

AUTOMORPHISMS OF FINITE ORDER OF THE AFFINE LIE ALGEBRA $A_l^{(1)}$

By

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Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

0. Introduction

We will classify all automorphisms of prime order of the *affine Lie algebra* $A_l^{(1)}$ up to conjugacy in the group of all automorphisms of $A_l^{(1)}$. To do this, we will use non abelian *group cohomology* of some finite cyclic group acting on $PGL_{l+1}(C[t, t^{-1}])$.

The authors wishes to express his appreciation to Professor E. Abe, Professor K. Kato, Mr. Terasoma for their encouragement and help with the present work.

1. Preliminary

Let \mathcal{G} be the *affine Lie algebra* over C of type $A_{n-1}^{(1)}$ ($n \geq 2$), i.e. the Lie algebra over C generated by e_i, h_i, f_i ($1 \leq i \leq n$) with the following defining relations; for $n > 2$

$$\begin{aligned} [h_i, h_j] &= 0, [e_i, f_j] = \delta_{ij} h_i \quad \text{for all } i, j. \\ [h_i, e_j] &= \begin{cases} 2e_i & \text{if } i=j, \\ -e_j & \text{if } |i-j|=1 \text{ or } n-1, \\ 0 & \text{otherwise,} \end{cases} \\ [h_i, f_j] &= \begin{cases} -2f_i & \text{if } i=j, \\ f_j & \text{if } |i-j|=1 \text{ or } n-1, \\ 0 & \text{otherwise,} \end{cases} \\ [e_i, [e_i, e_j]] &= [f_i, [f_i, f_j]] = 0 \quad \text{if } |i-j|=1 \text{ or } n-1, \\ [e_i, e_j] &= [f_i, f_j] = 0 \quad \text{if } |i-j| \neq 1 \text{ and } n-1, \end{aligned}$$

and for $n=2$

$$\begin{aligned} [h_i, h_j] &= 0, [e_i, f_j] = \delta_{ij} h_i \quad \text{for all } i, j, \\ [h_i, e_j] &= \begin{cases} 2e_i & \text{if } i=j, \\ -2e_j & \text{if } i \neq j, \end{cases} \\ [h_i, f_j] &= \begin{cases} -2f_i & \text{if } i=j, \\ 2f_j & \text{if } i \neq j, \end{cases} \\ [e_i, [e_i, [e_i, e_j]]] &= [f_i, [f_i, [f_i, f_j]]] = 0 \quad \text{if } i \neq j. \end{aligned}$$

Let E_{ij} be the matrix unit with 1 in the i, j position and 0 elsewhere, and let $C[t, t^{-1}]$ be the ring of Laurent polynomials in t with the coefficients in C .

Then we have a universal central extension over C :

$$(1) \quad 0 \longrightarrow Cz \longrightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{sl}_n(C[t, t^{-1}]) \longrightarrow 0,$$

where $z = h_1 + h_2 + \cdots + h_n$, $\pi(e_i) = E_{i, i+1}$, $\pi(e_n) = tE_{n1}$, $\pi(h_i) = E_{ii} - E_{i+1, i+1}$, $\pi(h_n) = E_{11} - E_{nn}$, $\pi(f_i) = E_{i+1, i}$, and $\pi(f_n) = t^{-1}E_{1n}$ for all $i = 1, \dots, n-1$.

By the universality of (1) and Theorem 2 [4],

$$(2) \quad \text{Aut}(\mathfrak{g}) \simeq \text{Aut}_C(\mathfrak{sl}_n(C[t, t^{-1}])) \\ \simeq \begin{cases} \text{Aut}_{C\text{-alg}}(C[t, t^{-1}]) \rtimes PGL_2(C[t, t^{-1}]) & \text{when } n=2 \\ (\langle \tau \rangle \times \text{Aut}_{C\text{-alg}}(C[t, t^{-1}])) \rtimes PGL_n(C[t, t^{-1}]) & \text{when } n \geq 3, \end{cases}$$

where τ is the involutive automorphism induced by the Dynkin diagram automorphism of A_{n-1} . More precisely, τ is defined by $\tau(e_i) = -e_{n-i}$ and $\tau(f_i) = -f_{n-i}$ for $i = 1, \dots, n-1$, $\tau(e_n) = -e_n$ and $\tau(f_n) = -f_n$. $PGL_n(C[t, t^{-1}])$ acts on $\mathfrak{sl}_n(C[t, t^{-1}])$ by conjugation (cf. [3]).

To classify all the automorphisms of finite order of \mathfrak{g} up to conjugacy is equivalent to classifying the elements of finite order of $\text{Aut}_C(\mathfrak{sl}_n(C[t, t^{-1}]))$ up to the conjugacy. Put $G = \text{Aut}_C(\mathfrak{sl}_n(C[t, t^{-1}]))$, $G_1 = \langle \tau \rangle \times \text{Aut}_{C\text{-alg}}(C[t, t^{-1}])$ and $G_2 = PGL_n(C[t, t^{-1}])$. Then elements of order k ($k \geq 2$) of $G = G_1 \cdot G_2$ have the expression $g_1 g_2$ ($g_i \in G_i$), where $g_1^k = 1$ and $\prod_{j=0}^{k-1} g_1^{-j} g_2 g_1^j = 1$ and if g_1 is conjugate to g'_1 in G then $g_1 g_2$ is conjugate in G to an element which has the expression $g'_1 g'_2$ for some $g'_2 \in G_2$.

2. Group cohomology

From now on we will let $R = C[t, t^{-1}]$. Let θP be an element of $\text{Aut}_C(\mathfrak{sl}_n(R))$, where $\theta \in \langle \tau \rangle \times \text{Aut}_{C\text{-alg}}(R)$ and $P \in PGL_n(R)$. θP is of prime order k if and only if $\theta^k = 1$ and $(\theta^{k-1} \cdot P) \cdots (\theta \cdot P) P = I_n$ (in $PGL_n(R)$), where $\theta \cdot$ notes the action of θ on $PGL_n(R)$. Let θP_1 and θP_2 be elements of order k . θP_1 is conjugate to θP_2 under $PGL_n(R)$ i.e. $\theta P_1 = Q^{-1}(\theta P_2)Q$ for some $Q \in PGL_n(R)$, if and only if $P_1 = (\theta \cdot Q^{-1})P_2 Q$ for some $Q \in PGL_n(R)$. The condition $(\theta^{k-1} \cdot P) \cdots (\theta \cdot P) P = I_n$ (resp. $P_1 = (\theta \cdot Q^{-1})P_2 Q$) coincides with the *cocycle condition* (resp. the *coboundary condition*) of the *group cohomology* $H^1(\mathbf{Z}_k, PGL_n(R))$ under the action of θ (= a *generator* of \mathbf{Z}_k) on $PGL_n(R)$.

Let σ (resp. ε_2) be the automorphism of R induced by $t \mapsto t^{-1}$ (resp. $t \mapsto -t$), then the set $\{\sigma, \varepsilon_2, \tau, \tau\sigma, \tau\varepsilon_2\}$ is a set of representatives of the conjugacy classes of order 2 of $\langle \tau \rangle \times \text{Aut}_{C\text{-alg}}(R)$. Let ε_k be the automorphism of R induced by $t \mapsto \zeta_k t$ ($\zeta_k =$ fixed k -th primitive root of unity), then the set $\{\varepsilon_k, (\varepsilon_k)^2, \dots, (\varepsilon_k)^{(k-1)/2}\}$ is a set of representatives of the conjugacy classes of odd prime order k of $\langle \tau \rangle \times \text{Aut}_{C\text{-alg}}(R)$.

In the following section, we will determine some *cohomologies* $H^1(\mathbf{Z}_k, PGL_n(R))$ in the following situation:

- (1) $k=2$
 - (a) trivial action

- (b) σ -action
 - (c) ε_2 -action
 - (d) τ -action
 - (e) $\tau\sigma$ -action
 - (f) $\tau\varepsilon_2$ -action.
- (2) $k \geq 3$
- (a) trivial action
 - (b) ε_k -action

For the rest of this section, we will determine $H^1(\mathbf{Z}_k, GL_n(R))$ with the above actions. We begin with the case when $k=2$.

- (1) $k=2$
- (a) trivial action.

PROPOSITION 1.

Under the trivial action on \mathbf{Z}_2 on $GL_n(R)$,

$$H^1(\mathbf{Z}_2, GL_n(R)) = \left\{ I_{a,b} = \begin{bmatrix} I_a & \\ & -I_b \end{bmatrix} \mid a+b=n \right\}$$

where I_k is the $k \times k$ unit matrix.

PROOF.

In this case, the cocycle condition is $P^2 = I_n P \in GL_n(R)$ and the coboundary condition is $P_1 \sim P_2 \iff P_1 = Q^{-1} P_2 Q$ for some $Q \in GL_n(R)$. Let M be a free R module of rank n , and let ϕ be a R module endomorphism satisfying $\phi^2 = \text{identity}$. We define free R modules M_1 and M_2 as follows $M_1 = \{m \in M \mid \phi(m) = m\}$, $M_2 = \{m \in M \mid \phi(m) = -m\}$. Then $M = M_1 \oplus M_2$ (direct sum) because $(1/2) \in R$. q.e.d.

- (b) σ -action.

PROPOSITION 2.

Under the σ -action of \mathbf{Z}_2 on $GL_n(R)$,

$$H^1(\mathbf{Z}_2, GL_n(R)) = \left\{ J_{a,b,c} = \begin{bmatrix} I_a & & \\ & tI_b & \\ & & -I_c \end{bmatrix}, J'_{a,b,c} = \begin{bmatrix} I_a & & \\ & -tI_b & \\ & & -I_c \end{bmatrix} \mid a+b+c=n \right\}.$$

In this case, the cocycle condition is $(\sigma \cdot P)P = I_n$, $P \in GL_n(R)$ and the coboundary condition is $P_1 \sim P_2 \iff P_1 = (\sigma \cdot Q)P_2Q^{-1}$ for some $Q \in GL_n(R)$. We first prove several lemmas.

LEMMA 1.

If P satisfies the cocycle condition and $P(1) = I_n$, $P(-1) = I_n$ ($P(a)$ is the specialization of

$P \in \mathcal{A}_n(R)$ at $a \in C^\times$, then $P = (\sigma \cdot Q)Q^{-1}$ for some $Q \in GL_n(R)$.

PROOF.

Let R^σ be the invariant subring of σ . Then R is a finite projective R^σ -algebra. Using Grothendieck's "theory of descent", we can prove $P = (\sigma \cdot Q)Q^{-1}$ for some $Q \in GL_n(R)$ (cf. Theorem 4.3[1]).

LEMMA 2.

If P satisfies the cocycle condition, then there exists $Q \in GL_n(R)$ s.t.

- (1) Q satisfies the cocycle condition
- (2) Q is cohomologous to P
- (3) $Q(1) = \begin{bmatrix} I_a & \\ & -I_b \end{bmatrix}$, $Q(-1) = \begin{bmatrix} I_c & \\ & -I_d \end{bmatrix}$
for some a, b, c, d .

PROOF.

Since $P(1)^2 = I_n$, there exists $\alpha \in GL_n(C)$ s.t.

$$\alpha P(1) \alpha^{-1} = \begin{bmatrix} I_a & \\ & -I_b \end{bmatrix}.$$

We may assume:

$$P(1) = \begin{bmatrix} I_a & \\ & -I_b \end{bmatrix}.$$

Let $\beta \in SL_n(C)$ s.t.

$$\beta P(-1) \beta^{-1} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & -1 \end{bmatrix},$$

then there exists $Z \in SL_n(R)$ s.t. $Z(1) = I_n$, $Z(-1) = \beta$. Set $Q = ZPZ^{-1}$, then

$$Q(1) = \begin{bmatrix} I_a & \\ & -I_b \end{bmatrix} \quad \text{and} \quad Q(-1) = \begin{bmatrix} I_c & \\ & -I_d \end{bmatrix}. \quad \text{q.e.d.}$$

We will call (a, b, c, d) the *invariant* of P . If P_1 is cohomologous to P_2 , then P_1, P_2 have the same *invariant*.

LEMMA 3.

If P_1, P_2 satisfy the cocycle condition and have the same invariant (a, b, c, d) , then P_1 is cohomologous to P_2 .

PROOF.

We will only prove this when $a > c$. The cases $a = c$ and $a < c$ are similar.

$$I_{c,p,b} = \begin{bmatrix} I_c & & \\ & tI_p & \\ & & -I_b \end{bmatrix},$$

then the *invariant* of $I_{c,p,b}$ is (a, b, c, d) . From the above lemma, We may assume

$$P_1(1) = \begin{bmatrix} I_a & \\ & -I_b \end{bmatrix} \quad \text{and} \quad P_1(-1) = \begin{bmatrix} I_c & \\ & -I_d \end{bmatrix}.$$

We will prove P_1 is cohomologous to $I_{c,p,b}$.

Set

$$\Gamma = \left\{ P \in GL_n(R) \mid P(1) = \begin{bmatrix} I_a & \\ & -I_b \end{bmatrix}, P(-1) = \begin{bmatrix} I_c & \\ & -I_d \end{bmatrix} \right\}$$

and

$$U = \begin{bmatrix} I_c & & \\ & \frac{1}{1+t} I_p & \\ & & \frac{1-t}{1+t} I_b \end{bmatrix}.$$

Then:

$$(1) \quad U^{-1}(\sigma \cdot U) = \begin{bmatrix} I_c & & \\ & tI_p & \\ & & -I_b \end{bmatrix}$$

$$(2) \quad (\sigma \cdot U) U^{-1} \in GL_n(R).$$

Let $P \in \Gamma$, then

$$((\sigma \cdot U) P U^{-1})(1) = I_n$$

$$((\sigma \cdot U) P U^{-1})(-1) = I_n.$$

Since P_1 satisfies the cocycle condition, $\sigma \cdot ((\sigma \cdot U) P_1 U^{-1}) ((\sigma \cdot U) P_1 U^{-1}) = I_n$.

From the above lemma, $(\sigma \cdot U) P_1 U^{-1} = (\sigma \cdot Q) Q^{-1}$ for some $Q \in GL_n(R)$.

By the following lemma, we may assume $Q(1) = I_n$, and $Q(-1) = \lambda I_n$ for some $\lambda \in C^\times$.

Then $U^{-1} Q (\sigma \cdot U) \in GL_n(R)$ and

$$\begin{aligned} P_1 &= (\sigma \cdot U)^{-1} (\sigma \cdot Q) Q^{-1} U \\ &= \sigma \cdot (U^{-1} Q (\sigma \cdot U)) U^{-1} (\sigma \cdot U) (U^{-1} Q (\sigma \cdot U))^{-1}. \end{aligned}$$

Since

$$U^{-1}(\sigma \cdot U) = \begin{bmatrix} I_c & & \\ & tI_b & \\ & & -I_b \end{bmatrix},$$

P_1 is cohomologous to $I_{c,p,b}$.

LEMMA 4.

Let U, Q be as above. There exists $Q_1 \in GL_n(R^\sigma)$ s.t. $QQ_1(1) = I_n$ and $QQ_1(-1) = \lambda I$ for some $\lambda \in C^\times$, where R^σ is the σ -invariant subring of R . In particular,

$$\begin{aligned} (\sigma \cdot (QQ_1))(QQ_1)^{-1} &= (\sigma \cdot Q)(\sigma \cdot Q_1)Q_1^{-1}Q^{-1} \\ &= (\sigma \cdot Q)Q^{-1} \\ &= (\sigma \cdot U)P_1U^{-1}. \end{aligned}$$

PROOF.

Set $\gamma = Q(1)^{-1} \in GL_n(C)$. Let $\lambda \in C^\times$ s.t. $\det(Q(-1)\gamma) = \lambda^n$ and $A \in SL_n(R^\sigma)$ s.t. $A(-1) = Q(-1) \cdot \lambda^{-1} I_n \gamma$ and $A(1) = I_n$. Set $Q_1 = \gamma A^{-1}$, then $Q(1)Q_1(1) = I_n$ and $Q(-1)Q_1(-1) = \lambda I_n$.

PROOF OF PROP. 2.

From the above lemmas, $\{J_{a,b,c}, J'_{a,b,c} | a+b+c=n\}$ is a set of representatives of $H^1(\mathbf{Z}_2, GL_n(R))$. q.e.d.

(c) ε_2 -action

PROPOSITION 3.

Under the ε_2 -action of \mathbf{Z}_2 on $GL_n(R)$, $H^1(\mathbf{Z}_2, GL_n(R)) = \{I_n\}$.

PROOF.

Let R^{ε_2} be the invariant subring of ε_2 . Then $R^{\varepsilon_2} \subset R$ is a galois extension (cf. P. 44 [1]). Using Grothendieck's "theory of descent", we can prove $H^1(\mathbf{Z}_2, GL_n(R)) = \{I_n\}$ (cf. Theorem 5.1 [1]).

(d) τ -action.

PROPOSITION 4.

Under the τ -action of \mathbf{Z}_2 on $GL_n(R)$,

$$H^1(\mathbf{Z}_2, GL_n(R)) = \left\{ K_1 = \begin{bmatrix} & & 1 \\ & \cdot & \cdot \\ 1 & & \end{bmatrix}, K_2 = \begin{bmatrix} & & 1 \\ & \cdot & \cdot \\ t & \cdot & \cdot \end{bmatrix} \right\}.$$

Let $P = (p_{ij}) \in GL_n(R)$, then $\tau \cdot P = (p_{n+1-j, n+1-i})^{-1}$. In this case, the cocycle condition is $(p_i) = (p_{n+1-j, n+1-i})$, $P = (p_{ij}) \in GL_n(R)$ and the coboundary condition is $P_1 \sim P_2 \Leftrightarrow P_1 = (q_{n+1-j, n+1-i})P_2Q$ for some $Q = (q_{ij}) \in GL_n(R)$. When we put $\tilde{P} = K_1P$, these conditions are equivalent to $\tilde{P} = {}^t\tilde{P}$ and $\tilde{P}_1 \sim \tilde{P}_2 \Leftrightarrow \tilde{P}_1 = {}^tQ\tilde{P}_2Q$ for some $Q \in GL_n(R)$, where tP is the transposed matrix of P .

We first prove several lemmas.

LEMMA 5.

Let $K = C(t)$ and let $X \in GL_n(K)$ s.t. ${}^tX = X$.

Let ψ_X be the bilinear form on K^n defined by $\psi_X(x, y) = {}^t x X y$ for $x, y \in K^n$. Then

(1) When n is even, the dimension of a maximal anisotropic subspace is $n/2$ or $(n/2) - 1$.

(2) When n is odd, the dimension of a maximal anisotropic subspace is $(n - 1)/2$, where an anisotropic subspace is a subspace V of K^n s.t. $\psi_X(x, y) = 0$ for all $x, y \in V$.

PROOF.

Let V be a maximal anisotropic subspace, and let $V^\perp = \{x \in K^n \mid \psi_X(x, v) = 0 \text{ for all } v \in V\}$. ψ_X defines a non-degenerate symmetric bilinear form $\tilde{\psi}_X$ on V^\perp/V . It is sufficient to prove $\dim_K V^\perp/V \leq 2$. Assume $\dim_K V^\perp/V \geq 3$ and take a three dimensional subspace W of V^\perp/V . Since K is a C_1 -field, there exists an element $w \in W$ s.t. $\tilde{\psi}_X(w, w) = 0$. Then $V \oplus wK$ is anisotropic, contradicting the maximality of V .

LEMMA 6.

Let $X \in GL_n(R)$ s.t. ${}^tX = X$, and let ψ_X satisfy one of the following conditions:

(1) $n = \text{odd}$,

(2) $n = \text{even}$ and the dimension of every maximal anisotropic subspace of K^n is $(n/2) - 1$.

Let V be a maximal anisotropic subspace of K^n , and let $W = V \cap R^n$. Put $W^\perp = V^\perp \cap R^n$. Then:

(1) The exact sequence $0 \rightarrow W \rightarrow W^\perp \rightarrow W^\perp/W \rightarrow 0$ is split.

(2) Let U be the image of a splitting of (1). Then $R^n = U \oplus U^\perp$ and the dimension of a maximal anisotropic subspace of $U \otimes_R K$ is $(n/2) - 1$ (resp. $(n - 1)/2$) when n is even (resp. odd).

PROOF.

(1) R^n/W is embedded in K^n/V and is torsion free. Then W^\perp/W is torsion free and free since R is a principal ideal domain.

(2) $W^\perp = U \oplus W$ and $R^n \supset U$. Let \mathfrak{x} be a maximal ideal of R , and let $k(\mathfrak{x})$ be the residual field. Then $\tilde{\psi}_X|(W^\perp/W) \otimes_R k(\mathfrak{x})$ is non-degenerate and $\psi_X|U \otimes_R k(\mathfrak{x})$ is also non-degenerate. Then $R^n = U \oplus U^\perp$ (cf. P. 5 [2]).

LEMMA 7.

Let $X \in GL_n(R)$ s.t. ${}^tX = X$, and let n be even. Assume there exists a maximal anisotropic subspace V of K^n whose dimension is $n/2$. Then there exists a free basis $\{e_1, \dots, e_n\}$ of R^n s.t.

$$(\psi_X(e_i, e_j)) = \begin{bmatrix} 0 & 1 & & & & \\ & 1 & 0 & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}.$$

PROOF.

Put $W = V \cap R^n$, $W^\perp = V^\perp \cap R^n$. Let $x \in W$ and let H be a submodule of W s.t. $W = xR \oplus H$. Then $H^\perp \supset W^\perp$ and H^\perp/W^\perp is torsion free. Let s be a *splitting* of the exact sequence $0 \rightarrow W^\perp \rightarrow H^\perp \rightarrow H^\perp/W^\perp \rightarrow 0$, and let $y \in s(H^\perp/W^\perp)$ s.t. $yR \oplus W^\perp = H^\perp$.

We define the map $f: R \rightarrow R$ by $f(r) = \psi_X(x, ry)$. For every maximal ideal \mathfrak{p} of R , there exists an element $r \in R$ s.t. $\psi_X(x, ry) \neq 0$ modulo \mathfrak{p} . Since the image of f is an ideal of R , f is surjective.

Then, there exists $z \in s(H^\perp/H)$ s.t. $\psi_X(x, z) = 1$, $\psi_X(x, x) = 0$. There exists $r \in R$ s.t. $\psi_X(x, rz) = \psi_X(z, z)$. Let $u = z - rx/2$, then $\psi_X(u, u) = 0$, $\psi_H(u, x) = 1$ and $H^\perp/H = xR \oplus uR$. Since $\tilde{\psi}_X|(H^\perp/H)$ is non-degenerate, $R^n = (H^\perp/H) \oplus (H^\perp/H)^\perp$ and $(H^\perp/H)^\perp \supset H$. By induction on n , we can prove this lemma.

LEMMA 8.

Let $X \in GL_2(R)$ s.t. ${}^tX = X$.

(1) If $\sqrt{\det X} \in R$, then there exists a basis $\{e_1, e_2\}$ of R^2 s.t.

$$(\psi_X(e_i, e_j)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(2) If $\sqrt{\det X} \notin R$, then there exists a basis $\{e_1, e_2\}$ of R^2 s.t.

$$(\psi_X(e_i, e_j)) = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}.$$

PROOF.

(1) If $\sqrt{\det X} \in R$, then there exists an element $v \in K^2$ s.t. $\psi_X(v, v) = 0$. Then there exists an element $u \in R^2$ s.t. $\psi_X(u, u) = 0$. We can prove (1) by Lemma 7.

(2) Let $\tilde{R} = C[u, u^{-1}]$ s.t. $u^2 = t$. Then $\tilde{R} \supset R$ and $\sqrt{\det X} \in \tilde{R}$. There exists $v \in \tilde{R}^2$ s.t. $\psi_X(v, v) = 0$. Let δ be the automorphism of \tilde{R} defined by $\delta(u) = -u$, then $\tilde{R}^\delta = R$. Let $e_1 = v + \delta(v)$ and $e_2 = uv - u\delta(v)$, then $e_i \in R^2$ and $\psi_X(e_1, e_2) = 0$. q.e.d.

PROOF OF PROP. 4.

From the above lemmas, under the transposing action of Z_2 on $GL_n(R)$,

$$H^1(Z_2, GL_n(R)) = \left\{ I_n, \begin{bmatrix} t & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix} \right\}.$$

Then $\{K_1, K_2\}$ is a representative set of $H^1(Z_2, GL_n(R))$ under the τ -action of Z_2 . q.e.d.

(e) $\tau\sigma$ -action.

PROPOSITION 5.

Under the $\tau\sigma$ -action of Z_2 on $GL_n(R)$, $H^1(Z_2, GL_n(R)) = \{K_1\}$.

(f) $\tau\varepsilon_2$ -action.

PROPOSITION 6.

Under the $\tau\varepsilon_2$ -action of Z_2 on $GL_n(R)$, $H^1(Z_2, GL_n(R)) = \{K_1\}$.

Proofs of Prop. 5 and Prop. 6 are analogous to that of Prop. 4.

(2) $k \geq 3$

(a) *trivial action.*

PROPOSITION 7.

Under the trivial action of Z_k on $GL_n(R)$,

$$H^1(Z_k, GL_n(R)) = \left\{ I_{a_0}, \dots, I_{a_{k-1}} = \begin{bmatrix} I_{a_0} & & & \\ & \zeta_k \cdot I_{a_1} & & \\ & & \ddots & \\ & & & \zeta_k^{k-1} \cdot I_{a_{k-1}} \end{bmatrix} \mid \sum_{m=0}^{k-1} a_m = n \right\}.$$

PROOF.

The proof is analogous to that of Prop. 1.

(b) ε_k -action.

PROPOSITION 8.

Under the ε_k -action of Z_k on $GL_n(R)$, $H^1(Z_k, GL_n(R)) = \{I_n\}$.

PROOF.

Proof is analogous to that of Prop. 3.

3. Determination of $H^1(Z_k, PGL_n(R))$.

In this section, we will determine $H^1(Z_k, PGL_n(R))$ with the previously mentioned actions.

THEOREM 1.

$H^1(Z_k, PGL_n(R))$ is:

(1) $k=2$

(a) *trivial action.*

(i) $\{I_{a,b} \mid a+b=n, a \geq b\}$ ($n = \text{odd}$).

(ii) $\left\{ I_{a,b} (a \geq b), \begin{bmatrix} 0 & t^{-1} & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 0 & t^{-1} \\ & & & 1 & 0 \end{bmatrix} \right\}$ ($n = \text{even}$).

(b) σ -action.

$\{J_{a,b,c}, J'_{d,e,f} \mid a \geq b+c, d \geq e+f, d \neq 0\}$

(c) ε_2 -action. $\{I_n\}$.

(d) τ -action.

(i) $\{K_1, K_2\}$ ($n=odd$).

$$(ii) \left\{ K_1, K_2, K_3 = \begin{bmatrix} & & -1 & 0 \\ & & 0 & 1 \\ & \ddots & & \\ -1 & 0 & & \\ 0 & 1 & & \end{bmatrix} \right\} \quad (n=even).$$

(e) $\tau\sigma$ -action.

(i) $\{K_1\}$ ($n=odd$).

$$(ii) \left\{ K_1, K_3, K_4 = \begin{bmatrix} & & 1 & 0 \\ & & 0 & t^{-1} \\ & \ddots & & \\ 1 & 0 & & \\ 0 & t^{-1} & & \end{bmatrix}, K_5 = \begin{bmatrix} & & -1 & 0 \\ & & 0 & t^{-1} \\ & \ddots & & \\ -1 & 0 & & \\ 0 & t^{-1} & & \end{bmatrix} \right\} \quad (n=even).$$

(f) $\tau\varepsilon_2$ -action. $\{K_1\}$.

(2) $k \geq 3$

(a) *trivial action.*

(i) *When k is not a divisor of n , $\{I_{a_0}, \dots, a_{k-1} \mid \sum_{m=0}^{k-1} a_m = n, (a_0, \dots, a_{k-1})$ runs a set of representatives of the equivalence relation generated by $(a_0, \dots, a_{k-1}) \sim (a'_0, \dots, a'_{k-1}) \Leftrightarrow a'_0 = a_1, \dots, a'_{k-2} = a_{k-1}, a'_{k-1} = a_0\}$.*

(ii) *When k is a divisor of n , $\{I_{a_0}, \dots, a_{k-1}, \begin{bmatrix} L & & \\ & \ddots & \\ & & L \end{bmatrix} \mid (a_0, \dots, a_{k-1})$: same condition as*

$$(i), L = \begin{bmatrix} & & & t^{-j} \\ 1 & & & \\ & \ddots & & \\ & & & 1 \end{bmatrix} : k \times k\text{-matrix and } j=1, \dots, (k-1)/2.$$

(b) ε_k -action. $\{I_n\}$.

PROOF.

(1) (a) and (b) are trivial.

(c) Since the sequence

$$1 \rightarrow R^\times \rightarrow GL_n(R) \rightarrow PGL_n(R) \rightarrow 1$$

is exact, the sequence

$$H^1(\mathbb{Z}_2, GL_n(R)) \rightarrow H^1(\mathbb{Z}_2, PGL_n(R)) \rightarrow H^2(\mathbb{Z}_2, R^\times)$$

is exact (cf. P. 125 [5]). Therefore $H^1(\mathbb{Z}_2, PGL_n(R)) = \{1\}$ follows from $H^2(\mathbb{Z}_2, R^\times) = \{1\}$.

(d) In this case, the cocycle condition is

$$(p_{ij}) = \pm (p_{n+1-j, n+1-i}), \quad P = (p_{ij}) \in GL_n(R).$$

When we put $\tilde{P} = K_1 P$, then this condition is equivalent to ${}^t \tilde{P} = \pm \tilde{P}$. If ${}^t \tilde{P} = -\tilde{P}$, then $\psi_{\tilde{P}}$ is a skew-symmetric bilinear form. Since R is a principal ideal domain, $\psi_{\tilde{P}}$ has a basis $\{e_1, \dots, e_n\}$ s.t.

$$(\psi_{\tilde{P}}(e_i, e_j)) = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix} \quad (\text{cf. p.7 [2]}).$$

(e) This follows from the exact sequence:

$$H^1(\mathbb{Z}_2, GL_n(R)) \rightarrow H^1(\mathbb{Z}_2, PGL_n(R)) \rightarrow H^2(\mathbb{Z}_2, R^\times)$$

and

$$|H^2(\mathbb{Z}_2, R^\times)| = \begin{cases} 1 & n = \text{odd} \\ 4 & n = \text{even} \end{cases}$$

(f) This follows from the above exact sequence and $H^2(\mathbb{Z}_2, R^\times) = \{1\}$.

(2) (a), (b) Proofs are analogous to those of (1) (a), (c).

4. Automorphisms of finite order of $A_{n-1}^{(1)}$

In this section, we will describe automorphisms of prime order of $A_{n-1}^{(1)}$ using the generator e_i, f_i ($1 \leq i \leq n$). Proofs follow from the results of the preceding section. Put

$$\begin{aligned} E_{\alpha_i + \alpha_{i+1} + \dots + \alpha_j} &= [\dots [e_i, e_{i+1}], e_{i+2}] \dots], e_j], \\ F_{\alpha_i + \alpha_{i+1} + \dots + \alpha_j} &= [\dots [f_j, f_{j-1}], f_{j-2}] \dots], f_i] \quad (i < j), \\ E_{\alpha_0} &= E_{\alpha_1 + \dots + \alpha_{n-1}} \quad \text{and} \quad F_{\alpha_0} = F_{\alpha_1 + \dots + \alpha_{n-1}}. \end{aligned}$$

THEOREM 2.

When $n \geq 3$, a complete set of representatives of automorphisms of prime order k up to conjugacy in $Aut(\mathcal{G})$ is the following.

$k=2$

(a) For $a = [\frac{n+1}{2}], \dots, n-1,$

$$\begin{aligned} e_i &\mapsto e_i, & f_i &\mapsto f_i \quad i \neq a, n, \\ e_a &\mapsto -e_a, & f_a &\mapsto -f_a, \\ e_n &\mapsto -e_n, & f_n &\mapsto -f_n. \end{aligned}$$

(a') When n is even,

$$\begin{aligned} e_1 &\mapsto [[e_n, E_{\alpha_0}], f_1], \\ f_1 &\mapsto [[f_n, F_{\alpha_0}], e_1], \\ e_2 &\mapsto [[f_n, F_{\alpha_0}], E_{\alpha_1+\alpha_2+\alpha_3}], \\ f_2 &\mapsto [[e_n, E_{\alpha_0}], F_{\alpha_1+\alpha_2+\alpha_3}], \\ &\dots \end{aligned}$$

(b) For $a = [\frac{n+1}{2}], \dots, n-1, a \geq b \geq 1$ and $a+b \leq n-1,$

$$\begin{aligned} e_i &\mapsto e_i, & f_i &\mapsto f_i \quad i \neq a, a+b, n, \\ e_a &\mapsto [[[f_n, F_{\alpha_1+\dots+\alpha_a}], F_{\alpha_1+\dots+\alpha_{n-1}}], e_a], \\ f_a &\mapsto -[[[e_n, E_{\alpha_1+\dots+\alpha_a}], E_{\alpha_{a+1}+\dots+\alpha_{n-1}}], f_a], \\ e_{a+b} &\mapsto -[[[e_n, E_{\alpha_1+\dots+\alpha_{a+b}}], E_{\alpha_{a+b+1}+\dots+\alpha_{n-1}}], e_{a+b}] \\ f_{a+b} &\mapsto [[[f_n, F_{\alpha_1+\dots+\alpha_{a+b}}], F_{\alpha_{a+b+1}+\dots+\alpha_{n-1}}], f_{a+b}], \\ e_n &\mapsto \frac{1}{2} [[f_n, F_{\alpha_0}], F_{\alpha_0}], \\ f_n &\mapsto \frac{1}{2} [[e_n, E_{\alpha_0}], E_{\alpha_0}]. \end{aligned}$$

(b') For $a = [\frac{n+1}{2}], \dots, n-1,$

$$\begin{aligned} e_i &\mapsto e_i, & f_i &\mapsto f_i \quad i \neq a, n, \\ e_a &\mapsto [[[f_n, F_{\alpha_1+\dots+\alpha_a}], F_{\alpha_{a+1}+\dots+\alpha_{n-1}}], e_a], \\ f_a &\mapsto -[[[e_n, E_{\alpha_1+\dots+\alpha_a}], E_{\alpha_{a+1}+\dots+\alpha_{n-1}}], f_a], \\ e_n &\mapsto F_{\alpha_0}, & f_n &\mapsto F_{\alpha_0}. \end{aligned}$$

(b'') For $a = [\frac{n+1}{2}], \dots, n-1,$

$$\begin{aligned} e_i &\mapsto e_i, & f_i &\mapsto f_i \quad i \neq a, n, \\ e_a &\mapsto -e_a, & f_a &\mapsto -f_a, \\ e_n &\mapsto \frac{1}{2} [[f_n, F_{\alpha_0}], F_{\alpha_0}], \\ f_n &\mapsto \frac{1}{2} [[e_n, E_{\alpha_0}], E_{\alpha_0}]. \end{aligned}$$

(b''')

$$\begin{aligned} e_i &\mapsto e_i, & f_i &\mapsto f_i \quad i \neq n, \\ e_n &\mapsto -\frac{1}{2} [[f_n, F_{\alpha_0}], F_{\alpha_0}], \end{aligned}$$

$$f_n \mapsto -\frac{1}{2} [[e_n, E_{\alpha_0}], E_{\alpha_0}].$$

(c)

$$\begin{aligned} e_i &\mapsto e_i, & f_i &\mapsto f_i \quad i \neq n, \\ e_n &\mapsto -e_n, & f_n &\mapsto -f_n. \end{aligned}$$

(d)

$$\begin{aligned} e_i &\mapsto -f_i, & f_i &\mapsto -e_i \quad i \neq n, \\ e_n &\mapsto \frac{1}{2} [[e_n, E_{\alpha_0}], E_{\alpha_0}], \\ f_n &\mapsto \frac{1}{2} [[f_n, F_{\alpha_0}], F_{\alpha_0}]. \end{aligned}$$

(d')

$$\begin{aligned} e_i &\mapsto -f_i, & f_i &\mapsto -e_i \quad i \neq n-1, n, \\ e_{n-1} &\mapsto -[[e_n, E_{\alpha_0}], f_{n-1}], \\ f_{n-1} &\mapsto -[[f_n, F_{\alpha_0}], e_{n-1}], \\ e_n &\mapsto -E_{\alpha_0}, & f_n &\mapsto -F_{\alpha_0}. \end{aligned}$$

(d'') *When n is even.*

$$\begin{aligned} e_1 &\mapsto e_1, & f_1 &\mapsto f_1, \\ e_2 &\mapsto F_{\alpha_1+\alpha_2+\alpha_3}, & f_2 &\mapsto E_{\alpha_1+\alpha_2+\alpha_3}, \\ &\dots\dots \\ e_n &\mapsto -[[e_n, E_{\alpha_1+\dots+\alpha_{n-1}}], E_{\alpha_2+\dots+\alpha_{n-1}}], \\ f_n &\mapsto -[[f_n, F_{\alpha_1+\dots+\alpha_{n-1}}], F_{\alpha_2+\dots+\alpha_{n-1}}]. \end{aligned}$$

(e)

$$e_i \mapsto -f_i, \quad f_i \mapsto -e_i.$$

(e') *When n is even,*

$$\begin{aligned} e_1 &\mapsto e_1, & f_1 &\mapsto f_1, \\ e_2 &\mapsto F_{\alpha_1+\alpha_2+\alpha_3}, & f_2 &\mapsto E_{\alpha_1+\alpha_2+\alpha_3}, \\ &\dots\dots \end{aligned}$$

$$\begin{aligned} e_n &\mapsto -[[f_n, f_1], f_{n-1}], \\ f_n &\mapsto -[[e_n, e_1], e_{n-1}]. \end{aligned}$$

(e'') *When n is even,*

$$\begin{aligned} e_1 &\mapsto [[e_n, E_{\alpha_0}], e_1], \\ f_1 &\mapsto [[f_n, F_{\alpha_0}], f_1], \\ e_2 &\mapsto [[f_n, F_{\alpha_0}], [[f_1, f_2], f_3]], \\ f_2 &\mapsto [[e_n, E_{\alpha_0}], [[e_1, e_2], e_3]], \\ &\dots\dots \end{aligned}$$

$$e_n \mapsto \frac{1}{2} [[[[f_n, E_{\alpha_0}], f_n], f_1], f_{n-1}],$$

$$f_n \mapsto \frac{1}{2} [[[[e_n, F_{\alpha_0}], e_n], e_1], e_{n-1}].$$

(f)

$$\begin{aligned} e_i &\mapsto -f_i, & f_i &\mapsto -e_i \quad i \neq n, \\ e_n &\mapsto -\frac{1}{2} [[e_n, E_{\alpha_0}], E_{\alpha_0}], \\ f_n &\mapsto -\frac{1}{2} [[f_n, F_{\alpha_0}], F_{\alpha_0}]. \end{aligned}$$

(2) $k \geq 3$ Put $\zeta = \zeta_k$,

(a) For a_0, \dots, a_{k-1} s.t. $\sum_{m=0}^{k-1} a_m = n$, (a_0, \dots, a_{k-1}) runs a set of representatives of the equivalence relation generated by $(a_0, \dots, a_{k-1}) \sim (a'_0, \dots, a'_{k-1}) \Leftrightarrow a'_0 = a_1, \dots, a'_{k-2} = a_{k-1}, a'_{k-1} = a_0$,

$$\begin{aligned} e_i &\mapsto e_i, & f_i &\mapsto f_i \quad i \neq a_m, n, \\ e_{a_m} &\mapsto \zeta e_{a_m}, & f_{a_m} &\mapsto \zeta^{-1} f_{a_m}, \\ e_n &\mapsto \zeta^{a_{m_0}} e_n, & f_n &\mapsto \zeta^{-a_{m_0}} f_n, \end{aligned}$$

where $m_0 = \max \{m \mid a_m \neq 0\}$.

(a') When k is a divisor of n , for $j=1, \dots, \frac{k-1}{2}$,

$$\begin{aligned} e_1 &\mapsto ad([e_n, E_{\alpha_0}])^j F_{\alpha_1 + \dots + \alpha_{k-1}}, \\ f_1 &\mapsto ad([f_n, F_{\alpha_0}])^j E_{\alpha_1 + \dots + \alpha_{k-1}}, \\ e_2 &\mapsto e_1, & f_2 &\mapsto f_1, & e_3 &\mapsto e_2, & f_3 &\mapsto f_2, \\ &\dots \dots \end{aligned}$$

(b) For $j=1, \dots, \frac{k-1}{2}$,

$$\begin{aligned} e_i &\mapsto e_i, & f_i &\mapsto f_i \quad i \neq n, \\ e_n &\mapsto \zeta^j e_n, & f_n &\mapsto \zeta^{-j} f_n. \end{aligned}$$

THEOREM 3.

When $n=2$, a complete set of representatives of automorphisms of prime order k up to conjugacy in $Aut(\mathcal{G})$ is the following.

(1) $k=2$

(a)

$$e_1 \mapsto -e_1, \quad f_1 \mapsto -f_1, \quad e_2 \mapsto -e_2, \quad f_2 \mapsto -f_2.$$

(a')

$$e_1 \mapsto e_2, \quad f_1 \mapsto f_2, \quad e_2 \mapsto e_1, \quad f_2 \mapsto f_1.$$

(b)

$$e_1 \mapsto e_1, \quad f_1 \mapsto f_1, \quad e_2 \mapsto -\frac{1}{2} [[f_2, f_1], f_1], \quad f_2 \mapsto -\frac{1}{2} [[e_2, e_1], e_1].$$

(b')

$$e_1 \mapsto -e_1, \quad f_1 \mapsto -f_1, \quad e_2 \mapsto \frac{1}{2} [[f_2, f_1], f_1], \quad f_2 \mapsto \frac{1}{2} [[e_2, e_1], e_1].$$

(b'')

$$e_1 \mapsto f_2, \quad f_1 \mapsto e_2, \quad e_2 \mapsto f_1, \quad f_2 \mapsto e_1.$$

(c)

$$e_1 \mapsto e_1, \quad f_1 \mapsto f_1, \quad e_2 \mapsto -e_2, \quad f_2 \mapsto -f_2.$$

(2) $k \geq 3$ Put $\zeta = \zeta_k$ (a) For $a = 1, \dots, \frac{k-1}{2}$,

$$e_1 \mapsto \zeta^a e_1, \quad f_1 \mapsto \zeta^{-a} f_1, \quad e_2 \mapsto \zeta^a e_2, \quad f_2 \mapsto \zeta^{-a} f_2.$$

(b) For $b = 1, \dots, \frac{k-1}{2}$,

$$e_1 \mapsto e_1, \quad f_1 \mapsto f_1, \quad e_2 \mapsto \zeta^b e_2, \quad f_2 \mapsto \zeta^{-b} f_2.$$

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