \mathcal{W})$.*

(i) *$\boldsymbol{r}(u)(\boldsymbol{gf})$ forms an approximative system map from $(\mathcal{X}, \mathcal{U})$ to $(\mathcal{Z}, \mathcal{W})$ in $\boldsymbol{C}$.*

(ii) *$\boldsymbol{r}(u)(\boldsymbol{gf}) \equiv : \boldsymbol{r}(u')(\boldsymbol{gf})$.*

(iii) *If $\boldsymbol{f} = : \boldsymbol{f}'$, then $\boldsymbol{r}(u)(\boldsymbol{gf}) = : \boldsymbol{r}(u)(\boldsymbol{gf}')$.*

(iv) *If $\boldsymbol{g} = : \boldsymbol{g}'$, then $\boldsymbol{r}(u)(\boldsymbol{gf}) = : \boldsymbol{r}(u)(\boldsymbol{g}'\boldsymbol{f})$.*

PROOF. We show (ii). By (2.5) there exists a 2-refinement function $u'': C \to C$ of $(\mathcal{Z}, \mathcal{W})$. We show that $\boldsymbol{r}(u'')(\boldsymbol{r}(u)(\boldsymbol{gf})) = : \boldsymbol{r}(u'')(\boldsymbol{r}(u')(\boldsymbol{gf}))$. Take any $c \in C$ and then

(1) $r^{-1}_{u''(c),c}\mathcal{W}_c > st^2\mathcal{W}_{u''(c)}$.

Take any $c' > uu''(c)$, $u'u''(c)$. Since $\boldsymbol{gf}: (\mathcal{X}, \mathcal{U}) \to st(\mathcal{Z}, \mathcal{W})$ is an approximative system map by (2.1), there exists $a > fg(c')$, $fguu''(c)$, $fgu'u''(c)$ such that

(2) $(g_{uu''(c)} f_{guu''(c)} p_{a, fguu''(c)}, r_{c', uu''(c)} g_{c'} f_{g(c')} p_{a, fg(c')}) < st\mathcal{W}_{uu''(c)}$ and

(3) $(g_{u'u''(c)} f_{gu'u''(c)} p_{a, fgu'u''(c)}, r_{c', u'u''(c)} g_{c'} f_{g(c')} p_{a, fg(c')}) < st\mathcal{W}_{u'u''(c)}$.

By (AI2) and (2.2) $r^{-1}_{uu''(c), u''(c)} st\mathcal{W}_{u''(c)} > st\mathcal{W}_{uu''(c)}$ and $r^{-1}_{u'u''(c), u''(c)} st\mathcal{W}_{u''(c)} > st\mathcal{W}_{u'u''(c)}$. Thus by (2) and (3)

(4) $(r_{uu''(c), u''(c)} g_{uu''(c)} f_{guu''(c)} p_{a, fguu''(c)},$
$\quad r_{u'u''(c), u''(c)} g_{u'u''(c)} f_{gu'u''(c)} p_{a, fgu'u''(c)}) < st^2\mathcal{W}_{u''(c)}$.

By (1) and (4)

(5) $(r_{uu''(c), c} g_{uu''(c)} f_{guu''(c)} p_{a, fguu''(c)}, r_{u'u''(c), c} g_{u'u''(c)} f_{gu'u''(c)} p_{a, fgu'u''(c)}) < \mathcal{W}_c$.

(5) means that $\boldsymbol{r}(u'')(\boldsymbol{r}(u)(\boldsymbol{gf})) = : \boldsymbol{r}(u'')(\boldsymbol{r}(u')(\boldsymbol{gf}))$. Hence by (2.7) $\boldsymbol{r}(u)(\boldsymbol{gf}) \equiv : \boldsymbol{r}(u')(\boldsymbol{gf})$. We have (ii). By similar ways as for (ii) we can prove the other assertions. ■

(2.9) COROLLARY. *If $\boldsymbol{f} \equiv : \boldsymbol{f}'$, $\boldsymbol{g} \equiv : \boldsymbol{g}'$ and $u, u': C \to C$ are 1-refinement functions of $(\mathcal{Z}, \mathcal{W})$, then $\boldsymbol{r}(u)(\boldsymbol{gf}) \equiv : \boldsymbol{r}(u')(\boldsymbol{g}'\boldsymbol{f}')$.* ■

Now we introduce a composition of equivalence classes of approximative system maps as follows: $[\boldsymbol{g}][\boldsymbol{f}] = [\boldsymbol{r}(u)(\boldsymbol{gf})]$ for a 1-refinement function u. By (2.9) this notion is well defined and does not depend on u. It is not difficult to show that

(2.10) LEMMA. *For any approximative system maps* $\boldsymbol{f}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$, $\boldsymbol{g}:(\mathcal{Y}, \mathcal{V})\to(\mathcal{Z}, \mathcal{W})$ *and* $\boldsymbol{h}:(\mathcal{Z}, \mathcal{W})\to(\mathcal{E}, \mathcal{K})$ *in* $\boldsymbol{C}$

(i) $[\boldsymbol{f}]=[\boldsymbol{f}][1_{(\mathcal{X},\mathcal{U})}]=[1_{(\mathcal{Y},\mathcal{V})}][\boldsymbol{f}]$ *and*

(ii) $([\boldsymbol{h}][\boldsymbol{g}])[\boldsymbol{f}]=[\boldsymbol{h}]([\boldsymbol{g}][\boldsymbol{f}])$. ■

We define the approximative pro-category of $\boldsymbol{C}$, in notation Appro-$\boldsymbol{C}$, as follows: Objects are all approximative inverse systems in $\boldsymbol{C}$. Morphisms from $(\mathcal{X}, \mathcal{U})$ to $(\mathcal{Y}, \mathcal{V})$ are equivalence classes of all approximative system maps from $(\mathcal{X}, \mathcal{U})$ to $(\mathcal{Y}, \mathcal{V})$ in $\boldsymbol{C}$. Obviously the collection of all morphisms from $(\mathcal{X}, \mathcal{U})$ to $(\mathcal{Y}, \mathcal{V})$ forms a set. The composition is defined above. This composition is associative and $[1_{(\mathcal{X},\mathcal{U})}]$ is the identity morphism of $(\mathcal{X}, \mathcal{U})$ by (2.10). Hence we may summarize the above results as follow.

(2.11) THEOREM. *Appro-$\boldsymbol{C}$ forms a category.*

Now we consider the properties of Appro-$\boldsymbol{C}$.

(2.12) PROPOSITION. *Let* $(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ *be an approximative inverse system in* $\boldsymbol{C}$. *If* A' *is a cofinal subset of* A; *then* $(\mathcal{X}, \mathcal{U})_{A'}=\{(\mathcal{X}_a, \mathcal{U}_a), p_{a',a}, A'\}$ *forms an approximative inverse system in* $\boldsymbol{C}$ *and is isomorphic to* $(\mathcal{X}, \mathcal{U})$ *in Appro-$\boldsymbol{C}$.*

(2.13) PROPOSITION. *If* $(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ *and* $(\mathcal{X}, \mathcal{V})=\{(X_a, \mathcal{V}_a), p_{a',a}, A\}$ *are approximative inverse systems in* $\boldsymbol{C}$, *then* $(\mathcal{X}, \mathcal{U})$ *and* $(\mathcal{X}, \mathcal{V})$ *are isomorphic in Appro-$\boldsymbol{C}$.*

Proofs of (2.12) and (2.13). We show (2.13). By (AI3) and (1.1) there exist increasing functions $m, n: A\to A$, $m, n>1_A$, such that $p^{-1}_{m(a),a}\mathcal{V}_a>\mathcal{U}_{m(a)}$ and $p^{-1}_{n(a),a}\mathcal{U}_a>\mathcal{V}_{n(a)}$ for $a\in A$. By these conditions $\boldsymbol{p}(m)=\{m, p_{m(a),a}: a\in A\}:(\mathcal{X}, \mathcal{U})\to(\mathcal{X}, \mathcal{V})$ and $\boldsymbol{p}(n)=\{n, p_{n(a),a}: a\in A\}:(\mathcal{X}, \mathcal{V})\to(\mathcal{X}, \mathcal{U})$ form approximative system maps. It is easy to show that $[\boldsymbol{p}(n)][\boldsymbol{p}(m)]=[1_{(\mathcal{X},\mathcal{U})}]$ and $[\boldsymbol{p}(m)][\boldsymbol{p}(n)]=[1_{(\mathcal{X},\mathcal{V})}]$. Hence we have (2.13). By a similar way we have (2.12). ■

In (2.1) we defined $st(\mathcal{X}, \mathcal{U})$. Inductively we define $st^n(\mathcal{X}, \mathcal{U})$ for each integer $n\geqq 0$ as follows: $st^0(\mathcal{X}, \mathcal{U})=(\mathcal{X}, \mathcal{U})$ and $st^{n+1}(\mathcal{X}, \mathcal{U})=st(st^n(\mathcal{X}, \mathcal{U}))$. By (2.1) and (2.13) we have that

(2.14) COROLLARY. *For each integer* $n\geqq 0$ $st^n(\mathcal{X}, \mathcal{U})$ *forms an approximative inverse system in* $\boldsymbol{C}$ *which is isomorphic to* $(\mathcal{X}, \mathcal{U})$ *in Appro-$\boldsymbol{C}$.* ■

We say that $(a, b)\in A\times B$ is an admissible pair of $\boldsymbol{f}$ provided that $a>f(b)$. Let (a', b') and (a, b) be admissible pairs of $\boldsymbol{f}$. We say that $(a', b')>(a, b)$ provided that both $a'>a$ and $b'>b$.

We say that $\boldsymbol{f}$ is a special approximative system map provided that $A=B$, $f=1_A: B=A\to A$ and it satisfies the following condition:

(SPAM) $(f_a p_{a',a}, q_{a',a} f_{a'})<\mathcal{V}_a$ for $a'>a$.

(2.15) THEOREM. *Let $\boldsymbol{f}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ be an approximative system map. Then there exist approximative inverse systems $(\mathcal{X}, \mathcal{U})'=\{(X'_c, \mathcal{U}'_c), p'_{c',c}, C\}$, $(\mathcal{Y}, \mathcal{V})'=\{(Y'_c, \mathcal{V}'_c), q'_{c',c}, C\}$, approximative system maps $\boldsymbol{s}:(\mathcal{X}, \mathcal{U})\to(\mathcal{X}, \mathcal{U})'$, $\boldsymbol{t}:(\mathcal{Y}, \mathcal{V})\to(\mathcal{Y}, \mathcal{V})'$, and an approximative special system map $\boldsymbol{g}=\{1_C, g_c: c\in C\}:(\mathcal{X}, \mathcal{U})'\to(\mathcal{Y}, \mathcal{V})'$ satisfying the following conditions:*

(i) $[\boldsymbol{g}][\boldsymbol{s}]=[\boldsymbol{t}][\boldsymbol{f}]$.

(ii) *$[\boldsymbol{s}]$ and $[\boldsymbol{t}]$ are isomorphisms in Appro-$\boldsymbol{C}$.*

(iii) *all $(X'_c, \mathcal{U}'_c)$, $p'_{c',c}$, $(Y'_b, \mathcal{V}'_b)$ and $q'_{b',b}$ are some $(X_a, \mathcal{U}_a)$, $p_{a',a}$, $(Y_b, \mathcal{V}_b)$ and $q_{b',b}$, respectively.*

(iv) *all g_c are composition of some $p_{a',a}$ and f_b.*

PROOF. Since B is cofinite, there exists an increasing function $g: B\to A$ such that

(1) $g>f$ and

(2) $(q_{b',b} f_{b'} p_{g(b'),f(b')}, f_b p_{g(b'),f(b)})<\mathcal{V}_b$ for $b'>b$.

We put $g'_b=f_b p_{g(b),f(b)}: X_{g(b)}\to Y_b$ for $b\in B$. Then by (2) $\boldsymbol{g}'=\{g, g'_b: b\in B\}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ forms an approximative system map and $\boldsymbol{f}=:\boldsymbol{g}'$.

We put $C=\{(a, b)\in A\times B: a>g(b)\}$ and define an order "$>$" in C as follows: $c'=(a', b')>c=(a, b)$ iff both $a'>a$ and $b'>b$. Then $(C, >)$ forms a cofinite directed set. Let $X'_c=X_a$, $\mathcal{U}'_c=\mathcal{U}_a$, $Y'_c=Y_b$ and $\mathcal{V}'_c=\mathcal{V}_b$ for $c=(a, b)\in C$. Let $p'_{c',c}=p_{a',a}$ and $q'_{c',c}=q_{b',b}$ for $c'=(a', b')>c=(a, b)$. It is easy to show that $(\mathcal{X}, \mathcal{U})'$ and $(\mathcal{Y}, \mathcal{V})'$ form approximative inverse systems. We put $g_c=f_b p_{a,f(b)}: X'_c=X_a\to Y_b=Y'_c$ for $c=(a, b)\in C$. By (2) $\boldsymbol{g}=\{1_C, g_c: c\in C\}:(\mathcal{X}, \mathcal{U})'\to(\mathcal{Y}, \mathcal{V})'$ forms an approximative special system map.

We define $\boldsymbol{s}=\{s, s_c: c\in C\}:(\mathcal{X}, \mathcal{U})\to(\mathcal{X}, \mathcal{U})'$ as follows: Define $s: C\to A$ by $s(c)=a$ for $c=(a, b)$ and $s_c=1_{X_a}: X_{s(c)}=X_a\to X'_c=X_a$ for $c=(a, b)$. Then clearly $\boldsymbol{s}$ forms an approximative system map.

We will now show the $\boldsymbol{s}$ induces an isomorphism in Appro-$\boldsymbol{C}$. To do so take any increasing function $d: A\to B$. Then $gd: A\to A$ is an increasing function. By (1.1) there exists an increasing function $e: A\to A$ such that $e>gd$ and $e>1_A$. We define an increasing function $u: A\to C$ by $u(a)=(e(a), d(a))$ and we put $u_a=p_{e(a),a}:X'_{u(a)}=X_{e(a)}\to X_a$ for $a\in A$. Then $\boldsymbol{u}=\{u,\ u_a:a\in A\}:(\mathcal{X},\ \mathcal{U})'\to(\mathcal{X},\ \mathcal{U})$ forms an approximative system map. It is easy to show that $\boldsymbol{us}=:1_{(\mathcal{X},\mathcal{U})}$ and $\boldsymbol{su}=:1_{(\mathcal{X},\mathcal{U})'}$. Hence $\boldsymbol{s}$ induces an isomorphism in Appro-$\boldsymbol{C}$.

We define $\boldsymbol{t}=\{t,\ t_c:c\in C\}:(\mathcal{Y},\ \mathcal{V})\to(\mathcal{Y},\ \mathcal{V})'$ as follows: Define $t:C\to B$ by $t(c)=b$ for $c=(a,\ b)$ and $t_c=1_{Y_b}:Y_{t(c)}=Y_b\to Y'_c=Y_b$ for $c=(a,\ b)$. Then $\boldsymbol{t}$ forms an approximative system map. In the same way as for $\boldsymbol{s}$, we see that $\boldsymbol{t}$ forms an isomorphism in Appro-$\boldsymbol{C}$. Since $\boldsymbol{tf}=:\boldsymbol{tg}'=:\boldsymbol{gs}$, it is easy to show that $[\boldsymbol{t}][\boldsymbol{f}]=[\boldsymbol{g}][\boldsymbol{s}]$. Hence $\boldsymbol{g}$ is the required one. ■

(2.16) THEOREM. *Let* $\boldsymbol{f}:(\mathcal{X},\ \mathcal{U})\to(\mathcal{Y},\ \mathcal{V})$ *be an approximative system map in* $\boldsymbol{C}$. *Then* $\boldsymbol{f}$ *induces an isomorphism in Appro-*$\boldsymbol{C}$ *iff it satisfies the following condition:*

(ISO) *For each admissible pair* $(a,\ b)$ *of* $\boldsymbol{f}$ *there exist an admissble pair* $(a',\ b')>(a,\ b)$ *and a map* $k:Y_{b'}\to X_a$ *in* $\boldsymbol{C}$ *such that*

(ISO1) $(p_{a',a},\ kf_{b'}p_{a',f(b')})<\mathcal{U}_a$,

(ISO2) $k^{-1}\mathcal{U}_a>\mathcal{V}_b$, *and*

(ISO3) $(q_{b',b},\ f_bp_{a,f(b)}k)<st\,\mathcal{V}_b$.

PROOF. First we assume that $\boldsymbol{f}$ induces an isomorphism in Appro-$\boldsymbol{C}$. Then there exists an approximative system map $\boldsymbol{h}=\{h,\ h_a:a\in A\}:(\mathcal{Y},\ \mathcal{V})\to(\mathcal{X},\ \mathcal{U})$ in $\boldsymbol{C}$ such that $[\boldsymbol{h}][\boldsymbol{f}]=[1_{(\mathcal{X},\mathcal{U})}]$ and $[\boldsymbol{f}][\boldsymbol{h}]=[1_{(\mathcal{Y},\mathcal{V})}]$. By the definition of composition and (2.7) there exist 1-refinement functions $s:A\to A,\ t:B\to B$ of $(\mathcal{X},\ \mathcal{U}),\ (\mathcal{Y},\ \mathcal{V})$, respectively, and increasing functions $u:A\to A,\ v:B\to B$ such that $u>1_A,\ v>1_B$,

(1) $\boldsymbol{p}(u)(\boldsymbol{p}(s)(\boldsymbol{hf}))=:\boldsymbol{p}(u)1_{(\mathcal{X},\mathcal{U})}$ and

(2) $\boldsymbol{q}(v)(\boldsymbol{q}(t)(\boldsymbol{fh}))=:\boldsymbol{q}(v)1_{(\mathcal{Y},\mathcal{V})}$.

We show (ISO). Take any admissible pair $(a,\ b)$ of f. By (AI3) and (2.5) there exist $a_1>a$ and $b_1>b$ such that $p_{a_1,a}^{-1}\mathcal{U}_a>st\,\mathcal{U}_{a_1}$ and $q_{b_1,b}^{-1}\mathcal{V}_b>st\,\mathcal{V}_{b_1}$. By (2) there exists $b_2>hftv(b_1),\ v(b_1)$ such that

(3) $(q_{tv(b_1),b_1}f_{tv(b_1)}h_{ftv(b_1)}q_{b_2,hftv(b_1)},\ q_{b_2,b_1})<\mathcal{V}_{b_1}$.

By (AM2) there exists $a_2>ftv(b_1),\ f(b_1),\ a_1$ such that

(4) $(f_bp_{a_2,f(b)},\ q_{b_1,b}f_{b_1}p_{a_2,f(b_1)})<\mathcal{V}_b$ and

(5) $(f_{b_1}p_{a_2,f(b_1)},\ q_{tv(b_1),b_1}f_{tv(b_1)}p_{a_2,ftv(b_1)})<\mathcal{V}_{b_1}$.

By (1) there exists $a_3>fhsu(a_2),\ u(a_2)$ such that

(6) $(p_{a_3,a_2},\ p_{su(a_2),a_2}h_{su(a_2)}f_{hsu(a_2)}p_{a_3,fhsu(a_2)})<\mathcal{U}_{a_2}$.

By (AM2) there exists $b_3>b_2,\ hsu(a_2)$ such that

(7) $(h_{ftv(b_1)}q_{b_3,hftv(b_1)},\ p_{su(a_2),ftv(b_1)}h_{su(a_2)}q_{b_3,hsu(a_2)})<\mathcal{U}_{ftv(b_1)}$.

By (AM2) there exists $a_4>a_3,\ f(b_3)$ such that

(8) $(f_{hsu(a_2)}p_{a_4,fhsu(a_2)},\ q_{b_3,hsu(a_2)}f_{b_3}p_{a_4,f(b_3)})<\mathcal{V}_{hsu(a_2)}$.

From (3)-(8) it is not difficult to show that the admissible pair (a_4, b_3) of $\boldsymbol{f}$ and the map $k=p_{su(a_2),a}h_{su(a_2)}q_{b_3,hsu(a_2)}:Y_{b_3}\to X_a$ satisfy (ISO1)-(ISO3) for (a, b). Hence we have (ISO).

Next we assume (ISO) and show that $\boldsymbol{f}$ induces an isomorphism in Appro-$\boldsymbol{C}$. We use the same notations as in the proof of (2.15). Since $\boldsymbol{f}$ satisfies (ISO), $\boldsymbol{g}$ in (2.15) satisfies the following Claim 1:

Claim 1. $\boldsymbol{g}$ satisfies the following condition:

(ISO)′ For each $c\in C$ there exist $m(c)>c$ and a map $k_{m(c),c}:Y'_{m(c)}\to X'_c$ satisfying

(ISO1)′ $(p'_{m(c),c},\ k_{m(c),c}g_{m(c)})<\mathcal{U}'_{m(c)}$,

(ISO2)′ $k^{-1}_{m(c),c}\mathcal{U}'_c>\mathcal{V}'_{m(c)}$.

(ISO3)′ $(q'_{m(c),c},\ g_c k_{m(c),c})<st\,\mathcal{V}'_c$.

Let $w:C\to C$ be a 3-refinement function of $(\mathcal{X}, \mathcal{U})'$ and put $k_c=p'_{w(c),c}k_{mw(c),w(c)}:Y'_{mw(c)}\to X'_c$ for $c\in C$. By straightforward computations and (ISO1)′-(ISO3)′ we have Claim 2:

Claim 2. $\boldsymbol{k}=\{mw,\ k_c:c\in C\}:(\mathcal{Y}, \mathcal{V})'\to(\mathcal{X}, \mathcal{U})'$ forms an approximative system map in $\boldsymbol{C}$.

Claim 3. $[\boldsymbol{k}][\boldsymbol{g}]=[1_{(\mathcal{X},\mathcal{U})'}]$ and $[\boldsymbol{g}][\boldsymbol{k}]=[1_{(\mathcal{Y},\mathcal{V})'}]$.

Take any 1-refinement function $i:C\to C$ of $(\mathcal{X}, \mathcal{U})'$ and any $c\in C$. Since $(p'_{mwii(c),c},\ p'_{wii(c),c}k_{mwii(c),wii(c)}g_{mwii(c)})<\mathcal{U}'_c$ by (ISO)′, $\boldsymbol{p}'(i)1_{(\mathcal{X},\mathcal{U})'}=:\boldsymbol{p}'(i)(\boldsymbol{p}'(i)(\boldsymbol{kg}))$ and hence $[1_{(\mathcal{X},\mathcal{U})'}]=[\boldsymbol{k}][\boldsymbol{g}]$. In the same way as above we have $[1_{(\mathcal{Y},\mathcal{V})'}]=[\boldsymbol{g}][\boldsymbol{k}]$. Thus we have Claim 3.

By Claims 2 and 3 $[\boldsymbol{g}]$ is an isomorphism in Appro-$\boldsymbol{C}$. Hence $[\boldsymbol{f}]$ is an isomorphism in Appro-$\boldsymbol{C}$ by (2.15). ■

(2.17) COROLLARY. *Let* $\boldsymbol{g}=\{1_A,\ g_a:a\in A\}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ *be an approximative special system map in* $\boldsymbol{C}$. *Then* $\boldsymbol{g}$ *induces an isomorphism in Appro-*$\boldsymbol{C}$ *iff it satisfies the following condition:*

(ISO)′ *For each* $a\in A$ *there exist* $a'>a$ *and a map* $k:Y_{a'}\to X_a$ *in* $\boldsymbol{C}$ *such that*

(ISO1)′ $(p_{a',a},\ kg_{a'})<\mathcal{U}_a$,

(ISO2)′ $k^{-1}\mathcal{U}_a>\mathcal{V}_{a'}$, *and*

(ISO3)′ $(q_{a',a},\ g_a k)<st\,\mathcal{V}_a$. ■

(2.18) COROLLARY. *Let* $\boldsymbol{f}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ *be an approximative system map in* $\boldsymbol{C}$. *Then* $\boldsymbol{f}$ *induces an isomorphism in Appro-*$\boldsymbol{C}$ *iff it satisfies the following two condition:*

(MO) *For each* $a\in A$ *there exist an admissible pair* (a_1, b_1) *of* $\boldsymbol{f}$ *with* $a_1>a$ *and a map* $k:Y_{b_1}\to X_a$ *in* $\boldsymbol{C}$ *such that*

(MO1) $(p_{a_1,a}, kf_{b_1}p_{a_1,f(b_1)})<\mathcal{U}_a$ *and*

(MO2) $k^{-1}\mathcal{U}_a>\mathcal{V}_{b_1}$.

(EP) *For each admissible pair* (a, b) *of* $\boldsymbol{f}$ *there exist* $b_1>b$ *and a map* $m:Y_{b_1}\to X_a$ *in* $\boldsymbol{C}$ *such that*

(EP1) $(q_{b_1,b}, f_b p_{a,f(b)}m)<st\,\mathcal{V}_b$.

PROOF. Trivially (ISO) implies (MO) and (EP). We assume (MO) and (EP), and show (ISO). Take any admissible pair (a, b) of $\boldsymbol{f}$. Then there exists $a_1>a$ such that $p_{a_1,a}^{-1}\mathcal{U}_a>st^3\mathcal{U}_{a_1}$. By (MO) there exist an admissible pair (a_2, b_1) of $\boldsymbol{f}$ with $a_2>a_1$ and a map $k:Y_{b_1}\to X_{a_1}$ in $\boldsymbol{C}$ such that

(1) $(p_{a_2,a_1}, kf_{b_1}p_{a_2,f(b_1)})<\mathcal{U}_{a_1}$ and

(2) $k^{-1}\mathcal{U}_{a_1}>\mathcal{V}_{b_1}$.

There exist $b_2>b_1$, b such that $q_{b_2,b}^{-1}\mathcal{V}_b>st\,\mathcal{V}_{b_2}$, and $a_3>a_2$, $f(b_2)$ such that

(3) $(f_{b_1}p_{a_3,f(b_1)}, q_{b_2,b_1}f_{b_2}p_{a_3,f(b_2)})<\mathcal{V}_{b_1}$ and

(4) $(f_b p_{a_3,f(b)}, q_{b_2,b}f_{b_2}p_{a_3,f(b_2)})<\mathcal{V}_b$.

By (EP) there exist $b_3>b_2$ and a map $m:Y_{b_3}\to X_{a_3}$ in $\boldsymbol{C}$ such that

(5) $(q_{b_3,b_2}, f_{b_2}p_{a_3,f(b_2)}m)<st\,\mathcal{V}_{b_2}$.

There exist $b_4>b_3$ such that $q_{b_4,b_3}^{-1}m^{-1}\mathcal{U}_{a_3}>\mathcal{V}_{b_4}$, and $a_4>a_3$, $f(b_4)$ such that

(6) $(f_{b_2}p_{a_4,f(b_2)}, q_{b_4,b_2}f_{b_4}p_{a_4,f(b_4)})<\mathcal{V}_{b_2}$.

From (1)-(6) it is not difficult to show that the admissible pair (a_4, b_4) and the map $r=p_{a_3,a}mq_{b_4,b_3}:Y_{b_4}\to X_a$ satisfy (ISO1)-(ISO3) for (a, b). Thus (MO) and (EP) imply (ISO) and hence by (2.15) we have (2.18). ■

We say that an approximative system map $\boldsymbol{f}=\{f, f_b : b\in B\}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ is commutative provided that it satisfies the following condition:

(CAM) For each $b, b'\in B$ with $b'>b$ there exists $a>f(b), f(b')$ such that $f_b p_{a,f(b)}=q_{b',b}f_{b'}p_{a,f(b')}$.

(2.19) COROLLARY. *Let* $\boldsymbol{f}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ *be an approximative commutative system map in* $\boldsymbol{C}$. *Then* $\boldsymbol{f}$ *induces an isomorphism in Appro-*$\boldsymbol{C}$ *iff it satisfies the* *following:*

(MO)′ *For each* $a \in A$ *there exist an admissible pair* (a_1, b_1) *of* $\boldsymbol{f}$ *with* $a_1 > a$ *and a map* $k: Y_{b_1} \to X_a$ *satisfying* (MO1).

(EP)′ *For each admissible pair* (a, b) *of* $\boldsymbol{f}$ *there exists* $b_1 > b$ *and a map* $m: Y_{b_1} \to X_a$ *such that*

(EP1)′ $(q_{b_1, b}, f_b p_{a, f(b)} m) < \mathcal{V}_b$.

PROOF. We will show that (MO) and (MO)′ are equivalent. Trivially (MO) implies (MO)′. We assume (MO)′ and show (MO). Take any $a \in A$ and then by (MO)′ there exist an admissible pair (a_1, b_1) of $\boldsymbol{f}$ and a map $k: Y_{b_1} \to X_a$ such that $a_1 > a$ and

(1) $(p_{a_1, a}, k f_{b_1} p_{a_1, f(b_1)}) < \mathcal{U}_a$.

By (AI3) there exists $b_2 > b_1$ such that $q_{b_2, b_1}^{-1} k^{-1} \mathcal{U}_a > \mathcal{V}_{b_2}$. By (CAM) there exists $a_2 > a_1$, $f(b_2)$ such that

(2) $f_{b_1} p_{a_2, f(b_1)} = q_{b_2, b_1} f_{b_2} p_{a_2, f(b_2)}$.

From (1) and (2) the admissible pair (a_2, b_2) and the map $r = k q_{b_2, b_1}$ satisfy (MO1) and (MO2) for a. Hence (MO) and (MO)′ are equivalent. In a similar way we can show that (EP) and (EP)′ are equivalent. Hence by (2.18) we have (2.19). ■

(2.20) COROLLARY. *Let* $\boldsymbol{f}: (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ *be an approximative commutative system map in* $\boldsymbol{C}$. *Then* $\boldsymbol{f}$ *induces an isomorphism in Appro-*$\boldsymbol{C}$ *iff it satisfies the following condition:*

(ISO)″ *For each admissible pair* (a, b) *of* $\boldsymbol{f}$ *there exist an admissible pair* $(a', b') > (a, b)$ *and a map* $k: Y_{b'} \to X_a$ *such that* $(p_{a', a}, k f_{b'} p_{a', f(b')}) < \mathcal{U}_a$ *and* $(q_{b', b}, f_b p_{a, f(b)} k) < \mathcal{V}_b$. ■

(2.21) REMARK. If $\boldsymbol{f}$ satisfies (MO), then $[\boldsymbol{f}]$ is a monomorphism in Appro-$\boldsymbol{C}$. If $\boldsymbol{f}$ satisfies (EP), then $[\boldsymbol{f}]$ is an epimorphism in Appro-$\boldsymbol{C}$. ■

(2.22) REMARK. Grothendieck introduced the notion of pro-categories (see MS [**18**, pp. 1-17]) and used it in algebraic geometry. Artin and Mazur used it to study etale homotopy. It plays a fundamental role in shape theory (see MS [**18**]). Artin and Mazur showed the re-indexing theorem (see MS [**18**, p. 12]) in pro-categories which corresponds to (2.15). In pro-categories Morita showed the diagonal theorem (see MS [**18**, p. 112]) which corresponds to (2.16).

§ 3. Approximative resolutions of spaces.

In this section we introduce the notion of an approximative resolution of a space. Mardešic [15] introduced the notion of a resolution of a space. Our notion improves his notion.

We say that a space X is an approximative polyhedron, in notation AP, provided that for each $\mathcal{U}\in \mathcal{C}ov(X)$ there exist a polyhedron P and maps $f:X\to P$, $g:P\to X$ such that $(gf, 1_X)<\mathcal{U}$. **AP** denotes the full subcategory of **TOP** consisting of all APs. Mardešic [15] introduced this notion and showed that

(3.1) LEMMA. (i) *Any ANR and any polyhedron are APs.*

(ii) *Let X be a paracompact space with* $\dim X=n<\infty$. *If X is LC^{n-1} (see* [11]) *then X is an AP.* ■

Let $\mathcal{X}=\{X_a, p_{a',a}, A\}$ be an inverse system in **TOP**. Let $\boldsymbol{p}=\{p_a: a\in A\}$ be a collection of maps $p_a: X\to X_a$, $a\in A$. We say that $\boldsymbol{p}: X\to\mathcal{X}$ is a system map provided that $p_a=p_{a',a}p_{a'}$ for $a'>a$. We say that a system map $\boldsymbol{p}:X\to\mathcal{X}$ is a resolution of X (see [15]) provided that it satisfies the following two conditions:

(R1) Let P be an AP, $\mathcal{V}\in\mathcal{C}ov(P)$ and $f:X\to P$ a map. Then there exist $a\in A$ and a map $f_a: X_a\to P$ such that $(f, f_a p_a)<\mathcal{V}$.

(R2) Let P be an AP and $\mathcal{V}\in\mathcal{C}ov(P)$. Then there exists $\mathcal{V}'\in\mathcal{C}ov(P)$ with the following property: If $a\in A$ and $f, f': X_a\to P$ are maps such that $(fp_a, f'p_a)<\mathcal{V}'$, then there exists $a'>a$ such that $(fp_{a',a}, f'p_{a',a})<\mathcal{V}$.

(3.2) LEMMA (Mardešic [15]). *$\boldsymbol{p}:X\to\mathcal{X}$ is a resolution of X iff* (R1) *and* (R2) *are fulfilled for all polyhedra P, or equivalently for all ANRs P.* ■

Let $(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ be an approximative inverse system in **TOP**. We say that $\boldsymbol{p}=\{p_a: a\in A\}: X\to(\mathcal{X}, \mathcal{U})$ is an approximative resolution of X provided that $\boldsymbol{p}:X\to\mathcal{X}=\{X_a, p_{a',a}, A\}$ is a system map and it satisfies the following two conditions:

(AR1) For each $\mathcal{U}\in\mathcal{C}ov(X)$ there exists $a\in A$ such that $p_a^{-1}\mathcal{U}_a<\mathcal{U}$.

(AR2) For each $a\in A$ there exists $a'>a$ such that $p_{a',a}(X_{a'})\subset st(p_a(X), \mathcal{U}_a)$.

(3.3) THEOREM. *$\boldsymbol{p}:X\to(\mathcal{X}, \mathcal{U})$ is an approximative resolution iff $\boldsymbol{p}:X\to\mathcal{X}$ is a resolution.*

(3.4) THEOREM. *$\boldsymbol{p}:X\to\mathcal{X}$ is a resolution iff it satisfies the following two conditions:*

(B1) *For each* $\mathcal{U}\in Cov(X)$ *there exist* $a\in A$ *and* $\mathcal{U}'\in Cov(X_a)$ *such that* $p_a^{-1}\mathcal{U}'<\mathcal{U}$.

(B4) *For each* $a\in A$ *and for each* $\mathcal{U}\in Cov(X_a)$ *there exists* $a'>a$ *such that* $p_{a',a}(X_{a'})\subset st(p_a(X), \mathcal{U})$.

We can easily show (3.3) by (3.4). The author [26] has proved (3.4). Our proof was a slight modification of Mardešic [15]. After that Mardešic [16] gave another simple proof of (3.4). His proof is already published and therefore we omit our proof. Recently Morita [23] showed that resolutions and proper inverse systems (see [21]) are equivalent.

Bacon [4] introduced the notion of complements. We say that a system map $\boldsymbol{p}: X\to\mathcal{X}$ is a complement of X provided that it satisfies (B1) and the following condition:

(B2) For each $a\in A$ and for each open set V in X_a with $p_a(X)\subset V$, there exists $a'>a$ such that $p_{a',a}(X_{a'})\subset V$.

Mardešic [15] considered the following condition:

(B3) For each $a\in A$ and for each open set V in X_a with $\overline{p_a(X)}\subset V$, there exists $a'>a$ such that $p_{a',a}(X_{a'})\subset V$.

(3.5) LEMMA. (i) (B2) *is stronger than* (B3), *and* (B3) *is stronger than* (B4).
(ii) (B3) *and* (B4) *are equivalent, when all* X_a *are normal* (*Hausdorff*) *spaces.*

PROOF. Since the first assertion in (i) is trivial, we show the second one in (i). Take any $a\in A$ and any $\mathcal{U}\in Cov(X_a)$. Since $\mathcal{U}$ is an open covering, $\overline{p_a(X)}\subset st(p_a(X), \mathcal{U})$. By (B3) there exists $a'>a$ such that $p_{a',a}(X_{a'})\subset st(p_a(X), \mathcal{U})$. Then (B4) holds and hence we have (i).

We show (ii). Take any $a\in A$ and open set V in X_a such that $\overline{p_a(X)}\subset V$. Since X_a is normal, by Theorem 1 of MS [18, p. 324] $\mathcal{W}=\{V, X_a-\overline{p_a(X)}\}$ is a normal open covering of X_a. Since $st(p_a(X), \mathcal{W})=V$, by (B4) there exists $a'>a$ such that $p_{a',a}(X_{a'})\subset st(p_a(X), \mathcal{W})=V$. Then (B3) holds and hence we have (ii). ■

(3.6) COROLLARY (Mardešic [15]). (i) *Any complement is a resolution.*
(ii) *When all* X_a *are normal spaces,* $\boldsymbol{p}: X\to\mathcal{X}$ *is a resolution iff it satisfies* (B1) *and* (B3). ■

Now we construct approximative resolutions from resolutions.

(3.7) PROPOSITION. *Let* $\boldsymbol{q}=\{q_b: b\in B\}: X\to\mathcal{Y}=\{Y_b, q_{b',b}, B\}$ *be a resolution. Then there exist an approximative resolution* $\boldsymbol{p}=\{p_a: a\in A\}: X\to(\mathcal{X}, \mathcal{U})=$

$\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ and an increasing function $s: A \to B$ satisfying the following conditions:

(i) A is cofinite, directed and antisymmetric.

(ii) $X_a = Y_{s(a)}$, $p_a = q_{s(a)}$ for $a \in A$ and $P_{a',a} = q_{s(a'),s(a)}$ for $a' > a$.

(iii) For any $b \in B$ and any $\mathcal{V} \in \mathcal{C}ov(Y_b)$ there exists $a \in A$ such that $s(a) = b$ and $\mathcal{U}_a = \mathcal{V}$.

PROOF. Let $F(B) = \{(b, \mathcal{V}) : b \in B$ and $\mathcal{V} \in \mathcal{C}ov(Y_b)\}$ and $M(B) = \{K \subset F(B) : K$ is finite and $K \neq \emptyset\}$. The set $A = M(B)$ is ordered by inclusion and trivially satisfies (i). Take a function $t: A \to B$ such that

(1) $t(a) = b$ for $a = \{(b, \mathcal{V})\} \in A$.

Since A is cofinite, by (1) and (1.1) there exists an increasing function $s: A \to B$ such that

(2) $s > t$ and $s(a) = b$ for $a = \{(b, \mathcal{V})\} \in A$.

We put $X_a = Y_{s(a)}$, $p_a = q_{s(a)}$ for $a \in A$ and $p_{a',a} = q_{s(a'),s(a)}$ for $a' > a$. Since s is an increasing function, $\mathcal{X} = \{X_a, p_{a',a}, A\}$ forms an inverse system. From the definitions (ii) is trivial and $\boldsymbol{p} = \{p_a : a \in A\} : X \to \mathcal{X}$ forms a system map.

Claim 1. $\boldsymbol{p}: X \to \mathcal{X}$ is a resolution of X.

We show (R1). Take any AP P, any $\mathcal{V} \in \mathcal{C}ov(P)$ and any map $f: X \to P$. By (R1) for $\boldsymbol{q}$ there exist $b \in B$ and a map $f_b: Y_b \to P$ such that $(f, f_b q_b) < \mathcal{V}$. Put $a = \{(b, \{Y_b\})\} \in A$ and then $X_a = Y_b$ and $p_a = q_b$ by (2). When we put $h = f_b : X_a = Y_b \to P$, $(f, hp_a) < \mathcal{V}$. This means (R1) for $\boldsymbol{p}$.

We show (R2). Take any AP P and any $\mathcal{V} \in \mathcal{C}ov(P)$. There exists $\mathcal{V}' \in \mathcal{C}ov(P)$ satisfying property (R2) for $\boldsymbol{q}$ and $\mathcal{V}$. Take any $a \in A$ and maps $f, f': X_a \to P$ such that $(fp_a, f'p_a) < \mathcal{V}'$. Then $(fq_{s(a)}, f'q_{s(a)}) < \mathcal{V}'$. By the choice of $\mathcal{V}'$ there exists $b' > s(a)$ such that $(fq_{b',s(a)}, f'q_{b',s(a)}) < \mathcal{V}$. Put $a' = a \cup \{(b', \{Y_{b'}\})\} \in A$ and then $s(a') > s(\{(b', \{Y_{b'}\})\}) = b'$ by (2). Thus $(fq_{s(a'),s(a)}, f'q_{s(a'),s(a)}) < \mathcal{V}$, that is, $(fp_{a',a}, f'p_{a',a}) < \mathcal{V}$. This means (R2) for $\boldsymbol{p}$. Hence $\boldsymbol{p}$ is a resolution.

We define coverings as follows: Take any $a = \{(b_1, \mathcal{V}_1), \cdots, (b_n, \mathcal{V}_n)\} \in A$. Since $s(a) > s(\{(b_i, \mathcal{V}_i)\}) = b_i$ by (2), we may put $\mathcal{U}_a = q_{s(a),b_1}^{-1}\mathcal{V}_1 \wedge \cdots \wedge q_{s(a),b_n}^{-1}\mathcal{V}_n \in \mathcal{C}ov(X_a)$.

Claim 2. $(\mathcal{X}, \mathcal{U}) = \{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ forms an approximative inverse system.

We show (AI1)-(AI3). (AI1) is trivial. We show (AI2). Take any $a' > a$ and put $a' = a \cup \{(b_{n+1}, \mathcal{V}_{n+1}), \cdots, (b_m, \mathcal{V}_m)\}$. Then

$$p_{a',a}^{-1}\mathcal{U}_a=q_{s(a'),s(a)}^{-1}(q_{s(a),b_1}^{-1}\mathcal{V}_1\wedge\cdots\wedge q_{s(a),b_n}^{-1}\mathcal{V}_n)$$
$$=q_{s(a'),b_1}^{-1}\mathcal{V}_1\wedge\cdots\wedge q_{s(a'),b_n}^{-1}\mathcal{V}_n>q_{s(a'),b_1}^{-1}\mathcal{V}_1\wedge\cdots$$
$$\wedge q_{s(a'),b_n}^{-1}\mathcal{V}_n\wedge\cdots\wedge q_{s(a'),b_m}^{-1}\mathcal{V}_m=\mathcal{U}_{a'}.$$

This means (AI2). We show (AI3). Take any $a\in A$ and any $\mathcal{U}\in\mathcal{C}ov(X_a)$. Put $a'=a\cup\{(s(a),\mathcal{U})\}\in A$. Then

$$\mathcal{U}_{a'}=q_{s(a'),b_1}^{-1}\mathcal{V}_1\wedge\cdots\wedge q_{s(a'),b_n}^{-1}\mathcal{V}_n\wedge q_{s(a'),s(a)}^{-1}\mathcal{U}<q_{s(a'),s(a)}^{-1}\mathcal{U}=p_{a',a}^{-1}\mathcal{U}.$$

This means (AI3). Hence we have Claim 2.

By (3.3), Claims 1 and 2 $\boldsymbol{p}:X\to(\mathcal{X},\mathcal{U})$ is an approximative resolution. For each $b\in B$ and $\mathcal{V}\in\mathcal{C}ov(Y_b)$ we put $a=\{(b,\mathcal{V})\}\in A$. By definition $s(a)=b$ and $\mathcal{U}_a=\mathcal{V}$. Then $\boldsymbol{p}$ satisfies (iii). ■

(3.8) PROPOSITION. *Let $\mathcal{X}=\{X_a, p_{a',a}, A\}$ be an inverse system. If all X_a are compact metric spaces, and A is infinite and cofinite, then there exist coverings $\mathcal{U}_a$ of X_a such that $(\mathcal{X},\mathcal{U})=\{(X_a,\mathcal{U}_a), p_{a',a}, A\}$ forms an approximative inverse system.*

PROOF. Since X_a is compact metric, there exist coverings $\mathcal{V}_{a,i}$ of X_a, $i=1, 2, \cdots$, such that

(1) $\mathcal{V}_{a,i}>\mathcal{V}_{a,i+1}$ for $i=1, 2, \cdots$, and

(2) for each $\mathcal{V}'\in\mathcal{C}ov(X_a)$ there exists i such that $\mathcal{V}'>\mathcal{V}_{a,i}$.

Since A is cofinite, $P(a)=\{a'\in A: a'<a\}$ is a finite set for $a\in A$. Put $P(a)=\{a_1, a_2, \cdots, a_n\}$ and define $\mathcal{U}_a=p_{a,a_1}^{-1}\mathcal{V}_{a_1,n}\wedge\cdots\wedge p_{a,a_n}^{-1}\mathcal{V}_{a_n,n}$ for $a\in A$. We show that $\mathcal{U}_a$ have all the required properties. We show (AI2). Take any $a'>a$ and put $P(a')=P(a)\cup\{a_{n+1}, \cdots, a_m\}$. By (1)

$$p_{a',a}^{-1}\mathcal{U}_a=p_{a',a_1}^{-1}\mathcal{V}_{a_1,n}\wedge\cdots\wedge p_{a',a_n}^{-1}\mathcal{V}_{a_n,n}>p_{a',a_1}^{-1}\mathcal{V}_{a_1,m}\wedge\cdots$$
$$\wedge p_{a',a_n}^{-1}\mathcal{V}_{a_n,m}>p_{a',a_1}^{-1}\mathcal{V}_{a_1,m}\wedge\cdots\wedge p_{a',a_m}^{-1}\mathcal{V}_{a_m,m}$$
$$=\mathcal{U}_{a'}.$$

This means (AI2).

We show (AI3). Take any $a\in A$ and any $\mathcal{U}\in\mathcal{C}ov(X_a)$. By (2) there exists n' such that $\mathcal{U}>\mathcal{V}_{a,n'}$. Since A is infinite, there exists $a'>a$ such that the cardinality of $P(a')=m\geqq n'$. Put $P(a')=\{a, a_1, \cdots, a_{m-1}\}$. Then we have that

$$\mathcal{U}_{a'}=p_{a',a}^{-1}\mathcal{V}_{a,m}\wedge p_{a',a_1}^{-1}\mathcal{V}_{a_1,m}\wedge\cdots\wedge p_{a',a_{m-1}}^{-1}\mathcal{V}_{a_{m-1},m}$$
$$<p_{a',a}^{-1}\mathcal{V}_{a,m}<p_{a',a}^{-1}\mathcal{V}_{a,n'}<p_{a',a}^{-1}\mathcal{U}.$$

This means (AI3). Hence $(\mathcal{X},\mathcal{U})$ forms an approximative inverse system. ■

(3.9) LEMMA. *Let $\boldsymbol{p}: X \to \mathcal{X} = \{X_a, p_{a',a}, A\}$ be a resolution of X. If all X_a are compact metric, and A is infinite and cofinite, then there exist coverings $\mathcal{U}_a$ of X_a such that $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U}) = \{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ forms an approximative resolution of X.*

(3.10) LEMMA. *Let $\boldsymbol{p} = \{p_a ; a \in A\}: X \to (\mathcal{X}, \mathcal{U})$ be an approximative resolu tion. If A' is a cofinal subset of A, then $\boldsymbol{p}_{A'} = \{p_a : a \in A'\} : X \to (\mathcal{X}, \mathcal{U})_{A'}$ forms an approximative resolution of X.*

(3.9) follows from (3.3) and (3.8). (3.10) follows from (2.12) and (3.3). ■

Let $\boldsymbol{C}$ be a subcategory of **TOP**. Let $\mathcal{K}$ be a collection of spaces. We say that a resolution $\boldsymbol{p}: X \to \mathcal{X}$ and an approximative resolution $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$ are a $\boldsymbol{C}$-resolution and an approximative $\boldsymbol{C}$-resolution provided that $\mathcal{X}$ is an inverse system in $\boldsymbol{C}$, respectively. We say that $\boldsymbol{p}: X \to \mathcal{X}$ and $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$ are rigid for $\mathcal{K}$ provided that they satisfy the following condition:

(R1)* For any map $f: X \to P$, where $P \in \mathcal{K}$, there exist $a \in A$ and a map $h: X_a \to P$ with $f = hP_a$.

When $\mathcal{K} = \mathrm{Ob}\,\boldsymbol{C}$, we say that they are rigid for $\boldsymbol{C}$. When we take **ANR**, **AP** and **POL** as $\boldsymbol{C}$, we have **POL**-resolutions, approximative **AP**-resolutions, rigidness for **ANR** and so on.

We quote some results on resolutions and inverse limits.

(3.11) LEMMA (Bacon [4] and Mardešic [15]). (i) *Any space X admits a polyhedral complement $\boldsymbol{p}: X \to \mathcal{X}$.*

(ii) *Any space admits an* **ANR**-*resolution which is rigid for* **ANR**. ■

Let X be a subset of a space M. Let $\mathcal{U}(X, M)$ be the inverse system consisting of all neighborhoods of X in M and inclusion maps as bonding maps. Let $\boldsymbol{p}: X \to \mathcal{U}(X, M)$ be the system map consisting of all inclusion maps. We say that $\boldsymbol{p}: X \to \mathcal{U}(X, M)$ is the complete neighborhoods system of X in M. By (3.4) we easily show that

(3.12) LEMMA. *If either X is P-embedded in M or M is hereditarily paracompact, then the complete neighborhoods system $\boldsymbol{p}: X \to \mathcal{U}(X, M)$ is a resolution.* ■

(3.13) LEMMA (Mardešic [15]). *Let $\mathcal{X}$ be an inverse system of compact spaces. Then any inverse limit $\boldsymbol{p}: X \to \mathcal{X}$ is a resolution.* ■

(3.14) LEMMA (Freudenthal [9], Eilenberg-Steenrod [7] and Mardešic [14]).

(i) *Any compact metric space X is an inverse limit of a finite polyhedral inverse sequence $\mathcal{X}$.*

(ii) *Any compact space X is an inverse limit of a finite polyhedral inverse system $\mathcal{X}$.*

In (i) *and* (ii) *we can achieve that dimensions of all spaces in* $\mathcal{X} \leqq \dim X$. ■

The following theorem gives existences of various approximative resolutions of spaces.

(3.15) THEOREM. (i) *Any space X admits an approximative* **POL**-*resolution* $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{V})$.

(ii) *Any space admits an approximative* **ANR**-*resolution, which is rigid for* **ANR.**

(iii) *Any compact space X admits an approximative* $\mathbf{POL}_f$-*resolution* $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{V})$.

(iv) *Any compact metric space X admits an approximative* $\mathbf{POL}_f$-*resolution* $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{V})$ *such that $\mathcal{X}$ is an inverse sequence.*

In (i), (iii) *and* (iv) *we can achieve that dimensions of all spaces in* $\mathcal{X} \leqq \dim X$.

PROOF. We show (i). Let $\mathcal{C}ov_1(X) = \{\mathcal{V} \in \mathcal{C}ov(X): \text{order of } \mathcal{V} \leqq \dim X + 1\}$. Since $\mathcal{C}ov_1(X)$ is cofinal in $\mathcal{C}ov(X)$, by the same way as in Bacon [**4**] we can show (i) in (3.11) with the property: Dimensions of all spaces in $\mathcal{X} \leqq \dim X$. Thus by (3.6) and (3.7) we have the required polyhedral resolution. Hence we have (i). (ii) follows from (3.7) and (ii) in (3.11). (iii) and (iv) follows from (3.7) and (3.14). ■

(3.16) REMARK. MS [**18**] introduced resolutions for pairs and showed (i) in (3.11) for pairs. Mardešic [**16**] characterized resolutions for pairs in a way similar to (3.4) and showed (ii) in (3.11) for pairs. Since (3.7) is true for resolutions for pairs, (3.15) holds for pairs (see Watanabe [**28**]).

(3.17) EXAMPLE. Let **PM** be the full subcategory of **TOP** consisting of all paracompact M-space (see Arhangelskiǐ [**2, 3**] and Morita [**20**]). Nagata [**24**] gave a characterization of these spaces as follows: A space X is a paracompact M-space iff X embeds as a closed subset in $M \times C$, where M is metric and C is compact. Metric spaces and compact spaces are paracompact M-spaces. AR(**PM**) and ANR(**PM**) denote the full subcategories of **TOP** consisting of all absolute retracts and all absolute neighborhood retracts for **PM**, respectively. Mardešic and Šostak [**17**] showed that

(i) any paracompact M-space X embeds as a closed set in an $M \in \mathrm{AR}(\mathbf{PM})$,

(ii) if X is a closed subset of an $M \in \mathrm{AR}(\mathbf{PM})$, then any neighborhood U of X in M contains an open neighborhood $V \in \mathrm{ANR}(\mathbf{PM})$ of X in M, and

(iii) any $X \in \mathrm{ANR}(\mathbf{PM})$ has the homotopy type of a polyhedron.

Modifying their proof of (iii) (see the proof of (5.7) in § **5**) we easily show that

(iv) any $X \in \mathrm{ANR}(\mathbf{PM})$ is an AP.

Let X be a paracompact M-space. By (i) X is a closed subset of an $M \in \mathrm{AR}(\mathbf{PM})$. Since X is P-embedded in M, by (3.12) the complete neighborhoods system $\boldsymbol{p}: X \to \mathcal{U}(X, M)$ is a resolution. Let $\mathcal{A}\mathcal{U}(X, M)$ be the inverse system consisting of all neighborhoods $V \in \mathrm{ANR}(\mathbf{PM})$ of X in M. By (ii) $\mathcal{A}\mathcal{U}(X, M)$ is a cofinal inverse sub-system of $\mathcal{U}(X, M)$. Then $\boldsymbol{p}$ induces an $\mathrm{ANR}(\mathbf{PM})$-resolution $\boldsymbol{p}: X \to \mathcal{A}\mathcal{U}(X, M)$. By (3.7) $\boldsymbol{p}$ induces an approximative $\mathrm{ANR}(\mathbf{PM})$-resolution $\boldsymbol{p}: X \to (\mathcal{A}\mathcal{U}(X, M), \mathcal{U})$ consisting of $\mathrm{ANR}(\mathbf{PM})$-neighborhoods of X in M and inclusion maps. Obviously $\boldsymbol{p}: X \to \mathcal{A}\mathcal{U}(X, M)$ and $\boldsymbol{p}: X \to (\mathcal{A}\mathcal{U}(X, M), \mathcal{U})$ are rigid for $\mathrm{ANR}(\mathbf{PM})$.

Let X be a metric space. By the Kuratowski-Wojdislawski Theorem (see Hu [**11**]) we may assume that X is a closed subset of an AR M. By (3.6) and (3.12) the complete neighborhoods system $\boldsymbol{p}: X \to \mathcal{U}(X, M)$ is a resolution. Let $\mathcal{O}\mathcal{U}(X, M)$ be the inverse system of all open neighborhoods of X in M. Then $\boldsymbol{p}$ induces an **ANR**-resolution $\boldsymbol{p}: X \to \mathcal{O}\mathcal{U}(X, M)$ and an approximative **ANR**-resolution $\boldsymbol{p}: X \to (\mathcal{O}\mathcal{U}(X, M), \mathcal{U})$. Obviously these are rigid for **ANR**.

Let X be a compact space with weight m. Then X is embedded in I^m. Here I^m is the product space of m-copies of the unit interval $I = [0, 1]$. By (3.6) and (3.12) the complete neighborhoods system $\boldsymbol{p}: X \to \mathcal{U}(X, I^m)$ is a resolution. We say that a subset K of I^m is a prism provided that K is homeomorphic to $P \times I^m$, where P is a finite polyhedron. We easily show that

(v) any prism is an $\mathrm{ANR}(\mathbf{COM})$ and an AP.

Let $\mathcal{P}\mathcal{U}(X, I^m)$ be the inverse system consisting of all prism neighborhoods of X in I^m. Then $\boldsymbol{p}$ induces an $\mathrm{ANR}(\mathbf{COM})$-resolution $\boldsymbol{p}: X \to \mathcal{P}\mathcal{U}(X, I^m)$ and an approximative $\mathrm{ANR}(\mathbf{COM})$-resolution $\boldsymbol{p}: X \to \mathcal{P}\mathcal{U}(X, I^m), \mathcal{U})$. These are rigid for $\mathrm{ANR}(\mathbf{COM})$).

When X is compact metric, X is embedded in the Hilbert cube $Q = I^\infty$. In this case $\boldsymbol{p}: X \to \mathcal{O}\mathcal{U}(X, Q)$, $\boldsymbol{p}: X \to (\mathcal{O}\mathcal{U}(X, Q), \mathcal{U})$, $\boldsymbol{p}: X \to \mathcal{P}\mathcal{U}(X, Q)$ and $\boldsymbol{p}: X \to$

$(\mathcal{PU}(X, Q), \mathcal{U})$ are **ANR**-resolutions and approximative **ANR**-resolutions, which are rigid for **ANR**. ∎

These special resolutions and approximative resolutions for special spaces are usefull in the sequel.

§4. Approximative resolutions of maps.

In this section we introduce the notion of an approximative resolution of a map and study its fundamental properties. Mardešic [15] introduced the notion of resolutions of maps. Our notion improves his notion.

Let X, Y be spaces and $f: X\to Y$ a map. Let $\boldsymbol{p}=\{p_a : a\in A\} : X\to(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ and $\boldsymbol{q}=\{q_b : b\in B\} : Y\to(\mathcal{Y}, \mathcal{V})=\{(Y_b, \mathcal{V}_b), q_{b',b}, B\}$ be approximative resolutions. Let $\boldsymbol{f}=\{f, f_b : b\in B\}$, $\boldsymbol{f}'=\{f', f'_b : b\in B\} : (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ be approximative system maps. We say that $\boldsymbol{f} : (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ is an approximative resolution of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$ provided that for each $b\in B$ $(q_b f, f_b p_{f(b)})<\mathcal{V}_b$.

(4.1) LEMMA. *Let $\boldsymbol{q} : Y\to(\mathcal{Y}, \mathcal{V})$ be an approximative* **AP**-*resolution. If $\boldsymbol{f}, \boldsymbol{f}' : (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ are approximative resolutions of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$, then $\boldsymbol{f}\equiv:\boldsymbol{f}'$.*

To prove (4.1) we need (4.2), which follows from (AI2), (AI3) and (1.1).

(4.2) LEMMA. *Let $(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ be an approximative inverse system. Let $\mathcal{U}'_a\in Cov(X_a)$ for $a\in A$. Then there exists an increasing function $s: A\to A$ such that $s>1_A$ and $p^{-1}_{s(a),a}\mathcal{U}'_a>\mathcal{U}_{s(a)}$ for $a\in A$.* ∎

PROOF OF (4.1). Since $\boldsymbol{q}$ is an approximative **AP**-resolution, all Y_b are APs. By (3.3) $\boldsymbol{p} : X\to\mathcal{X}$ is a resolution. For each $b\in B$ there exists $\mathcal{V}'_b\in Cov(Y_b)$ satisfying the property of (R2) for $\boldsymbol{p}$ and $\mathcal{V}_b$. By (4.2) there exists an increasing function $t: B\to B$ such that $t>1_B$ and

(1) $q^{-1}_{t(b),b}\mathcal{V}'_b>\mathcal{V}_{t(b)}$ for $b\in B$.

Take any 1-refinement function $u: B\to B$ of $(\mathcal{Y}, \mathcal{V})$ and any $b\in B$. Since $\boldsymbol{f}$ and $\boldsymbol{f}'$ are approximative resolutions of f, we have that $(q_{ut(b)}f, f_{ut(b)}p_{fut(b)})<\mathcal{V}_{ut(b)}$ and $(q_{ut(b)}f, f'_{ut(b)}p_{f'ut(b)})<\mathcal{V}_{ut(b)}$. Then $(f_{ut(b)}p_{fut(b)}, f'_{ut(b)}p_{f'ut(b)})<st\,\mathcal{V}_{ut(b)}$. Since u is a·1-refinement function,

(2) $(q_{ut(b),t(b)}f_{ut(b)}p_{fut(b)}, q_{ut(b),t(b)}f'_{ut(b)}p_{f'ut(b)})<\mathcal{V}_{t(b)}$.

Take any $a>fut(b)$, $f'ut(b)$. (1) and (2) imply that

(3) $(q_{ut(b),b}f_{ut(b)}p_{a,fut(b)}p_a,\ q_{ut(b),b}f'_{ut(b)}p_{a,f'ut(b)}p_a)<\mathcal{V}'_b$.

By (3) and the choice of $\mathcal{V}'_b$ there exists $a'>a$ such that

(4) $(q_{ut(b),b}f_{ut(b)}p_{a',fut(b)},\ q_{ut(b),b}f'_{ut(b)}p_{a',f'ut(b)})<\mathcal{V}_b$.

(4) means that $\boldsymbol{q}(ut)\boldsymbol{f}=:\boldsymbol{q}(ut)\boldsymbol{f}'$. Hence by (2.7) $\boldsymbol{f}\equiv:\boldsymbol{f}'$. ■

(4.3) THEOREM. *Let* $\boldsymbol{p}:X\to(\mathcal{X},\mathcal{U})$ *and* $\boldsymbol{q}:Y\to(\mathcal{Y},\mathcal{V})$ *be approximative resolutions. If* $\boldsymbol{q}$ *is an approximative* **AP**-*resolution, then for any map* $f:X\to Y$ *there exists an approximative resolution* $\boldsymbol{f}:(\mathcal{X},\mathcal{U})\to(\mathcal{Y},\mathcal{V})$ *of* f *with respect to* $\boldsymbol{p}$ *and* $\boldsymbol{q}$.

To prove (4.3) we need (4.4).

(4.4) LEMMA. *Let* $(\mathcal{X},\mathcal{U})=\{(X_a,\mathcal{U}_a),p_{a',a},A\}$ *be an approximative inverse system. Let* $\mathcal{U}'_a\in Cov(X_a)$ *for each* $a\in A$. *Then there exist* $\mathcal{U}''_a\in Cov(X_a)$ *for* $a\in A$ *such that* $(\mathcal{X},\mathcal{U}'')=\{(X_a,\mathcal{U}''_a),p_{a',a},A\}$ *forms an approximative inverse system and* $\mathcal{U}''_a<\mathcal{U}_a\wedge\mathcal{U}'_a$ *for* $a\in A$.

PROOF. $\|T\|$ denotes the cardinality of a set T. Let $P(a)=\{a'\in A:a'<a\}$ for each $a\in A$. For each positive integer n we put $A_n=\{a\in A:\|P(a)\|=n\}$. Since A is cofinite, $A=\cup\{A_i:i=1,2,\cdots\}$ and $A_i\cap A_j=\emptyset$ for $i\neq j$.

Inductively we construct $\mathcal{U}''_a\in Cov(X_a)$ for $a\in A(n)=\cup\{A_i:i=1,2,\cdots,n\}$ satisfying the following condition:

(P_n) $\mathcal{U}''_a<\mathcal{U}_a\wedge\mathcal{U}'_a$ and $p^{-1}_{a',a}\mathcal{U}''_a>\mathcal{U}''_{a'}$ for $a',a\in A(n)$ with $a'>a$.

First for any $a\in A_1$ we put $\mathcal{U}''_a=\mathcal{U}_a\wedge\mathcal{U}'_a$. Then clearly (P_1) holds. Next, we assume that for $a\in A(n-1)$ $\mathcal{U}''_a$ are already defined satisfying (P_{n-1}). Take any $a\in A_n$. Put $B(a)=P(a)\cap A(n-1)$ and $C(a)=P(a)-A(n-1)$. We define $\mathcal{U}''_a$ as follows:

(1) $\mathcal{U}''_a=(\wedge\{p^{-1}_{a,b}\mathcal{U}''_b:b\in B(a)\})\wedge(\wedge\{p^{-1}_{a,b}(\mathcal{U}_b\wedge\mathcal{U}'_b):b\in C(a)\})$.

Since $a\in C(a)$, by (1) $\mathcal{U}''_a<\mathcal{U}_a\wedge\mathcal{U}'_a$. We need to show the second property in (P_n). Take any $a',a\in A(n)$ with $a'>a$. Then there are four cases: (i) $a',a\in A(n-1)$, (ii) $a'\in A_n$ and $a\in A(n-1)$, (iii) $a'\in A(n-1)$ and $a\in A_n$ and (iv) $a',a\in A_n$. In the case (i) (P_{n-1}) implies the required condition. In the case (ii) $a\in B(a')$. Then by (1) $\mathcal{U}''_{a'}<p^{-1}_{a',a}\mathcal{U}''_a$. We consider the case (iii). Since $a'>a$, $P(a')\supset P(a)$. Since $a\in A_n$, $\|P(a')\|\geqq\|P(a)\|=n$. Since $a'\in A(n-1)$, $\|P(a')\|\leqq n-1$. This is a contradiction. Hence (iii) does not happen. We consider the case (iv). Since $a'>a$, $P(a')\supset P(a)$. Since a', $a\in A_n$, $\|P(a')\|=\|P(a)\|=n$. Thus $P(a')=P(a)$. Since $B(a')=B(a)$ and $C(a')=C(a)$, from (1) we have that $p_{a',a}^{-1}\mathcal{U}_a''=\mathcal{U}_{a'}''$. This is the required condition. Hence we have (P_n).

By the inductive construction we obtain the coverings $\mathcal{U}_a''$ for all $a\in A$. Since $(\mathcal{X}, \mathcal{U})$ satisfies (AI1)-(AI3), by (P_n) we easily show that $(\mathcal{X}, \mathcal{U}'')$ satisfies (AI1)-(AI3). ∎

Note. In the proof of (4.4) when A is antisymmetric, $B(a)=P(a)-\{a\}$ and $C(a)=\{a\}$. Then our proof is reduced to a simple one. However we do not assume the antisymmetric condition for A.

PROOF OF (4.3). By (3.3) $\boldsymbol{p}: X\to\mathcal{X}$ is a resolution. Then it satisfies (R1) and (R2). Since each Y_b is an AP, there exists $\mathcal{V}_b'\in \mathcal{C}ov(Y_b)$ satisfying the property in (R2) for $\boldsymbol{p}$ and $\mathcal{V}_b$. By (4.4) there exist $\mathcal{V}_b''\in \mathcal{C}ov(Y_b)$ such that $(\mathcal{Y}, \mathcal{V}'')$ is an approximative inverse system and

(1) $st\,\mathcal{V}_b''<\mathcal{V}_b\wedge\mathcal{V}_b'$ for $b\in B$.

By (R1) for $\boldsymbol{p}$ there exist a function $t: B\to A$ and maps $g_b: X_{t(b)}\to Y_b$ for $b\in B$ such that

(2) $(q_b f, g_b p_{t(b)})<\mathcal{V}_b''$ for $b\in B$.

By (AI3) for $(\mathcal{X}, \mathcal{U})$ there exists a function $f: B\to A$ such that $f>t$ and

(3) $p_{f(b),t(b)}^{-1}(g_b^{-1}\mathcal{V}_b)>\mathcal{U}_{f(b)}$ for $b\in B$.

Claim. $\boldsymbol{f}=\{f, f_b: b\in B\}: (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ is an approximative system map. Here $f_b=g_b p_{f(b),t(b)}: X_{f(b)}\to Y_b$ for $b\in B$.

We need to show (AM1) and (AM2). (AM1) follows from (3). We show (AM2). Take any $b'>b$. (2) implies that

(4) $(q_b f, f_b p_{f(b)})<\mathcal{V}_b''$ and $(q_{b'} f, f_{b'} p_{f(b')})<\mathcal{V}_{b'}''$.

By (AI2) for $(\mathcal{Y}, \mathcal{V}'')$ and (4)

(5) $(q_b f, q_{b',b} f_{b'} p_{f(b')})<\mathcal{V}_b''$.

Take any $a>f(b)$, $f(b')$. By (1), (4) and (5)

(6) $(f_b p_{a,f(b)} p_a, q_{b',b} f_{b'} p_{a,f(b')} p_a)<st\,\mathcal{V}_b''<\mathcal{V}_b'$.

By the choice of $\mathcal{V}_b'$ and (6) there exists $a'>a$ such that

(7) $(f_b p_{a',f(b)}, q_{b',b} f_{b'} p_{a',f(b')})<\mathcal{V}_b$.

(7) means (AM2) for $\boldsymbol{f}$. Hence we have our Claim.

(1) and (4) imply that $(q_b f, f_b p_{f(b)}) < \mathcal{V}_b$ for $b \in B$. This means that $\boldsymbol{f}$ is an approximative resolution of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. ■

The next assertion follows from (3.3), (3.8), (3.13) and (4.3).

(4.5) COROLLARY. *Let $\mathcal{X}=\{X_a, p_{a',a}, A\}$ and $\mathcal{Y}=\{Y_b, q_{b',b}, B\}$ be inverse systems of compact metric spaces. Let $\boldsymbol{p}: X \to \mathcal{X}$ and $\boldsymbol{q}: Y \to \mathcal{Y}$ be inverse limits. If A, B are infinite, cofinite sets and all Y_b are* APs, *then there exist coverings $\mathcal{U}_a \in \mathcal{C}ov(X_a)$ and $\mathcal{V}_b \in \mathcal{C}ov(Y_b)$ such that $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U}) = \{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ and $\boldsymbol{q}: Y \to (\mathcal{Y}, \mathcal{V}) = \{(Y_b, \mathcal{V}_b), q_{b',b}, B\}$ are approximative resolutions with the property: For any map $f: X \to Y$ there exists an approximative resolution of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$.* ■

Let $g: Y \to Z$ be a map. Let $\boldsymbol{r}: Z \to (\mathcal{Z}, \mathcal{W})$ be an approximative resolution. Let $\boldsymbol{g}: (\mathcal{Y}, \mathcal{V}) \to (\mathcal{Z}, \mathcal{W})$ be an approximative system map. In a straightforward manner we can show the following:

(4.6) LEMMA. *If $\boldsymbol{f}: (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ and $\boldsymbol{g}: (\mathcal{Y}, \mathcal{U}) \to (\mathcal{Z}, \mathcal{W})$ are approximative resolutions of f and g with respect to $\boldsymbol{p}$, $\boldsymbol{q}$, and with respect to $\boldsymbol{q}$, $\boldsymbol{r}$, respectively, then $\boldsymbol{r}(u)(\boldsymbol{gf}): (\mathcal{X}, \mathcal{U}) \to (\mathcal{Z}, \mathcal{W})$ is an approximative resoluton of gf with respect to $\boldsymbol{p}$ and $\boldsymbol{r}$ for each* 1*-refinement function u of $(\mathcal{Z}, \mathcal{W})$.* ■

Mardešic [**15**] introduced the notion of resolution for maps. Let $f: X \to Y$ be a map. Let $\boldsymbol{p} = \{p_a : a \in A\}: X \to \mathcal{X} = \{X_a, p_{a',a}, A\}$ and $\boldsymbol{q} = \{q_b : b \in B\}: Y \to \mathcal{Y} = \{Y_b, q_{b',b}, B\}$ be resolutions. Let $\boldsymbol{f} = \{f, f_b : b \in B\}$ be a collection consisting of a function $f: B \to A$ and of maps $f_b: X_{f(b)} \to Y_b$ for $b \in B$. We say that $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ is a resolution of f provided that it satisfies the following two conditions:

(RM1) For each $b' > b$ there exists $a > f(b'), f(b)$ such that $f_b p_{a,f(b)} = q_{b',b} f_{b'} p_{a,f(b')}$.

(RM2) $q_b f = f_b p_{f(b)}$ for $b \in B$.

Sometimes we say that $\boldsymbol{f}: \mathcal{X} \to \mathcal{Y}$ is a resolution of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$.

(4.7) LEMMA (Mardešic [**15**] and Haxhibeqiri [**10**]). (i) *Any map $f: X \to Y$ admits an* ANR*-resolution.*

(ii) *Any map f admits a polyphedral resolution.* ■

(4.8) LEMMA. *Let $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ be a resolution of f. Then there exist approximative resolutions $\boldsymbol{p}': X \to (\mathcal{X}, \mathcal{U})'$, $\boldsymbol{q}': Y \to (\mathcal{Y}, \mathcal{V})'$ and an approximative resolution $\boldsymbol{f}': (\mathcal{X}, \mathcal{U})' \to (\mathcal{T}, \mathcal{V})'$ of f with respect to $\boldsymbol{p}'$ and $\boldsymbol{q}'$ satisfying the following:*

(i) *$\boldsymbol{p}'$ and $\boldsymbol{q}'$ are constructed from $\boldsymbol{p}$ and $\boldsymbol{q}$ in the same way as in* (3.7), *respectively.*

(ii) *$(\boldsymbol{f}', \boldsymbol{p}', \boldsymbol{q}')$ is a resolution of f;*

(iii) *Each map in $\boldsymbol{f}'$ is a map in $\boldsymbol{f}$*

PROOF. Let $F(A)$, $F(B)$, $M(A)$ and $M(B)$ be the same as in the proof of (3.7). By (3.7) there exist approximative resolutions $\boldsymbol{p}'=\{p'_{a'}: a'\in M(A)\}: X\to(\mathcal{X}, \mathcal{U})'=\{(X'_{a'}, \mathcal{U}'_{a'}), p'_{a'',a'}, M(A)\}$, $\boldsymbol{q}'=\{q'_{b'}: b'\in M(B)\}: Y\to(\mathcal{Y}, \mathcal{V})'=\{(Y'_{b'}, \mathcal{V}'_{b'}), q'_{b'',b'}, M(B)\}$ and increasing functions $s: M(A)\to A$, $t: M(B)\to B$ satisfying (i)-(iii) in (3.7), respectively.

We define $\boldsymbol{f}'=\{f', f'_{b'}: b'\in M(B)\}:(\mathcal{X}, \mathcal{U})'\to(\mathcal{Y}, \mathcal{V})'$ as follows: Take any $b'\in M(B)$. Since $f_{t(b')}: X_{ft(b')}\to Y_{t(b')}=Y'_{b'}$, $f_{t(b')}^{-1}\mathcal{V}'_{b'}\in Cov(X_{ft(b')})$. By (iii) of (3.7) there exists $f'(b')\in M(A)$ such that

(1) $s(f'(b'))=ft(b')$ and $\mathcal{U}'_{f'(b')}=f_{t(b')}^{-1}\mathcal{V}'_{b'}$.

Then we have a function $f': M(B)\to M(A)$. By (ii) of (3.7) and (1) $X'_{f'(b')}=X_{sf'(b')}=X_{ft(b')}$ and $Y'_{b'}=Y_{t(b')}$. Thus we may define a map $f'_{b'}=f_{t(b')}: X'_{f'(b')}=X_{ft(b')}\to Y_{t(b')}=Y'_{b'}$ for $b'\in M(B)$.

Claim. $\boldsymbol{f}'$ satisfies (AM1), (RM1) and (RM2).

(1) implies that $f'^{-1}\mathcal{V}'_{b'}=f_{t(b')}^{-1}\mathcal{V}'_{b'}=\mathcal{U}'_{f'(b')}$. This means (AM1) for $\boldsymbol{f}'$. We show (RM2). Take any $b'', b'\in M(B)$ with $b''>b'$. Since t is increasing, $t(b'')>t(b')$. By (RM2) for $\boldsymbol{f}$ there exists an $a>ft(b'')$, $ft(b')$ such that

(2) $q_{t(b''),t(b')}f_{t(b'')}p_{a,ft(b'')}=f_{t(b')}p_{a,ft(b')}$.

Put $a'=f'(b')\cup f'(b'')\cup\{(a, \{X_a\})\}\in M(A)$. Since s is increasing, (2) in the proof of (3.7) and (1) imply that $s(a')>sf'(b'')=ft(b'')$, $s(a')>sf'(b')=ft(b')$ and $s(a')>a$. By (2)

(3) $q_{t(b''),t(b')}f_{t(b'')}p_{s(a'),ft(b'')}=f_{t(b')}p_{s(a'),ft(b')}$.

(1), (3) and (ii) in (3.7) imply that $q'_{b'',b'}f'_{b''}p'_{a',f'(b'')}=f'_{b'}p'_{a',f'(b')}$. This means (RM2) for $\boldsymbol{f}'$. We show (RM1). By (RM1) for $\boldsymbol{f}$ and (1) $f'_{b'}p'_{f'(b')}=f_{t(b')}p_{sf'(b')}=f_{t(b')}p_{ft(b')}=q_{t(b')}f=q'_{b'}f$. This means (RM1) for $\boldsymbol{f}'$. Hence we have the Claim.

Since (AM2) follows from (RM2), Claim means that $\boldsymbol{f}'$ has the required properties. ■

We say that an approximative resolution is commutative provided that it satisfies condition (RM1). By (4.7) and (4.8) we have the following:

(4.9) THFOREM. *For any map $f: X\to Y$ there exist approximative* **ANR**- *or* **POL**-*resolutions $\boldsymbol{p}: X\to(\mathcal{X}, \mathcal{U})$, $\boldsymbol{q}: Y\to(\mathcal{Y}, \mathcal{V})$ and a commutative approximative* *resolution* $\boldsymbol{f}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ *of* f *with respect to* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *such that* $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ *is a resolution of* f. ■

(4.10) EXAMPLE. Let X and Y be paracompact M-spaces. By (3.17) we have approximative ANR(**PM**)-resolutions $\boldsymbol{p}=\{p_a : a\in A\} : X\to(\mathcal{AU}(X, MX), \mathcal{U})=\{(U_a, \mathcal{U}_a), p_{a',a}, A\}$ and $\boldsymbol{q}=\{q_b : b\in B\} : Y\to(\mathcal{AU}(Y, MY), \mathcal{V})=\{(V_b, \mathcal{V}_b), q_{b',b}, B\}$. Here MX and MY are AR(**PM**)s containing X and Y as closed subsets, respectively. All U_a and all V_b are ANR(**PM**)-neighborhoods of X and Y in MX and MY, respectively, and all p_a, $p_{a',a}$, q_b, $q_{b',b}$ are inclusion maps.

Let $f: X\to Y$ be a map. We have an extension $F: MX\to MY$ of f. Take any $b\in B$. By (ii) of (3.17) there exists $g(b)\in A$ such that $F^{-1}(V_b)\supset U_{g(b)}$. By (AI2) there exists $f(b)\in A$ such that $f(b)>g(b)$ and $(Fp_{f(b),g(b)})^{-1}\mathcal{V}_b>\mathcal{U}_{f(b)}$. Thus we have a function $f: B\to A$ and maps $f_b=Fp_{f(b),g(b)}: U_{f(b)}\to V_b$ for $b\in B$. We have a commutative approximative resolution $\boldsymbol{f}=\{f, f_b : b\in B\}$: $(\mathcal{AU}(X, MX), \mathcal{U})\to(\mathcal{AU}(Y, MY), \mathcal{V})$ of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. Obviously this is also a resolution of f. We consider a special case of this method in § **0**. ■

(4.11) EXAMPLE. Let C be the Cantor set and $I=[0, 1]$ be the unit interval. Let $f: C\to I$ be an onto map. Then in § **0** we noticed that we have no expansion of f with respect to some inverse limits $\boldsymbol{p}: C\to\boldsymbol{C}$ and $\boldsymbol{q}: I\to\mathcal{I}$. By (3.13) $\boldsymbol{p}$ and $\boldsymbol{q}$ are resolutions of C and I. Hence f has no resolution with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$.

In the same way as in § **0** we can show that if $(\boldsymbol{f}', \boldsymbol{p}', \boldsymbol{q}')$ is an **POL**-resolution of f, then almost all spaces, appearing in $\boldsymbol{p}'$, have dimensions $\geqq 1$. This is curious, because $\dim C=0$. In fact when we embed C in I, by (4.10) we have a resolution $\boldsymbol{f}$ of f with respect to some $\boldsymbol{p}'$ and $\boldsymbol{q}$ such that almost all spaces, appearing in $\boldsymbol{p}'$, are 1-dimensional polyhedra.

On the other hand by (4.5) we can choose coverings $\mathcal{U}_i\in \mathcal{C}ov(X_i)$ and $\mathcal{V}_i\in \mathcal{C}ov(Y_i)$, which make approximative resolutions $\boldsymbol{p}: C\to(\mathcal{X}, \mathcal{U})=\{(X_i, \mathcal{U}_i), p_{i,j}, N\}$ and $\boldsymbol{q}: I\to(\mathcal{Y}, \mathcal{V})=\{(Y_i, \mathcal{V}_i), q_{i,j}, N\}$. Hence by (4.3) for any map $f: X\to Y$ we have an approximative resolution $\boldsymbol{f}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. ■

The above observations ((4.10) and (4.11)) explain the difference between (4.3) and (4.9), that is, the difference between approximative resolutions and resolutions. Approximative resolutions for maps have many advantages over resolutions for maps.

(4.12) REMARK. Mioduszewski [**19**] studied approximative expansions of maps into inverse sequences of polyhedra. His discription is neither simple nor categorical. However he essentially proved (4.5) for compact metric spaces. In the latter section we shall show that our treatment is natural and categorical.

§ 5. The approximative shape category.

In this section we introduce the approximative shape category and some natural functors.

Let X, Y and Z be spaces. Let $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$, $\boldsymbol{q}: Y \to (\mathcal{Y}, \mathcal{V})$ and $\boldsymbol{r}: Z \to (\mathcal{Z}, \mathcal{W})$ be approximative **AP**-resolutions. Let $f: X \to Y$ and $g: Y \to Z$ be maps. By (4.3) there exists an approximative resolution $\boldsymbol{f}: (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. By (4.1) its equivalence class $[\boldsymbol{f}]$ is unique, that is, $[\boldsymbol{f}]$ does not depend on the choice of approximative resolutions of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. Therefore we denote it by $[f]_{\boldsymbol{p},\boldsymbol{q}}$. From (4.6) we have the following:

(5.1) LEMMA. (i) $[g]_{\boldsymbol{q},\boldsymbol{r}}[f]_{\boldsymbol{p},\boldsymbol{q}} = [gf]_{\boldsymbol{p},\boldsymbol{r}}$.

(ii) $[1_X]_{\boldsymbol{p},\boldsymbol{p}} = [1_{(\mathcal{X},\mathcal{U})}]$.

(iii) *If $f: X \to Y$ is a homeomorphism, theu $[f]_{\boldsymbol{p},\boldsymbol{q}}$ is an isomorphism iu Appro-***AP**. ∎

We define $E(X) = \{\boldsymbol{p} : \boldsymbol{p}$ is an approximative **AP**-resolution of $X\}$. For $\boldsymbol{p} \in E(X)$ and $\boldsymbol{q} \in E(Y)$ we define $E(\boldsymbol{p}, \boldsymbol{q}) = (\text{Appro-}\mathbf{AP})((\mathcal{X}, \mathcal{U}), (\mathcal{Y}, \mathcal{V}))$. We define $E(X, Y) = \cup\{E(\boldsymbol{p}, \boldsymbol{q}) : \boldsymbol{p} \in E(X)$ and $\boldsymbol{q} \in E(Y)\}$ (disjoint sum). We define a relation on $E(X, Y)$ as follows: Let $m, m' \in E(X, Y)$. There are $\boldsymbol{p}, \boldsymbol{p}' \in E(X)$ and $\boldsymbol{q}, \boldsymbol{q}' \in E(Y)$ such that $m \in E(\boldsymbol{p}, \boldsymbol{q})$ and $m' \in E(\boldsymbol{p}', \boldsymbol{q}')$. We say that m is equivalent to m', in notation $m \equiv m'$, provided that $[1_Y]_{\boldsymbol{q},\boldsymbol{q}'} m = m' [1_X]_{\boldsymbol{p},\boldsymbol{p}'}$ in Appro-**AP**. By (5.1) we can show the following:

(5.2) LEMMA. *The above relation $\equiv$ is an equivalence relation on $E(X, Y)$.* ∎

$\langle m \rangle$ denotes the equivalence class of $m \in E(X, Y)$ by the relation $\equiv$. Put $\langle E(X, Y) \rangle = \{\langle m \rangle : m \in E(X, Y)\}$. We define the composition $\langle n \rangle \langle m \rangle$ for $m \in E(X, Y)$ and $n \in E(Y, Z)$ as follows: $\langle n \rangle \langle m \rangle = \langle n [1_Y]_{\boldsymbol{q},\boldsymbol{q}'} m \rangle$ where $\boldsymbol{p} \in E(X)$, $\boldsymbol{q}, \boldsymbol{q}' \in E(Y)$, $\boldsymbol{r}' \in E(Z)$, $m \in E(\boldsymbol{p}, \boldsymbol{q})$ and $n \in E(\boldsymbol{q}', \boldsymbol{r}')$. By (2.10) and (5.1) we can show the following:

(5.3) LEMMA. (i) *The above composition is well defined.*

(ii) $\langle m \rangle \langle [1_X]_{\boldsymbol{p},\boldsymbol{p}} \rangle = \langle m \rangle = \langle [1_Y]_{\boldsymbol{q},\boldsymbol{q}} \rangle \langle m \rangle$ *for* $m \in E(X, Y)$.

(iii) $\langle w\rangle(\langle n\rangle\langle m\rangle)=(\langle w\rangle\langle n\rangle)\langle m\rangle$ *for* $m\in E(X, Y)$, $n\in E(Y, Z)$ *and* $w\in E(Z, K)$.

(iv) $\langle[f]_{\boldsymbol{p},\boldsymbol{q}}\rangle=\langle[f]_{\boldsymbol{p}',\boldsymbol{q}'}\rangle$.

(v) $\langle[g]_{\boldsymbol{q},\boldsymbol{r}}\rangle\langle[f]_{\boldsymbol{p},\boldsymbol{q}}\rangle=\langle[gf]_{\boldsymbol{p},\boldsymbol{r}}\rangle$, where $\boldsymbol{r}\in E(Z)$. ■

We define a function $\Phi(\boldsymbol{p}, \boldsymbol{q}): E(\boldsymbol{p}, \boldsymbol{q})\rightarrow\langle E(\boldsymbol{p}, \boldsymbol{q})\rangle$ for $\boldsymbol{p}\in E(X)$ and $\boldsymbol{q}\in E(Y)$ as follows: $\Phi(\boldsymbol{p}, \boldsymbol{q})(m)=\langle m\rangle$ for $m\in E(\boldsymbol{p}, \boldsymbol{q})$.

(5.4) LEMMA. *For* $\boldsymbol{p}\in E(X)$ *and* $\boldsymbol{q}\in E(Y)$ $\Phi(\boldsymbol{p}, \boldsymbol{q}): E(\boldsymbol{p}, \boldsymbol{q})\rightarrow\langle E(X, Y)\rangle$ *is bijective.*

PROOF. Take any $m'\in E(\boldsymbol{p}', \boldsymbol{q}')$ for $\boldsymbol{p}'\in E(X)$ and $\boldsymbol{q}'\in E(Y)$. We put $m=[1_Y]_{\boldsymbol{q}',\boldsymbol{q}}m'[1_X]_{\boldsymbol{p},\boldsymbol{p}'}$. Then $m\in E(\boldsymbol{p}, \boldsymbol{q})$ and $[1_Y]_{\boldsymbol{q},\boldsymbol{q}'}m=[1_Y]_{\boldsymbol{q},\boldsymbol{q}'}[1_Y]_{\boldsymbol{q}'\boldsymbol{q}}m'[1_X]_{\boldsymbol{p},\boldsymbol{p}'}=[1_Y]_{\boldsymbol{q}',\boldsymbol{q}'}m'[1_X]_{\boldsymbol{p},\boldsymbol{p}'}=m'[1_X]_{\boldsymbol{p},\boldsymbol{p}'}$ by (5.1). This means that $\langle m\rangle=\langle m'\rangle$ and hence $\Phi(\boldsymbol{p}, \boldsymbol{q})$ is onto. Trivially it follows from (ii) of (5.1) that $\Phi(\boldsymbol{p}, \boldsymbol{q})$ is injective. ■

Now, we define the approximative shape category, in notation **ASh**, as follows: Objects of **ASh** are all spaces. For spaces X and Y $\mathbf{ASh}(X, Y)=\langle E(X, Y)\rangle$. The composition of morphisms is defined in the above. Since $E(\boldsymbol{p}, \boldsymbol{q})$ is a set, $\mathbf{ASh}(X, Y)$ forms a set by (5.4) and the axiom of substitution in set theory. Note that $\langle E(X, Y)\rangle$ forms a set, but $E(X)$ and $E(X, Y)$ do not form sets. By (5.3) **ASh** forms a category. We call morphisms in **ASh** approximative shape morphisms, or approximative shapings.

We define an approximative shape functor AS: **TOP**→**ASh** as follows: For each space X we put $\mathrm{AS}(X)=X$. For a map $f: X\rightarrow Y$ we put $\mathrm{AS}(f)=\langle[f]_{\boldsymbol{p},\boldsymbol{q}}\rangle$ for some $\boldsymbol{p}\in E(X)$ and $\boldsymbol{q}\in E(Y)$. By (5.3) AS is well defined and forms a functor.

(5.5) LEMMA. *Let X be an* ANR(**PM**) *or a polyhedron. For each* $\mathcal{U}\in\mathcal{C}ov(X)$ *there exists* $\mathcal{U}'\in\mathcal{C}ov(X)$ *satisfying*

(*) *any two* $\mathcal{U}'$*-near maps* $f, g: Y\rightarrow X$ *are* $\mathcal{U}$*-homotopic, where Y is an arbitrary space.*

PROOF. We show also (iv) in (3.17). Let X be an ANR(**PM**). By Nagata [**24**] and by the Kuratowski-Wojdislawski Theorem (see [**11**]) we may assume that X is a closed subset of $C\times I^{\tau}$, where C is a convex set in a normed vector space M and τ is an arbitrary cardinal. Take any $\mathcal{U}$, $\mathcal{U}_1\in\mathcal{C}ov(X)$ with $st^2\mathcal{U}_1<\mathcal{U}$. Since X is an ANR(**PM**), there exist a neighborhood U of X in $C\times I^{\tau}$ and a retraction $r: U\rightarrow X$. By Theorem 4 of Mardešic and Šostak [**17**] there exists an open paracompact neighborhood V of X in U with the property; each point x of V has a neighborhood $K(x)\subset V$ such that $K(x)$ is convex in $M\times R^{\tau}$. Here R is the real line. By a theorem of Palais [**25**, p. 5] there exist a simplicial complex K, maps $h:V\to|K|$, $k:|K|\to V$ and a $r^{-1}\mathcal{V}_1$-homotopy $H^1:U\times I\to U$ such that $H_0^1=1_U$ and $H_1^1=kh$. Since $(rkhw, 1_X)<\mathcal{V}$, X is an AP. Here $w:X\to V$ is the inclusion map. By Theorem 4 of MS [**18**, p. 292] there exists a subdivision L of K such that $\overline{st(L)}<k^{-1}r^{-1}\mathcal{V}_1$. Here $st(v, L)$ denotes the open star at a vertex v in L, $st(L)=\{st(v, L): v$ is a vertex of $L\}$ and $\overline{st(L)}=\{\overline{st(v, L)}: v$ is a vertex of $L\}$. By Theorem 9 and Remark 1 of MS [**18**, pp. 302-303] there exist maps $i:|K|=|L|\to|L|_m$, $j:|L|_m\to|L|=|K|$ and a $\overline{st(L)}$-homotopy $H^2:|K|\times I\to|K|$ such that $H_0^2=1_{|K|}$ and $H_1^2=ji$. $|L|_m$ denotes the realization of L with the metric topology. By Theorem 11 of MS [**18**, p. 304] $|L|_m$ is an ANR and then by Theorem 11 of Hu [**11**, p. 111] there exists $\mathcal{W}\in Cov(|L|_m)$ satisfying (*) for $|L|_m$ and $j^{-1}st(L)$. From the above facts it is easy to show that $\mathcal{V}=(jhw)^{-1}\mathcal{W}\in Cov(X)$ satisfies the required condition (*). Obviously the above argument also contains a proof for polyhedra. ■

(5.6) LEMMA. *Any space X admits an approximative* **ANR**-*resolution* $\boldsymbol{p}:X\to(\mathcal{X}, \mathcal{U})$ *and an approximative* **POL**-*resolution* $\boldsymbol{p}:X\to(\mathcal{X}, \mathcal{U})$ *with the property:*

(**) *any two $\mathcal{U}_a$-near maps $f, g:Y\to X_a$ are homotopic for $a\in A$, where Y is an arbitrary space.*

PROOF. By (3.15) there exists an approximative **ANR**-resolution $\boldsymbol{p}:X\to(\mathcal{X}, \mathcal{U}')=\{(X_a, \mathcal{U}_a'), p_{a'a'}, A\}$ of a space X. Since all X_a are ANRs, by (5.5) there exist $\mathcal{U}_a''\in Cov(X_a)$ with property (*) for $\mathcal{U}_a'$. By (4.4) we make coverings $\mathcal{U}_a\in Cov(X_a)$ such that $(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a'a}, A\}$ forms an approximative inverse system and $\mathcal{U}_a<\mathcal{U}_a'\wedge\mathcal{U}_a''$ for $a\in A$. By (3.3) $\boldsymbol{p}:X\to(\mathcal{X}, \mathcal{U})$ is an approximative resolution. Since $\mathcal{U}_a<\mathcal{U}_a''$, it has the required property. In the same way we construct a required approximative **POL**-resolution. ■

We recall that H: **TOP**→**HTOP** and S: **HTOP**→**Sh** are the homotopy functor and the shape functor, respectively. Then $H(f)$ denotes the homotopy class of the map f, and $H(\mathcal{X})=\{X_a, H(p_{a'a}), A\}$ is an inverse system in **HTOP**. $H(\boldsymbol{p})=\{H(\boldsymbol{p}_a): a\in A\}: X\to H(\mathcal{X})$ is a morphism of inverse systems from X to $H(\mathcal{X})$ (see MS [**18**, p. 4]). We say that $H(\boldsymbol{p}): X\to H(\mathcal{X})$ is an **HTOP**-expansion (see MS [**18**]) provided that it satisfies the following two conditions:

(E1) For each ANR P and a map $h:X\to P$ there exist $a\in A$ and a map $h_a:X_a\to P$ such that $h\simeq h_a p_a$.

(E2) For each $a\in A$ and for maps $h, h': X_a\to P\in$ ANR such that $hp_a\simeq h'p_a$ there exists $a'>a$ such that $hp_{a',a}\simeq h'p_{a',a}$.

(5.7) LEMMA (Mardešic [15]). *If* $p=\{p_a: a\in A\}: X\to\mathcal{X}=\{X_a, p_{a',a}, A\}$ *is a resolution of* X, *then* $H(p)=\{H(p_a): a\in A\}: X\to H(\mathcal{X})=\{X_a, H(p_{a',a}), A\}$ *is a* **HTOP**-*expansion.* ■

Let $r: Z\to(\mathcal{T}, \mathcal{W})\in E(Z)$. *Let* $f=\{f, f_b: b\in B\}$, $f'=\{f', f'_b: b\in B\}: (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ and $g: (\mathcal{Y}, \mathcal{U})\to(\mathcal{T}, \mathcal{W})$ be approximative system maps.

(5.8) LEMMA. *If* $\mathcal{V}$ *and* $\mathcal{W}$ *satisfy property* (**) *in* (5.6), *we have the following:*

(i) $H(f)=\{f, H(f_b): b\in B\}: H(\mathcal{X})\to H(\mathcal{Y})$ *is a morphism of inverse systems in* **HPOL**.

(ii) *If* $f\equiv: f'$, *then* $H(f)$ *and* $H(f')$ *are equivalent, i.e.,* $H(f)\sim H(f')$ (see [18, p. 6]).

(iii) *If* f *is an approximative resolution of* f *with respect to* p *and* q, *then* $H(q)H(f)=H(f)H(p)$.

(iv) *For each* 1-*refinement function* u *of* $(\mathcal{T}, \mathcal{W})$ $H(r(u)(gf))\sim H(g)H(f)$.

PROOF. We show (ii). It is sufficient to show that $f=: f'$ implies $H(f)\sim H(f')$. We assume that $f=: f'$. Take any $b\in B$. Then there exists $a>f(b)$, $f'(b)$ such that $(f_b p_{a,f(b)}, f'_b p_{a,f'(b)})<\mathcal{V}_b$. By property (**) of (5.6) $H(f_b)H(p_{a,f(b)})=H(f'_b)H(p_{a,f'(b)})$. This means that $H(f)\sim H(f')$. Hence we have (ii). In a similar way we can show the other claims. ■

Since we have (5.6), hereafter we consider only approximative **POL**-resolutions of spaces with property (**) of (5.6). By (5.7) $H(p): X\to H(\mathcal{X})$, $H(q): Y\to H(\mathcal{Y})$ are **HPOL**-expansions. By (i) of (5.8) $H(f)$ is a morphism of inverse systems. Then $H(f)$ determines an equivalence class $\alpha H(f)$ given by the equivalence relation $\sim$, that is, $\alpha H(f): H(\mathcal{X})\to H(\mathcal{Y})$ is a morphism of pro-**HPOL** (see MS [18]). $\alpha H(f)$ determines a shape morphism $s\alpha H(f): X\to Y$ (see MS [18]).

Let $[f]=[f']$. Since $f\equiv: f'$, by (ii) of (5.8) $H(f)\sim H(f')$, that is, $\alpha H(f)=\alpha H(f')$. Thus we may define $\tilde{\alpha}([f])=\alpha H(f)$. From (iv) of (5.8) we have that $\tilde{\alpha}([g])\tilde{\alpha}([f])=\tilde{\alpha}([g][f])$. Since $H(1_{(\mathcal{X},\mathcal{U})}): H(\mathcal{X})\to H(\mathcal{X})$ is the identity, $\tilde{\alpha}([1_{(\mathcal{X},\mathcal{U})}])=\alpha(1_{H(\mathcal{X})})$. Let $f: (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ be an approximative resolution of f with respect to p and q. By (iii) of (5.7) $\alpha H(f): H(\mathcal{X})\to H(\mathcal{Y})$ is a morphism in pro-**HPOL** with $\alpha H(f)\alpha H(p)=\alpha H(q)\alpha H(f)$. Thus $s\alpha H(f)$ is the shape morphism induced by f. Hence $s\tilde{\alpha}([f]_{p,q})=SH(f)$.

Let $\langle[\boldsymbol{f}]\rangle=\langle[\boldsymbol{f}'']\rangle\in\langle E(X, Y)\rangle$. Let $[\boldsymbol{f}]\in E(\boldsymbol{p}, \boldsymbol{q})$ and $[\boldsymbol{f}'']\in E(\boldsymbol{p}'', \boldsymbol{q}'')$. Since $[\boldsymbol{f}]\equiv[\boldsymbol{f}'']$, $[1_Y]_{\boldsymbol{q},\boldsymbol{q}'}[\boldsymbol{f}]=[\boldsymbol{f}''][1_X]_{\boldsymbol{p},\boldsymbol{p}'}$. Thus $\tilde{\alpha}([1_Y]_{\boldsymbol{q},\boldsymbol{q}'})\tilde{\alpha}([\boldsymbol{f}])=\tilde{\alpha}([\boldsymbol{f}''])\tilde{\alpha}([1_X]_{\boldsymbol{p},\boldsymbol{p}'})$. Since $\tilde{\alpha}([1_X]_{\boldsymbol{p},\boldsymbol{p}'})$ and $\tilde{\alpha}([1_Y]_{\boldsymbol{q},\boldsymbol{q}'})$ are morphisms in pro-**HPOL** induced by identities, then $\tilde{\alpha}([\boldsymbol{f}])$ and $\tilde{\alpha}([\boldsymbol{f}''])$ induce the same shape morphism, i.e., $s\tilde{\alpha}([\boldsymbol{f}])=s\tilde{\alpha}([\boldsymbol{f}''])$.

Now we define a functor $\mathrm{ASS}:\mathbf{ASh}\to\mathbf{Sh}$ as follows: For each space X $\mathrm{ASS}(X)=X$ and for $\langle[\boldsymbol{f}]\rangle\in\langle E(X, Y)\rangle=\mathrm{ASh}(X, Y)$ $\mathrm{ASS}(\langle[\boldsymbol{f}]\rangle)=s\tilde{\alpha}([\boldsymbol{f}])$. From the above facts we easily show that it is well defined and forms a functor with $\mathrm{S}\circ\mathrm{H}=\mathrm{ASS}\circ\mathrm{AS}$. We summarize results in this section as follows:

(5.9) THEOREM. **ASh** *forms a category and* $\mathrm{AS}:\mathbf{TOP}\to\mathbf{ASh}$, $\mathrm{ASS}:\mathbf{ASh}\to\mathbf{Sh}$ *are functors with the following commutative diagram:*

$$\begin{array}{ccc} \mathbf{TOP} & \xrightarrow{\ \mathrm{H}\ } & \mathbf{HTOP} \\ \downarrow{\scriptstyle \mathrm{AS}} & & \downarrow{\scriptstyle \mathrm{S}} \\ \mathbf{ASh} & \xrightarrow{\ \mathrm{ASS}\ } & \mathbf{Sh}. \end{array} \quad \blacksquare$$

We say that X and Y have the same approximative shape type, in notation $\mathrm{ASh}(X)=\mathrm{ASh}(Y)$, provided that X and Y are isomorphic in **ASh**. $\mathrm{ASh}(X)<\mathrm{ASh}(Y)$ denotes that X is dominated by Y in **ASh**.

(5.10) COROLLARY. (i) *If X is dominated by Y in* **TOP**, *then* $\mathrm{ASh}(X)<\mathrm{ASh}(Y)$.

(ii) *If X is homeomorphic to Y, then* $\mathrm{ASh}(X)=\mathrm{ASh}(Y)$. ■

§ 6. The Tychonoff functor and the completion functor.

In this section we investigate the influence of the Tychonoff functor and the completion functor on **ASh**.

Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be full subcategories of **TOP**. Let $F:\boldsymbol{C}\to\boldsymbol{D}$ be a covariant functor. Let $j:\boldsymbol{C}\to\mathbf{TOP}$ and $j':\boldsymbol{D}\to\mathbf{TOP}$ be the inclusion functors. Let $\boldsymbol{t}:j\to j'F$ be a natural transformation. We say that $\boldsymbol{t}$ is dense provided that for each $X\in\mathrm{Ob}\,\boldsymbol{C}$ the image of $t_X:j(X)=X\to j'F(X)=F(X)$ is a dense subset of $F(X)$. Let $\boldsymbol{K}$ be a subcategory of **TOP**. We say that $\boldsymbol{t}$ is rigid for $\boldsymbol{K}$ provided that it satisfies the following condition:

$(R)^*$ For each $X\in\mathrm{Ob}\,\boldsymbol{C}$, each $K\in\mathrm{Ob}\,\boldsymbol{K}$ and each map $f:X\to K$ there exists a map $f':F(X)\to K$ such that $f't_X=f$.

(6.1) LEMMA. *Let* $\boldsymbol{t}: j \to j'F$ *be dense and rigid for* **POL**. *Let* $X \in \mathrm{Ob}\,\boldsymbol{C}$ *and* $\mathcal{X} = \{X_a, p_{a'a}, A\}$ *be an inverse system in* $\boldsymbol{C}$. *Then* $\boldsymbol{p} = \{p_a : a \in A\} : X \to \mathcal{X}$ *is a resolution of* X *iff* $F(\boldsymbol{p}) = \{F(p_a) : a \in A\} : F(X) \to F(\mathcal{X}) = \{F(X_a), F(p_{a',a}), A\}$ *is a resolution of* $F(X)$.

To prove (6.1) we need the following which is easy to show.

(6.2) LEMMA. *Let* $k : X \to Y$ *and* $f, g : Y \to Z$ *be maps. Let* $k(X)$ *be dense in* Y. *For each* $\mathcal{W} \in \mathcal{C}ov(Z)$ *if* $(fk, gk) < \mathcal{W}$, *then* $(f, g) < st\,\mathcal{W}$. ■

PROOF OF (6.1). First we assume that $\boldsymbol{p} : X \to \mathcal{X}$ is a resolution. Then it satisfies (R1) and (R2) for polyhedra. We show that $F(\boldsymbol{p})$ satisfies (R1) and (R2) for polyhedra.

We show (R1). Let $P \in \mathrm{Ob}\,\mathbf{POL}$ and $\mathcal{U} \in \mathcal{C}ov(P)$. Let $f' : F(X) \to P$ be a map. Take $\mathcal{U}' \in \mathcal{C}ov(P)$ such that $st\,\mathcal{U}' < \mathcal{U}$. By (R1) for $\boldsymbol{p}$ there exist $a \in A$ and a map $g : X_a \to P$ such that $(gp_a, f't_X) < \mathcal{U}'$. Since $\boldsymbol{t}$ is rigid for **POL**, there exists a map $g' : F(X_a) \to P$ such that $g't_{X_a} = g$. Since $gp_a = g'F(p_a)t_X$, $(g'F(p_a)t_X, f't_X) < \mathcal{U}'$. Thus by (6.2) $(g'F(p_a), f') < st\,\mathcal{U}' < \mathcal{U}$. Hence we have (R1).

We show (R2). Let $P \in \mathrm{Ob}\,\mathbf{POL}$ and $\mathcal{U} \in \mathcal{C}ov(P)$. Take $\mathcal{U}' \in \mathcal{C}ov(P)$ such that $st\,\mathcal{U}' < \mathcal{U}$. By (R2) for $\boldsymbol{p}$ there exists $\mathcal{V} \in \mathcal{C}ov(P)$ satisfying the property in (R2) for $\boldsymbol{p}$, P and $\mathcal{U}'$. Take any $a \in A$ and maps $f', g' : F(X_a) \to P$ such that $(f'F(p_a), g'F(p_a)) < \mathcal{V}$. Since $F(p_a)t_X = t_{X_a}p_a$, $(f't_{X_a}p_a, g't_{X_a}p_a) < \mathcal{V}$. By the choice of $\mathcal{V}$ there exists $a' > a$ such that $(f't_{X_a}p_{a',a}, g't_{X_a}p_{a',a}) < \mathcal{U}'$. Since $t_{X_a}p_{a',a} = F(p_{a',a})t_{X_{a'}}$, $(f'F(p_{a',a})t_{X_{a'}}, g'F(p_{a',a})t_{X_{a'}}) < \mathcal{U}'$. By (6.2) $(f'F(p_{a',a}), g'F(p_{a',a})) < st\,\mathcal{U}' < \mathcal{U}$. Hence we have (R2).

Since $F(\boldsymbol{p})$ satisfies (R1) and (R2) for polyhedra, by (3.1) it is a resolution.

Next, we assume that $F(\boldsymbol{p})$ is a resolution. Thus it satisfies (R1) and (R2) for polyhedra. We show that $\boldsymbol{p}$ satisfies (R1) and (R2) for polyhedra.

We show (R1). Let $P \in \mathrm{Ob}\,\mathbf{POL}$ and $\mathcal{U} \in \mathcal{C}ov(P)$. Let $f : X \to P$ be a map. Since $\boldsymbol{t}$ is rigid for **POL**, there exists a map $f' : F(X) \to P$ such that $f = f't_X$. By (R1) for $F(\boldsymbol{p})$ there exists $a \in A$ and a map $g' : F(X_a) \to P$ such that $(g'F(p_a), f') < \mathcal{U}$. Thus $(g'F(p_a)t_X, f't_X) < \mathcal{U}$, and hence $(g't_{X_a}p_a, f) < \mathcal{U}$. This means that $g't_{X_a} : X_a \to P$ has the required one. Hence we have (R1).

We show (R2). Let $P \in \mathrm{Ob}\,\mathbf{POL}$ and $\mathcal{U} \in \mathcal{C}ov(P)$. There exists $\mathcal{V} \in \mathcal{C}ov(P)$ satisfying the condition (R2) for $F(\boldsymbol{p})$, P and $\mathcal{U}$. Take $\mathcal{W} \in \mathcal{C}ov(P)$ such that $st\,\mathcal{W} < \mathcal{V}$. Take any $a \in A$ and maps $f, g : X_a \to P$ such that $(fp_a, gp_a) < \mathcal{W}$. Since $\boldsymbol{t}$ is rigid for **POL**, there exist maps $f', g' : F(X_a) \to P$ satisfying $f = f't_{X_a}$ and $g = g't_{X_a}$. Since $fp_a = f'F(p_a)t_X$ and $gp_a = g'F(p_a)t_X$, $(f'F(p_a)t_X, g'F(p_a)t_X)$

$<\mathcal{W}$. By (6.2) $(f'F(p_a), g'F(p_a))<st\mathcal{W}<\mathcal{V}$. By the choice of $\mathcal{V}$ there exists $a'>a$ such that $(f'F(p_{a',a}), g'F(p_{a',a}))<\mathcal{U}$. Since $F(p_{a',a})t_{X_{a'}}=t_{X_a}p_{a',a}$, $(fp_{a',a}, gp_{a',a})<\mathcal{U}$. Hence we have (R2).

Since $\boldsymbol{p}$ satisfies (R1) and (R2) for polyhedra, by (3.1) it is a resolution. ∎

(6.3) LEMMA. *Let* $\boldsymbol{t}: j\to j'F$ *be dense and rigid for* **POL**. *If* $X\in \mathrm{Ob}\,\boldsymbol{C}$ *is an* AP, *then* $F(X)$ *is also an* AP.

PROOF. Take any $\mathcal{U}, \mathcal{U}_1\in \mathcal{C}ov(F(X))$ such that $st\,\mathcal{U}_1<\mathcal{U}$. Since X is an AP, there exist a polyhedron P and maps $f: X\to P$, $g: P\to X$ such that $(gf, 1_X)<t_X^{-1}(\mathcal{U}_1)$. Since $\boldsymbol{t}$ is rigid for **POL**, there exists a map $f': F(X)\to P$ such that $f=f't_X$. Since $(t_Xgf't_X, 1_{F(X)}t_X)<\mathcal{U}_1$, by (6.2) $(t_Xgf', 1_{F(X)})<st\,\mathcal{U}_1<\mathcal{U}$. Hence $F(X)$ is an AP. ∎

Hereafter we assume that $\boldsymbol{t}$ is dense and rigid for **POL** with the following two conditions:

(*) **POL** is a subcategory of $\boldsymbol{C}$.

(**) For each polyhedron P $F(P)=P$ and $t_P: P\to F(P)$ is the identity map.

Let $X\in \mathrm{Ob}\,\boldsymbol{C}$. By (3.15) there exists an approximative **POL**-resolution $\boldsymbol{p}=\{p_a: a\in A\}: X\to(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$. By (3.3) $\boldsymbol{p}: X\to\mathcal{X}$ is a resolution. By (6.1) $F(\boldsymbol{p})=\{F(p_a): a\in A\}: F(X)\to F(\mathcal{X})=\{F(X_a), F(p_{a',a}), A\}$ is a resolution. Since $\boldsymbol{t}$ is a natural transformation, $\boldsymbol{t}=\{1_A, t_{X_a}: a\in A\}: \mathcal{X}\to F(\mathcal{X})$ is a resolution of $t_X: X\to F(X)$ with respect to $\boldsymbol{p}$ and $F(\boldsymbol{p})$. Since $F(X_a)=X_a$ and $t_{X_a}=1_{X_a}$ by (*) and (**), $F(p_{a',a})=p_{a',a}$ for $a'>a$. Thus $F(\mathcal{X})=\mathcal{X}$ and $\boldsymbol{t}=1_{\mathcal{X}}$. Since $F(\boldsymbol{p}): F(X)\to F(\mathcal{X})=\mathcal{X}$ is a resolution, by (3.3) $F(\boldsymbol{p}): F(X)\to(\mathcal{X}, \mathcal{U})$ is an approximative resolution. Thus $\boldsymbol{t}=1_{(\mathcal{X},\mathcal{U})}: (\mathcal{X}, \mathcal{U})\to(\mathcal{X}, \mathcal{U})$ is an approximative resolution of t_X with respect to $\boldsymbol{p}$ and $F(\boldsymbol{p})$. Hence $\mathrm{AS}(t_X)=\langle[1_{(\mathcal{X},\mathcal{U})}]\rangle$, which is an isomorphism in **ASh**. We have the following:

(6.4) LEMMA. *For* $X\in \mathrm{Ob}\,\boldsymbol{C}$ $t_X: X\to F(X)$ *induces an isomorphism* $\mathrm{AS}(t_X): X\to F(X)$ *in* **ASh**. ∎

Let $Y\in \mathrm{Ob}\,\boldsymbol{C}$ and let $\boldsymbol{q}=\{q_b: b\in B\}: Y\to(\mathcal{Y}, \mathcal{V})=\{(Y_b, \mathcal{V}_b), q_{b',b}, B\}$ be an approximative **POL**-resolution. Since $F(\boldsymbol{p}): F(X)\to(\mathcal{X}, \mathcal{U})$ and $F(\boldsymbol{q}): F(Y)\to(\mathcal{Y}, \mathcal{V})$ are approximative **POL**-resolutions, $E(\boldsymbol{p}, \boldsymbol{q})=\text{Appro-}\mathbf{AP}((\mathcal{X}, \mathcal{U}), (\mathcal{Y}, \mathcal{V}))=E(F(\boldsymbol{p}), F(\boldsymbol{q}))$. Then we may define a bijective function $\Psi(\boldsymbol{p}, \boldsymbol{q}): E(\boldsymbol{p}, \boldsymbol{q})\to E(F(\boldsymbol{p}), F(\boldsymbol{q}))$ as follows: $\Psi(\boldsymbol{p}, \boldsymbol{q})([m])=[m]$ for $[m]\in E(\boldsymbol{p}, \boldsymbol{q})$.

(6.5) LEMMA. $\Psi(\boldsymbol{p}, \boldsymbol{q})([f]_{\boldsymbol{p},\boldsymbol{q}})=[F(f)]_{F(\boldsymbol{p}),F(\boldsymbol{q})}$ *for a map* $f: X\to Y$ *in* $\boldsymbol{C}$.

PROOF. Let $\boldsymbol{f}=\{f, f_b : b\in B\} : (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ be an approximative resolution of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. Thus $[\boldsymbol{f}]=[f]_{\boldsymbol{p},\boldsymbol{q}}$. Let $u : B\to B$ be a 1-refinement function of $(\mathcal{Y}, \mathcal{V})$. For each $b\in B$ $(q_{u(b)}f, f_{u(b)}p_{fu(b)})<\mathcal{V}_{u(b)}$. Since $F(q_{u(b)})F(f)t_X=q_{u(b)}f$, $F(f_{u(b)})=f_{u(b)}$ and $F(f_{u(b)})F(p_{fu(b)})t_X=f_{u(b)}p_{fu(b)}$ by (**), $(F(q_{u(b)})F(f)t_X, f_{u(b)}F(p_{fu(b)})t_X)<\mathcal{V}_{u(b)}$. By (6.2) $(F(q_{u(b)})F(f), f_{u(b)}F(p_{fu(b)}))$ $<st\,\mathcal{V}_{u(b)}$. Since u is a 1-refinement function, $(F(q_b)F(f), q_{u(b),b}f_{u(b)}F(p_{fu(b)}))$ $<\mathcal{V}_b$. This means that $\boldsymbol{q}(u)\boldsymbol{f} : (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ is an approximative resolution of $F(f)$ with respect to $F(\boldsymbol{p})$ and $F(\boldsymbol{q})$. Thus $[\boldsymbol{q}(u)\boldsymbol{f}]=[F(f)]_{F(\boldsymbol{p}),F(\boldsymbol{q})}$. Since $[\boldsymbol{q}(u)\boldsymbol{f}]=[\boldsymbol{f}]$ by (2.6), $\Psi(\boldsymbol{p}, \boldsymbol{q})([f]_{\boldsymbol{p},\boldsymbol{q}})=\Psi(\boldsymbol{p}, \boldsymbol{q})([\boldsymbol{f}])=[\boldsymbol{f}]=[\boldsymbol{q}(u)\boldsymbol{f}]=$ $[F(f)]_{F(\boldsymbol{p}),F(\boldsymbol{q})}$. ■

(6.6) COROLLARY. *$\Psi(\boldsymbol{p}, \boldsymbol{q})([1_X]_{\boldsymbol{p},\boldsymbol{p}'})=[1_{F(X)}]_{F(\boldsymbol{p}),F(\boldsymbol{p}')}$ for approximative* **POL**-*resolutions $\boldsymbol{p}$, $\boldsymbol{p}'$ of $X\in \mathrm{Ob}\,\boldsymbol{C}$.* ■

We define a function $\check{F}(\boldsymbol{p}, \boldsymbol{q}) : \mathbf{ASh}(X, Y)\to\mathbf{ASh}(F(X), F(Y))$ as follows: $\check{F}(\boldsymbol{p}, \boldsymbol{q})=\Phi(F(\boldsymbol{p}), F(\boldsymbol{q}))\Psi(\boldsymbol{p}, \boldsymbol{q})\Phi(\boldsymbol{p}, \boldsymbol{q})^{-1}$, where $\Phi(\boldsymbol{p}, \boldsymbol{q}) : E(X, Y)\to\langle E(X, Y)\rangle=$ $\mathbf{ASh}(X, Y)$ is defined in (5.4). By (6.6) and the definition $\equiv$ we easily show that $\check{F}(\boldsymbol{p}, \boldsymbol{q})=\check{F}(\boldsymbol{p}', \boldsymbol{q}')$ for approximative **POL**-resolutions $\boldsymbol{p}$, $\boldsymbol{p}'$ and $\boldsymbol{q}$, $\boldsymbol{q}'$ of X and Y, respectively. Thus we may put $\check{F}=\check{F}(\boldsymbol{p}, \boldsymbol{q}) : \mathbf{ASh}(X, Y)\to\mathbf{ASh}(F(X),$ $F(Y))$. Since $\Phi(\boldsymbol{p}, \boldsymbol{q})$ and $\Psi(\boldsymbol{p}, \boldsymbol{q})$ are bijection, so is $\check{F}$. By (6.6) and the definition of composition we have that $\check{F}(n)\check{F}(m)=\check{F}(nm)$ for $m\in\mathbf{ASh}(X, Y)$ and $n\in\mathbf{ASh}(Y, Z)$. Hence we have a functor $\check{F} : \mathbf{ASh}(\boldsymbol{C})\to\mathbf{ASh}(\boldsymbol{D})$, when $\check{F}(X)=$ $F(X)$ for $X\in\mathrm{Ob}\,\boldsymbol{C}$. Here $\mathbf{ASh}(\boldsymbol{C})$ denotes the full subcategory of **ASh** consisting of $\mathrm{Ob}\,\boldsymbol{C}$. (6.5) means that $\mathrm{AS}\circ F=\check{F}\circ\mathrm{AS}$. By definitions $\mathrm{ASS}=\mathrm{ASS}\circ\check{F}$. We summarize our results as follows:

(6.7) THEOREM. *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be full subcategories of* **TOP**. *Let $j : \boldsymbol{C}\to$* **TOP** *and $j' : \boldsymbol{D}\to$* **TOP** *be the inclusion functors. Let $F : \boldsymbol{C}\to\boldsymbol{D}$ and $\boldsymbol{t} : j\to j'F$ be a functor and a natural transformation, respectively. If $\boldsymbol{t}$ is dense and rigid for* **POL** *with (*) and (**), then F induces a functor $\check{F} : \mathbf{ASh}(\boldsymbol{C})\to\mathbf{ASh}(\boldsymbol{D})$ with the following properties:*

(i) *The following diagram is commutative:*

$$\begin{tikzcd}
\boldsymbol{C} \arrow[rr, "F"] \arrow[d, "\mathrm{AS}"] & & \boldsymbol{D} \arrow[d, "\mathrm{AS}"] \\
\mathbf{ASh}(\boldsymbol{C}) \arrow[rr, "\check{F}"] \arrow[dr, "\mathrm{ASS}"'] & & \mathbf{ASh}(\boldsymbol{D}) \arrow[dl, "\mathrm{ASS}"] \\
& \mathbf{Sh} &
\end{tikzcd}$$

(ii) $\mathrm{AS}(t_X): X \to F(X)$ *is an isomorphism in* $\mathbf{ASh}$ *for* $X \in \mathrm{Ob}\,\boldsymbol{C}$.

(iii) $\check{F}: \mathbf{ASh}(X, Y) \to \mathbf{ASh}(F(X), F(Y))$ *is bijective for* $X, Y \in \mathrm{Ob}\,\boldsymbol{C}$. ■

Tychonoff spaces are completely regular Hausdorff spaces. A Tychonoff space is topologically complete provided that it is complete with respect to some uniformities. $\mathbf{TOP}_{3.5}$ and $\mathbf{CTOP}_{3.5}$ denote the full subcategories of $\mathbf{TOP}$ consisting of all Tychonoff spaces and of all topologically complete Tychonoff spaces, respectively.

Morita [22] introduced the Tychonoff functor $T: \mathbf{TOP} \to \mathbf{TOP}_{3.5}$ and showed the following properties: For each space X there exists an onto map $t_X: X \to T(X)$ such that

(T1) if X is a Tychonoff space, then $T(X)=X$ and $t_X=1_X$,

(T2) for any map $f: X \to Y$ $t_Y f = T(f) t_X$ and

(T3) for any Tychonoff space Y and for any map $f: X \to Y$ there exists a unique map $g: T(X) \to Y$ such that $g t_X = f$.

Let $j': \mathbf{TOP}_{3.5} \to \mathbf{TOP}$ be the inclusion functor. By (T2) $\boldsymbol{t} = \{t_X\}: 1_{\mathbf{TOP}} \to j'T$ is a natural transformation. By the above data $\boldsymbol{t}$ and T satisfy all the assumptions in (6.7). Thus by (6.1) and (6.7) we have the following:

(6.8) COROLLARY. $\boldsymbol{p}: X \to \mathscr{X}$ *is resolution of a space* X *iff* $T(\boldsymbol{p}): T(X) \to T(\mathscr{X})$ *is a resolution of* $T(X)$. *Moreover* $\boldsymbol{p}$ *is rigid for* $\mathbf{TOP}_{3.5}$ *iff so is* $T(\boldsymbol{p})$. ■

(6.9) COROLLARY. *The Tychonoff functor* $T: \mathbf{TOP} \to \mathbf{TOP}_{3.5}$ *induces a functor* $\check{T}: \mathbf{ASh} \to \mathbf{ASh}(\mathbf{TOP}_{3.5})$ *with the following properties:*

(i) *The following diagram is commutative:*

$$\begin{tikzcd}
\mathbf{TOP} \arrow[rr, "T"] \arrow[d, "\mathrm{AS}"] & & \mathbf{TOP}_{3.5} \arrow[d, "\mathrm{AS}"] \\
\mathbf{ASh} \arrow[rr, "\check{T}"] \arrow[dr, "\mathrm{ASS}"'] & & \mathbf{ASh}(\mathbf{TOP}_{3.5}) \arrow[dl, "\mathrm{ASS}"] \\
& \mathbf{Sh} &
\end{tikzcd}$$

(ii) $\mathrm{AS}(t_X): X \to T(X)$ *is an isomorphism in* $\mathbf{ASh}$ *for any space* X.

(iii) $\check{T}: \mathbf{ASh}(X, Y) \to \mathbf{ASh}(T(X), T(Y))$ *is bijective for spaces* X *and* Y. ■

Let X be a Tychonoff space. Then $\mathcal{C}ov(X)$ forms the finest uniformity of X. Let $C(X)$ be the completion of X with respect to $\mathcal{C}ov(X)$. Thus we have the completion functor $C: \mathbf{TOP}_{3.5} \to \mathbf{CTOP}_{3.5}$ with the following properties: We may consider X as a dense subset of $C(X)$. Let $j_X: X \to C(X)$ be the inclusion map.

(C1) If X is topologically complete, then $C(X)=X$ and $j_X=1_X$.

(C2) $j_Y f=C(f)j_X$ for a map $f:X\to Y$ in $\mathbf{TOP}_{3.5}$.

(C3) Let $X\in\mathrm{Ob}\,\mathbf{TOP}_{3.5}$ and $Y\in\mathrm{Ob}\,\mathbf{CTOP}_{3.5}$. For any map $f:X\to Y$ there exists a unique map $g:C(X)\to Y$ such that $gj_X=f$.

Let $j':\mathbf{CTOP}_{3.5}\to\mathbf{TOP}_{3.5}$ be the inclusion functor. By (C2) $\boldsymbol{j}=\{j_X\}:1_{\mathbf{TOP}_{3.5}}\to j'C$ forms a natural functor. By the above data $\boldsymbol{j}$ satisfies all the assumptions in (6.7). Hence by (6.1)' and (6.7) we have the following:

(6.10) Corollary. *$\boldsymbol{p}:X\to\mathcal{X}$ is a resolution of a Tychonoff space X iff $C(\boldsymbol{p}):C(X)\to C(\mathcal{X})$ is a resolution of $C(X)$. Moreover $\boldsymbol{p}$ is rigid for $\mathbf{CTOP}_{3.5}$ iff so is $C(\boldsymbol{p})$.* ■

(6.11) Corollary. *The completion functor $C:\mathbf{TOP}_{3.5}\to\mathbf{CTOP}_{3.5}$ induces a functor $\check{C}:\mathbf{ASh}(\mathbf{TOP}_{3.5})\to\mathbf{ASh}(\mathbf{CTOP}_{3.5})$ satisfying the following:*

(i) *The following diagram is commutative:*

$$\begin{tikzcd}
\mathbf{TOP}_{3.5} \arrow[rr, "C"] \arrow[d, "\mathrm{AS}"] & & \mathbf{CTOP}_{3.5} \arrow[d, "\mathrm{AS}"] \\
\mathbf{ASh}(\mathbf{TOP}_{3.5}) \arrow[rr, "\check{C}"] \arrow[dr, "\mathrm{ASS}"'] & & \mathbf{ASh}(\mathbf{CTOP}_{3.5}) \arrow[dl, "\mathrm{ASS}"] \\
& \mathbf{Sh} &
\end{tikzcd}$$

(ii) $\mathrm{AS}(j_X):X\to C(X)$ *is an isomorphism in* $\mathbf{ASh}$ *for a Tychonoff space X.*

(iii) $\check{C}:\mathbf{ASh}(X,Y)\to\mathbf{ASh}(C(X),C(Y))$ *is bijective for Tychonoff spaces X and Y.* ■

(6.13) Remark. Independently Morita [23] considered (6.8) and (6.10). He showed only one directions of (6.8) and (6.10). ■

§ 7. The realization functor.

In this section we introduce the realization functor and investigate its properties.

(7.1) Lemma. *Let $\boldsymbol{q}=\{q_b:b\in B\}:Y\to\mathcal{Y}=\{Y_b,q_{b'b},B\}$ be a resolution of a space Y. If $Y\in\mathrm{Ob}\,\mathbf{CTOP}_{3.5}$ and $Y_b\in\mathrm{Ob}\,\mathbf{TOP}_{3.5}$ for $b\in B$, then $\boldsymbol{q}:Y\to\mathcal{Y}$ is an inverse limit of $\mathcal{Y}$.*

The author [**26**] has proved (7.1). After a while, independently, Morita [**23**] proved it. His paper is already published, and therefore we omit our proof. ■

Let $\boldsymbol{p}=\{p_a : a\in A\} : X\to(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ and $\boldsymbol{q}=\{q_b : b\in B\} : Y \to(\mathcal{Y}, \mathcal{V})=\{(Y_b, \mathcal{V}_b), q_{b',b}, B\}$ be approximative resolutions of spaces X and Y, respectively.

(7.2) LEMMA. *Let $\boldsymbol{f}=\{f_b : b\in B\}$ be a collection of maps $f_b : X\to Y_b$. If Y, $Y_b\in \mathrm{Ob}\,\mathbf{CTOP}_{3.5}$ for all $b\in B$ and $(f_b, q_{b',b}f_{b'})<\mathcal{V}_b$ for $b'>b$ then there exists a unique map $r(\boldsymbol{f}) : X\to Y$ such that $(f_b, q_b r(\boldsymbol{f}))<st\,\mathcal{V}_b$ for $b\in B$.*

PROOF. Take any $x\in X$ and any $b_0\in B$. We put $C_{b_0}(x)=\{q_{b,b_0}f_b(x) : b\in B$ *with* $b>b_0\}$.

Claim 1. $C_{b_0}(x)$ is a Cauchy net in Y_{b_0} with respect to the finest uniformity $\mathcal{C}ov(Y_{b_0})$.

Take any $\mathcal{V}$, $\mathcal{V}_1\in\mathcal{C}ov(Y_{b_0})$ with $st\,\mathcal{V}_1<\mathcal{V}$. By (AI3) there exists $b_1>b_0$ such that $q_{b_1,b_0}^{-1}\mathcal{V}_1>\mathcal{V}_{b_1}$. There exist $V_1\in\mathcal{V}_1$ and $V_2\in\mathcal{V}$ such that

(1) $q_{b_1,b_0}f_{b_1}(x)\in V_1$ and $st(V_1, \mathcal{V}_1)\subset V_2$.

Take any $b>b_1$. By the property of $\boldsymbol{f}$ and the choice of b_1, $(q_{b_1,b_0}f_{b_1}, q_{b,b_0}f_b) <\mathcal{V}_1$. There exists $V_3\in\mathcal{V}_1$ such that $q_{b_1,b_0}f_{b_1}(x)$, $q_{b,b_0}f_b(x)\in V_3$ and then by (1)

(2) $q_{b,b_0}f_b(x)\in st(V_1, \mathcal{V}_1)\subset V_2$ for each $b>b_1$.

(2) is the required condition. Thus we have Claim 1.

Since Y_{b_0} is topologically complete, there exists a unique limit point $r(\boldsymbol{f})_{b_0}(x)$ of $C_{b_0}(x)$. Then we have a function $r(\boldsymbol{f})_{b_0} : X\to Y_{b_0}$. It is easy to show that

(3) $q_{b_0',b_0}r(\boldsymbol{f})_{b_0'}=r(\boldsymbol{f})_{b_0}$ for any $b_0'>b_0$.

Claim 2. For each $\mathcal{V}\in\mathcal{C}ov(Y_{b_0})$ there exists $b_2>b_0$ such that $(q_{b,b_0}f_b, r(\boldsymbol{f})_{b_0}) <\mathcal{V}$ for each $b>b_2$.

Take any $\mathcal{V}\in\mathcal{C}ov(Y_{b_0})$. By (AI3) there exists $b_2>b_0$ such that $q_{b_2,b_0}^{-1}\mathcal{V}> st\,\mathcal{V}_{b_2}$. Take any $b>b_2$ and any $x\in X$. Since $r(\boldsymbol{f})_{b_2}(x)=\lim C_{b_2}(x)$, there exists $b_2(x)>b_2$ such that $q_{b',b_2}f_{b'}(x)\in st(r(\boldsymbol{f})_{b_2}(x), \mathcal{V}_{b_2})$ for each $b'>b_2(x)$. Then there exists $V_1\in\mathcal{V}_{b_2}$ such that $q_{b_2(x),b_2}f_{b_2(x)}(x)$, $r(\boldsymbol{f})_{b_2}(x)\in V_1$. By the condition of $\boldsymbol{f}$ there exist V_2, $V_3\in\mathcal{V}_{b_2}$ such that $f_{b_2}(x)$, $q_{b_2(x),b_2}f_{b_2(x)}(x)\in V_2$ and $f_{b_2}(x)$, $q_{b,b_2}f_b(x) \in V_3$. Then $r(\boldsymbol{f})_{b_2}(x)$, $q_{b,b_2}f_b(x)\in st(V_2, \mathcal{V}_{b_2})$. By the choice of b_2 there exists $V\in\mathcal{V}$ such that $q_{b_2,b_0}r(\boldsymbol{f})_{b_2}(x)$, $q_{b,b_0}f_b(x)\in V$. Then $(q_{b_2,b_0}r(\boldsymbol{f})_{b_2}, q_{b,b_0}f_b)<\mathcal{V}$ and hence by (3) we have Claim 2.

Claim 3. $r(\boldsymbol{f})_{b_0}: X \to Y_{b_0}$ is continuous.

Take any $x \in X$ and any neighborhood G of $r(\boldsymbol{f})_{b_0}(x)$ in Y_{b_0}. Since Y_{b_0} is a Tychonoff space, there exist $\mathcal{V}, \mathcal{V}_1 \in \mathcal{C}ov(Y_{b_0})$ such that $st\,\mathcal{V}_1 < \mathcal{V}$ and

(4) $st(r(\boldsymbol{f})_{b_0}(x), \mathcal{V}) \subset G.$

By Claim 2 there exists $b_2 > b_0$ such that

(5) $(q_{b,b_0} f_b, r(\boldsymbol{f})_{b_0}) < \mathcal{V}_1$ for $b > b_2$.

Since $q_{b_2,b_0} f_{b_2}: X \to Y_{b_0}$ is continuous, there exists an open neighborhood H of x in X such that

(6) $q_{b_2,b_0} f_{b_2}(H) \subset st(q_{b_2,b_0} f_{b_2}(x), \mathcal{V}_1).$

Take any $x' \in H$. By (5) there exist $V_1, V_2 \in \mathcal{V}_1$ such that $q_{b_2,b_0} f_{b_2}(x')$, $r(\boldsymbol{f})_{b_0}(x') \in V_1$ and $q_{b_2,b_0} f_{b_2}(x)$, $r(\boldsymbol{f})_{b_0}(x) \in V_2$. By (6) there exists $V_3 \in \mathcal{V}_1$ such that $q_{b_2,b_0} f_{b_2}(x')$, $q_{b_2,b_0} f_{b_2}(x) \in V_3$. Thus $r(\boldsymbol{f})_{b_0}(x')$, $r(\boldsymbol{f})_{b_0}(x) \in V_1 \cup V_2 \subset st(V_3, \mathcal{V}_1) \subset V_4$ for some $V_4 \in \mathcal{V}$, because $st\,\mathcal{V}_1 < \mathcal{V}$. By (4) $r(\boldsymbol{f})_{b_0}(x') \in st(r(\boldsymbol{f})_{b_0}(x), \mathcal{V}) \subset G$. This means that $r(\boldsymbol{f})_{b_0}(H) \subset G$. Hence it is continuous.

By (3.3) and (7.1) $\boldsymbol{q}: Y \to \mathcal{Y}$ is an inverse limit. By Claim 3 and (3) there exists a map $r(\boldsymbol{f}): X \to Y$ such that

(7) $q_b r(\boldsymbol{f}) = r(\boldsymbol{f})_b$ for $b \in B$.

Claim 4. $(f_b, q_b r(\boldsymbol{f})) < st\,\mathcal{V}_b$ for $b \in B$.

Take any $b_0 \in B$. By Claim 2 there exists $b_2 > b_0$ satisfying $(q_{b_2,b_0} f_{b_2}, r(\boldsymbol{f})_{b_0}) < \mathcal{V}_{b_0}$. Since $(q_{b_2,b_0} f_{b_2}, f_{b_0}) < \mathcal{V}_{b_0}$ by the property of $\boldsymbol{f}$, by (7) $(f_{b_0}, q_{b_0} r(\boldsymbol{f})) < st\,\mathcal{V}_{b_0}$. Hence we have Claim 4.

Claim 5. If $g, h: X \to Y$ are maps such that $(q_b g, q_b h) < st^2 \mathcal{V}_b$ for $b \in B$, then $g = h$.

We assume that $g \neq h$. There exists $x \in X$ such that $g(x) \neq h(x)$. Since $\boldsymbol{q}: Y \to \mathcal{Y}$ is an inverse limit by (7.1), there exists $b \in B$ such that $q_b g(x) \neq q_b h(x)$. Since Y_b is Tychonoff, there exists $\mathcal{V} \in \mathcal{C}ov(Y_b)$ such that $st(q_b g(x), \mathcal{V}) \cap st(q_b h(x), \mathcal{V}) = \emptyset$. By (AI3) there exists $b' > b$ such that $q_{b',b}^{-1} \mathcal{V} > st^2 \mathcal{V}_{b'}$. This and the assumption imply that $(q_b g, q_b h) < \mathcal{V}$. Then there exists $V \in \mathcal{V}$ such that $q_b g(x), q_b h(x) \in V$. This means that $st(q_b g(x), \mathcal{V}) \cap st(q_b h(x), \mathcal{V}) \neq \emptyset$. This is a contradiction. Hence $g = h$. We have Claim 5.

From Claims 4 and 5 we have the uniqueness of $r(\boldsymbol{f})$. Hence we have completed the proof. ∎

(7.3) LEMMA. *Let $Y, Y_b \in \mathrm{Ob}\,\mathbf{CTOP}_{3.5}$ for $b \in B$. For any approximative system map $\boldsymbol{f}: (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ there exists a unique map $r(\boldsymbol{f}): X \to Y$ such that*

$(f_b p_{f(b)}, q_b r(\boldsymbol{f})) < st\, \mathcal{V}_b$ *for each* $b \in B$.

When we apply (7.2) to the collection $\{f_b p_{f(b)} : b \in B\}$ of maps, we have (7.3). ■

(7.4) LEMMA. *Let* $f : X \to Y$ *be a map. Under the same conditions as in* (7.3), *if* $\boldsymbol{f}$ *is an approximative resolution of* f *with respect to* $\boldsymbol{p}$ *and* $\boldsymbol{q}$, *then* $r(\boldsymbol{f}) = f$.

(7.5) LEMMA. *Let* $\boldsymbol{f}' : (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ *be an approximative system map. Under the same conditions as in* (7.3), *if* $[\boldsymbol{f}] = [\boldsymbol{f}']$, *then* $r(\boldsymbol{f}) = r(\boldsymbol{f}')$.

(7.4) and (7.5) follow from Claim 5 in the proof of (7.2). ■

Let $\boldsymbol{k} = \{k, k_c : c \in C\} : Z \to (\mathcal{Z}, \mathcal{W}) = \{(Z_c, \mathcal{W}_c), k_{c',c}, C\}$ be an approximative resolution of a space Z. Let $\boldsymbol{g} = \{g, g_c : c \in C\} : (\mathcal{Y}, \mathcal{V}) \to (\mathcal{Z}, \mathcal{W})$ be an approximative system map.

(7.6) LEMMA. *Let* $Y, Y_b, Z, Z_c \in \mathrm{Ob}\,\mathbf{CTOP}_{3.5}$. *For each* 1-*refinement function* $u : C \to C$ *of* $(\mathcal{Z}, \mathcal{W})$ $r(\boldsymbol{g}) r(\boldsymbol{f}) = r(\boldsymbol{k}(u)(\boldsymbol{g}\boldsymbol{f}))$.

PROOF. Take any $c \in C$. Since $(g_{u(c)} q_{gu(c)}, k_{u(c)} r(\boldsymbol{g})) < st\, \mathcal{W}_{u(c)}$ by (7.3), $(g_{u(c)} q_{gu(c)} r(\boldsymbol{f}), k_{u(c)} r(\boldsymbol{g}) r(\boldsymbol{f})) < st\, \mathcal{W}_{u(c)}$. Since $(f_{gu(c)} p_{fgu(c)}, q_{gu(c)} r(\boldsymbol{f})) < st\, \mathcal{V}_{gu(c)}$ by (7.3), by (AM1) and (2.2) $(g_{u(c)} f_{gu(c)} p_{fgu(c)}, g_{u(c)} q_{gu(c)} r(\boldsymbol{f})) < st\, \mathcal{W}_{u(c)}$. Then $(g_{u(c)} f_{gu(c)} p_{fgu(c)}, k_{u(c)} r(\boldsymbol{g}) r(\boldsymbol{f})) < st^2 \mathcal{W}_{u(c)}$. Since u is a 1-refinement function, we have that

(1) $(k_{u(c),c} g_{u(c)} f_{gu(c)} p_{fgu(c)}, k_c r(\boldsymbol{g}) r(\boldsymbol{f})) < st\, \mathcal{W}_c$.

By (7.3)

(2) $(k_{u(c),c} g_{u(c)} f_{gu(c)} p_{fgu(c)}, k_c r(\boldsymbol{k}(u)(\boldsymbol{g}\boldsymbol{f}))) < st\, \mathcal{W}_c$.

By (1) and (2) $(k_c r(\boldsymbol{g}) r(\boldsymbol{f}), k_c r(\boldsymbol{k}(u)(\boldsymbol{g}\boldsymbol{f}))) < st^2 \mathcal{W}_c$. Hence by Claim 5 in the proof of (7.2) $r(\boldsymbol{g}) r(\boldsymbol{f}) = r(\boldsymbol{k}(u)(\boldsymbol{g}\boldsymbol{f}))$. ■

(7.7) LEMMA. *Let* $\boldsymbol{q} : Y \to (\mathcal{Y}, \mathcal{V})$ *be an approximative* **AP**-*resolution. Under the same conditions as in* (7.3) $[\boldsymbol{f}] = [r(\boldsymbol{f})]_{\boldsymbol{p},\boldsymbol{q}}$ *for each approximative system map* $\boldsymbol{f} : (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$.

PROOF. Take any 1-refinement function u of $(\mathcal{Y}, \mathcal{V})$ and any $b \in B$. By (7.3) $(f_{u(b)} p_{fu(b)}, q_{u(b)} r(\boldsymbol{f})) < st\, \mathcal{V}_{u(b)}$ and then $(q_{u(b),b} f_{u(b)} p_{fu(b)}, q_b r(\boldsymbol{f})) < \mathcal{V}_b$. This means that $\boldsymbol{q}(u)\boldsymbol{f}$ is an approximative resolution of $r(\boldsymbol{f})$ with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. Thus $[r(\boldsymbol{f})]_{\boldsymbol{p},\boldsymbol{q}} = [\boldsymbol{q}(u)\boldsymbol{f}]$. Since $[\boldsymbol{f}] = [\boldsymbol{q}(u)\boldsymbol{f}]$ by (2.6), $[\boldsymbol{f}] = [r(\boldsymbol{f})]_{\boldsymbol{p},\boldsymbol{q}}$. ■

We assume that all spaces are completely Tychonoff spaces. Let $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{p}' : X \to (\mathcal{X}, \mathcal{U})'$, $\boldsymbol{q}' : Y \to (\mathcal{Y}, \mathcal{V})'$, $\boldsymbol{k} : Z \to (\mathcal{Z}, \mathcal{W})$ be approximative **POL**-resolutions.

Let $\boldsymbol{f} : (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ be an approximative system map. By (7.3) there exists a unique map $r(\boldsymbol{f}) : X \to Y$. By (7.5) $r(\boldsymbol{f})$ does not depend on representations of the equivalence class $[\boldsymbol{f}]$. Thus we may define $r([\boldsymbol{f}]) = r(\boldsymbol{f})$.

Let $\boldsymbol{g} : (\mathcal{Y}, \mathcal{V}) \to (\mathcal{Z}, \mathcal{W})$ be an approximative system map. (7.6) means that

(i) $r([\boldsymbol{g}])r([\boldsymbol{f}]) = r([\boldsymbol{g}][\boldsymbol{f}])$.

By (7.4) we have that

(ii) $r([1_X]_{\boldsymbol{p},\boldsymbol{p}'}) = 1_X$.

By (i) and (ii) we can easily show that for an approximative system maps $\boldsymbol{f}' : (\mathcal{X}, \mathcal{U})' \to (\mathcal{Y}, \mathcal{V})'$

(iii) if $\langle[\boldsymbol{f}]\rangle = \langle[\boldsymbol{f}']\rangle$, then $r([\boldsymbol{f}]) = r([\boldsymbol{f}'])$.

(iii) means that $r([\boldsymbol{f}])$ does not depend on representations of the equivalence class $\langle[\boldsymbol{f}]\rangle$. Thus we may define $r(\langle[\boldsymbol{f}]\rangle) = r([\boldsymbol{f}])$. By (i) and (ii) we easily show that

(iv) $r(\langle[\boldsymbol{g}']\rangle)r(\langle[\boldsymbol{f}]\rangle) = r(\langle[\boldsymbol{g}']\rangle\langle[\boldsymbol{f}]\rangle)$, where $\boldsymbol{g}' : (\mathcal{Y}, \mathcal{V})' \to (\mathcal{Z}, \mathcal{W})'$ is an approximative system map.

By (ii) we have that

(v) $r(\langle[1_X]_{\boldsymbol{p},\boldsymbol{p}'}\rangle) = 1_X$.

Now we define the realization functor $R : \mathbf{ASh}(\mathbf{CTOP}_{3.5}) \to \mathbf{CTOP}_{3.5}$ as follows: $R(X) = X$ for $X \in \mathrm{Ob}\,\mathbf{CTOP}_{3.5}$ and $R(m) = r(\Phi(\boldsymbol{p}, \boldsymbol{q})^{-1}(m))$ for $m \in \mathbf{ASh}(\mathbf{CTOP}_{3.5})(X, Y)$. Here $\boldsymbol{p} : X \to (\mathcal{X}, \mathcal{U})$ and $\boldsymbol{q} : Y \to (\mathcal{Y}, \mathcal{V})$ are approximative **POL**-resolutions and $\Phi(\boldsymbol{p}, \boldsymbol{q}) : E(\boldsymbol{p}, \boldsymbol{q}) \to \mathbf{ASh}(X, Y)$ is defined in (5.4). By (iii) R is well defined. By (iv) and (v) R forms a functor. (7.4) and (iii) mean that $R \circ \mathrm{AS} = 1$. (7.7) and (iii) mean that $\mathrm{AS} \circ R = 1$. Hence we summarize as follows:

(7.8) THEOREM. *There exists a realization functor* $R : \mathbf{ASh}(\mathbf{CTOP}_{3.5}) \to \mathbf{CTOP}_{3.5}$ *with the following commutative diagram:*

$$\begin{tikzcd}
\mathbf{CTOP}_{3.5} \arrow[rr, "1"] \arrow[dr, "\mathrm{AS}"'] & & \mathbf{CTOP}_{3.5} \arrow[dr, "\mathrm{AS}"] & \\
& \mathbf{ASh}(\mathbf{CTOP}_{3.5}) \arrow[ur, "R"'] \arrow[rr, "1"] & & \mathbf{ASh}(\mathbf{CTOP}_{3.5})
\end{tikzcd} \quad \blacksquare$$

Let $\boldsymbol{P}$ be the full subcategory of **TOP** consisting of all paracompact spaces. Note that paracompact spaces are topologically complete Tychonoff spaces.

(7.9) COROLLARY. (i) *R : $\mathbf{ASh}(\mathbf{CTOP}_{3.5}) \to \mathbf{CTOP}_{3.5}$ is a categorical isomorphism.*

(ii) *R induces a categorical isomorphism R : $\mathbf{AShP}(\boldsymbol{P}) \to \boldsymbol{P}$.* ■

(7.10) COROLLARY. *Let $f: X \to Y$ be a map. Let $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$ and $\boldsymbol{q}: Y \to (\mathcal{Y}, \mathcal{V})$ be approximative* **AP**-*resolutions. Let $\boldsymbol{f}: (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ be an approximative resolution of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. Then the following assertions are equivalent:*

(i) *$\boldsymbol{f}$ satisfies* (ISO) *in Appro-***AP**.

(ii) *$[\boldsymbol{f}]$ is an isomorphism in Appro-***AP**.

(iii) *AS(f) is an isomorphism in* **ASh**.

(iv) *CT(f) : CT(X) $\to$ CT(Y) is a homeomorphism.*

(7.11) COROLLARY. *Spaces X and Y have the same approximative shape type iff* CT(X) *and* CT(Y) *are homeomorphic.*

PROOFS OF (7.10) AND (7.11). We show (7.10). (i) and (ii) are equivalent by (2.16). From the definition of **ASh** it is easy to show that (ii) and (iii) are equivalent. By (6.9), (6.11) and (7.8) (iii) and (iv) are equivalent. (7.11) follows from (6.9), (6.11) and (7.10). ■

Shape theory is a generalization of homotopy theory on **POL**. The principle of shape theory is to " investigated bad spaces and bad maps by means of the good category **HPOL** ". (7.9) gives us a new description of $\mathbf{CTOP}_{3.5}$. Thus we can study **TOP** throughout **ASh**. The principle of approximative shape theory is to " *Investigate bad spaces and bad maps by means of the good category* **POL** ". Our theory and shape theory are similar in ideas. We say that *our approximative shape theory is a shape theory without homotopies.* In the papers which will follow we will show that **ASh** has richer structures than **TOP** and has many applications in topology.

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