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HOMOGENEOUS TUBES OVER ONE-POINT EXTENSIONS

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Introduction

Let A be a finite dimensional algebra over a field k, and M a finite dimensional left A-module. We denote by $R=R(A, M)$ the one-point extension of A by M, namely,

$$R=\begin{bmatrix} A & M \\ 0 & k \end{bmatrix}.$$

V. Dlab and C. M. Ringel looked into the indecomposable representations of tame hereditary algebras [3]. As a result, they found that stable tubes, in particular homogeneous tubes, play an important role in their Auslander-Reiten quivers. Here a connected component Γ of the Auslander-Reiten quiver is called a stable tube if Γ is of the form $\mathbf{Z}A_\infty/n$ for some $n\in\mathrm{N}$, and called a homogeneous tube if Γ is a stable tube with $n=1$ [6]. Recently, in case of the base field being algebraically closed, C. M. Ringel generalized their results in terms of the one-point extension, and gave conditions on A and M that make $R(A, M)$ have stable separating tubular families [6].

We are interested in stable tubes, and in this paper we characterize broader parts of DTr-invariant R-modules in terms of the one-point extension, and construct the homogeneous tubes which contain them.

Throughout this paper, we deal only with finite dimensional algebras over a field k, and finite dimensional (usually left) modules. We denote by $P(X)$, the projective cover of X, and by $E(Y)$, the injective hull of Y. The k-dual $\mathrm{Hom}_k(-, k)$ is denoted by D, and the A-dual $\mathrm{Hom}_A(-, A)$ (resp. the R-dual $\mathrm{Hom}_R(-, R)$) is denoted by $-^*$ (resp. $-^\#$). Further we freely use the results of [1], [2] and [5], and denote DTr by τ.

1. The Auslander-Reiten Translation over One-point Extensions

In this section, we calculate the Auslander-Reiten translation of $R(A, M)$-modules. Given $R=R(A, M)$, it is well known that the category of left R-modules is equivalent to the category $\mathcal{M}({}_AM_k)$. Recall that the category $\mathcal{M}({}_AM_k)$ of representations of the bimodule ${}_AM_k$ has as objects the triples $({}_kU, {}_AX, \phi)$ with an A-homomorphism ϕ: ${}_AM\otimes_k U\to{}_AX$, and a morphism from $({}_kU, {}_AX, \phi)$ to $({}_kU', {}_AX', \phi')$ is given by a pair (α, β) of a k-linear map α:

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${}_kU \to {}_kU'$, and an A-homomorphism β: ${}_AX \to {}_AX'$, satisfying $\beta\phi = \phi'(1 \otimes \alpha)$. After this, we write $(\dim_k U, X, \phi)$ for (U, X, ϕ) and we will call $V = (\dim_k U, X, \phi)$ just an R-module.

Now, for an R-module $V = (n, X, \phi)$, we consider the following commutative diagram with exact rows:

$$\text{(A)} \qquad \begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} \upsilon \arrow[r, "\iota"] \arrow[d, "\chi"', "\wr"] & Y \arrow[r, "\upsilon"] \arrow[d, "\mu"'] \arrow[rd, phantom, "\text{exact}"] & P(\operatorname{Cok}\phi) \arrow[r, "\varepsilon"] \arrow[d, "\rho"] & \operatorname{Cok}\phi \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Ker} \phi \arrow[r, "\lambda"'] & M^n \arrow[r, "\phi"'] & X \arrow[r, "\pi"'] & \operatorname{Cok}\phi \arrow[r] & 0
\end{tikzcd} \; .$$

This construction is as follows. In the bottom row morphisms are canonical. Since $P(\operatorname{Cok} \phi) \xrightarrow{\varepsilon} \operatorname{Cok} \phi \to 0$ is the projective cover, we can take $\rho \in \operatorname{Hom}_A (P(\operatorname{Cok} \phi), X)$ such that $\varepsilon = \pi\rho$. For the pair (ϕ, ρ), we take the pull-back $(Y; \mu, \upsilon)$. Then this square is exact, and $\operatorname{Ker} \upsilon$ is isomorphic to $\operatorname{Ker} \phi$.

PROPOSITION 1.1. *Let $V = (n, X, \phi)$ be a non-projective indecomposable R-module. Then $\tau_R V$ is isomorphic to the R-module* $(\dim_k \operatorname{Hom}_A (M, \tau_A(\operatorname{Cok} \phi) \oplus I_V) - n, \tau_A(\operatorname{Cok} \phi) \oplus I_V, \tilde{\phi})$ *with some $\tilde{\phi}$. Here I_V is the injective A-module $D(Q^*)$ where Q is the direct summand of $P(Y)$ such that $P(Y) = Q \oplus P(\operatorname{Ker} \varepsilon)$.*

PROOF. It is easy to see that an indecomposable projective R-module has the form $(0, P, 0)$ with an indecomposable projective A-module P, or the form $(1, M, 1_M)$. Applying $-^{\#}$, $(0, P, 0)^{\#} \simeq (\dim_k P^*, \operatorname{Hom}_A (P, M), \eta(P))$ where $\eta(P)$ is the canonical isomorphism $(\eta(P)(m \otimes f))(p) = f(p)m$, $m \in M$, $f \in P^*$ and $p \in P$, or $(1, M, 1_M)^{\#} \simeq (0, k, 0)$. (For right R-modules, we use the similar notations.) Now the minimal projective presentation of V has the following form:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & M^n \arrow[r, equal] \arrow[d, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] & M^n \arrow[r] \arrow[d, "\phi"] & 0 \\
P(Y) \arrow[r, "\begin{bmatrix} \mu\alpha \\ \upsilon\alpha \end{bmatrix}"'] & M^n \oplus P(\operatorname{Cok} \phi) \arrow[r, "{(\phi, -\rho)}"'] & X \arrow[r] & 0
\end{tikzcd}$$

where α is the projective cover $P(Y) \overset{\alpha}{\to} Y \to 0$, and each row is exact. According to the difinition of the transpose, applying $-^{\#}$ to the above, we obtain the following diagram with exact columns:

$$\begin{tikzcd}[column sep=huge]
(P(\operatorname{Cok} \phi)^* \otimes_A M \arrow[r, "\begin{bmatrix} 0 \\ \eta(P(\operatorname{Cok} \phi)) \end{bmatrix}"] \arrow[d, "(\upsilon\alpha)^* \otimes 1"] & k^n \oplus \operatorname{Hom}_A(P(\operatorname{Cok} \phi), M)) \arrow[d, "\kappa"'] \\
{} & {}
\end{tikzcd}$$

$$\begin{tikzcd}
(P(Y)^*\otimes_A M \arrow[r, "\eta(P(Y))"] \arrow[d] & \mathrm{Hom}_A(P(Y), M)) \arrow[d] \\
((Tr_A(\mathrm{Cok}\,\phi)\oplus Q^*)\otimes_A M \arrow[r, "\xi"] \arrow[d] & \mathrm{Cok}\,\kappa) \arrow[d] \\
0 & 0
\end{tikzcd}$$

Here $\kappa=(\kappa_1, \kappa_2)$ where κ_1: $k^n\rightarrow\mathrm{Hom}_A\,(P(Y), M)$ and κ_2: $\mathrm{Hom}_A\,(P(\mathrm{Cok}\,\phi), M)\rightarrow\mathrm{Hom}_A\,(P(Y), M)$ with $\kappa_1((a_i))=\Sigma_{i=1}^n a_i\mu_i\alpha$, $\kappa_2(f)=fv\alpha$ where μ_i is the composition of μ and the i-th projection, and ξ is the induced morphism. We obtain $Tr_R\,V\simeq(\dim_k\,(Tr_A\,(\mathrm{Cok}\,\phi)\oplus Q^*)$, $\mathrm{Cok}\,\kappa$, $\xi)$. Consequently $\tau_R V\simeq(\dim_k D(\mathrm{Cok}\,\kappa)$, $\tau_A(\mathrm{Cok}\,\phi)\oplus I_V$, $\tilde{\phi})$ with some $\tilde{\phi}$. To complete the proof, it is sufficient to show $\dim_k D(\mathrm{Cok}\,\kappa)=\dim_k\mathrm{Hom}_A\,(M, \tau_A(\mathrm{Cok}\,\phi)\oplus I_V)-n$. Since $\mathrm{Hom}_A\,(M, \tau_A(\mathrm{Cok}\,\phi)\oplus I_V)\simeq D((Tr_A\,(\mathrm{Cok}\,\phi)\oplus Q^*)\otimes_A M)$, we will show $\dim_k\mathrm{Cok}\,\kappa=\dim_k\,((Tr_A\,(\mathrm{Cok}\,\phi)\oplus Q^*)\otimes_A M)-n$. This follows from the following two facts: (1) $\mathrm{Im}\,\kappa_1\cap\mathrm{Im}\,\kappa_2=0$ and (2) κ_1 is a monomorphism.

(1) Assume $\mathrm{Im}\,\kappa_1\cap\mathrm{Im}\,\kappa_2\neq 0$. Then there exists $(a_i)\in k^n$ and $f\in\mathrm{Hom}_A\,(P(\mathrm{Cok}\,\phi), M)$ such that $fv\alpha=\Sigma_{i=1}^n a_i\mu_i\alpha\neq 0$. Since the following diagram is push-out, we have $\delta\in\mathrm{Hom}_A\,(X, M)$ such that $\delta\phi=(a_i)$.

$$\begin{tikzcd}
P(Y) \arrow[r, "v\alpha"] \arrow[d, "\mu\alpha"'] & P(\mathrm{Cok}\,\phi) \arrow[d, "p"] \\
M^n \arrow[r, "\phi"'] & X
\end{tikzcd}$$

This means that V has a projective direct summand $(1, M, 1_M)$. It's a contradiction.

(2) Similarly.

COROLLARY 1.2. *Let $V=(n, X, \phi)$ be a non-projective indecomposable R-module. Then*

(1) *If ϕ is an epimorphism, $\tau_R V$ is isomorphic to*
$(\dim_k\mathrm{Hom}_A\,(M, E(\mathrm{top}\,(\mathrm{Ker}\,\phi)))-n, E(\mathrm{top}\,(\mathrm{Ker}\,\phi)), \tilde{\phi})$.

(2) *If ϕ is a monomorphism, $\tau_R V$ is isomorphic to*
$(\dim_k\mathrm{Hom}_A\,(M, \tau_A(\mathrm{Cok}\,\phi))-n, \tau_A(\mathrm{Cok}\,\phi), \tilde{\phi})$.

(3) *If* $\mathrm{proj.dim}_A\,\mathrm{Cok}\,\phi=1$, *$\tau_R V$ is isomorphic to*
$(\dim_k\mathrm{Hom}_A\,(M, \tau_A(\mathrm{Cok}\,\phi)\oplus E(\mathrm{top}\,(\mathrm{Ker}\,\phi)))-n, \tau_A(\mathrm{Cok}\,\phi)\oplus E(\mathrm{top}\,(\mathrm{Ker}\,\phi)), \tilde{\phi})$.

PROOF. By Proposition 1.1.

2. Homogeneous Tubes

In this section, we characterize some τ_R-invariant modules by using the previous proposition. And we construct homogeneous tubes which contain them.

LEMMA 2.1. *Let* $V=(n, X, \phi)$, $(n\neq 0)$ *be a non-projective indecomposable R-module. Then the Auslander-Reiten sequence which has the end-term V has the following form:*

$$\begin{tikzcd}
0 \arrow[r] & M^{m-n} \arrow[r, "\begin{bmatrix}1\\0\end{bmatrix}"] \arrow[d, "\begin{bmatrix}\tilde{\phi}_1\\ \tilde{\phi}_2\end{bmatrix}"] & M^{m-n}\oplus M^n \arrow[r, "(0\ 1)"] \arrow[d, "\begin{bmatrix}\tilde{\phi}_1 & \psi_1\\ \tilde{\phi}_2 & \psi_2\\ 0 & \phi\end{bmatrix}"] & M^n \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & \tau_A(\mathrm{Cok}\,\phi)\oplus I_V \arrow[r, "\begin{bmatrix}10\\01\\00\end{bmatrix}"'] & (\tau_A(\mathrm{Cok}\,\phi)\oplus I_V)\oplus X \arrow[r, "(001)"'] & X \arrow[r] & 0
\end{tikzcd}$$

with some $\tilde{\phi}_1$, $\tilde{\phi}_2$, ψ_1 *and* ψ_2, *where* $m=\dim_k \mathrm{Hom}_A(M, \tau_A(\mathrm{Cok}\,\phi)\oplus I_V)$.

PROOF. By Proposition 1.1, the Auslander-Reiten sequence has the following form:

$$\begin{tikzcd}
0 \arrow[r] & M^{m-n} \arrow[r, "\begin{bmatrix}1\\0\end{bmatrix}"] \arrow[d, "\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}"] & M^{m-n}\oplus M^n \arrow[r, "(0\ 1)"] \arrow[d] & M^n \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & \tau_A(\mathrm{Cok}\,\phi)\oplus I_V \arrow[r, "(\alpha_1\ \alpha_2)"'] & E \arrow[r, "\beta"'] & X \arrow[r] & 0
\end{tikzcd}$$

with some E, and some ϕ_1, ϕ_2, α_1, α_2 and β. Since the R-homomorphism

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & M^n \arrow[d, "\phi"] \\
X \arrow[r, equal] & X
\end{tikzcd}$$

is not a splitable epimorphism, it factors through $((0\ 1), \beta)$, and E has X as a direct summand.

THEOREM 2.2. *Let* $V=(1, X, \phi)$ *be a non-projective indecomposable R-module.*

(I) *If* ϕ *is an epimorphism, the following two statements are equivalent.*

(1) $\tau_R V\simeq V$.

(2) (a) ${}_AX\simeq E(\mathrm{top}\,(\mathrm{Ker}\,\phi))$.

(b) $\dim_k \mathrm{Hom}_A(M, X)=2$.

(II) *If ϕ is not an epimorphism, the following two statements are equivalent.*

(1) $\tau_R V \simeq V$.

(2) (a) ${}_A X \simeq \tau_A(\mathrm{Cok}\,\phi)$.

(b) $\dim_k \mathrm{Hom}_A(M, X) = 2$.

(c) *In the commutative diagram*(A), $\mathrm{Im}\,\iota \subset \mathrm{rad}\,Y$.

PROOF. (I) (2)→(1) By Proposition 1.1, $\tau_R V \simeq (1, X, \tilde{\phi})$ with some $\tilde{\phi}$. Then, by Lemma 2.1, the Auslander-Reiten sequence which has the end-term V has the following form:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "{\begin{bmatrix}1\\0\end{bmatrix}}"] \arrow[d, "\tilde{\phi}"] & M\oplus M \arrow[r, "(0\ 1)"] \arrow[d, "{\begin{bmatrix}\tilde{\phi} & \psi\\ 0 & \phi\end{bmatrix}}"] & M \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & X \arrow[r, "{\begin{bmatrix}1\\0\end{bmatrix}}"'] & X\oplus X \arrow[r, "(0\ 1)"'] & X \arrow[r] & 0
\end{tikzcd}$$

with some ψ. If ϕ and $\tilde{\phi}$ are linearly independent over k, this extension splits. It's a contradiction. Hence $\tau_R V \simeq V$. (1)→(2) Obviously.

(II) By the after remark, the proof is similar to (I).

COROLLARY 2.3. *Let $V = (1, X, \phi)$ be a non-projective indecomposable R-module.*

(I) *If ϕ is a monomorphism, the following two statements are equivalent.*

(1) $\tau_R V \simeq V$.

(2) (a) ${}_A X \simeq \tau_A(\mathrm{Cok}\,\phi)$.

(b) $\dim_k \mathrm{Hom}_A(M, X) = 2$.

(II) *If ϕ is not an epimorphism and* $\mathrm{proj.dim}_A\,\mathrm{Cok}\,\phi = 1$, *the following two statements are equivalent.*

(1) $\tau_R V \simeq V$.

(2) (a) *ϕ is a monomorphism.*

(b) ${}_A X \simeq \tau_A(\mathrm{Cok}\,\phi)$.

(c) $\dim_k \mathrm{Hom}_A(M, X) = 2$.

REMARK. *In case of $\tau_R V \simeq V$, X is indecomposable.* Otherwise, X decomposes as $X = X_1 \oplus X_2$, X_1, $X_2 \neq 0$, we have $\dim_k \mathrm{Hom}_A(M, X_1) = \dim_k \mathrm{Hom}_A(M, X_2) = 1$, and the Auslander-Reiten sequence which has the end-term V has the following form:

$$\begin{tikzcd}[column sep=large, row sep=huge]
0 \arrow[r] & M \arrow[r, "\begin{bmatrix}1\\0\end{bmatrix}"] \arrow[d, "\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}"] & M\oplus M \arrow[r, "(0\ 1)"] \arrow[d, "\begin{bmatrix}\phi_1 & b_1\phi_1\\ \phi_2 & b_2\phi_2\\ 0 & \phi_1\\ 0 & \phi_2\end{bmatrix}"] & M \arrow[r] \arrow[d, "\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}"] & 0 \\
0 \arrow[r] & X_1\oplus X_2 \arrow[r, "\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\end{bmatrix}"'] & (X_1\oplus X_2)\oplus(X_1\oplus X_2) \arrow[r, "\begin{bmatrix}0010\\ 0001\end{bmatrix}"'] & X_1\oplus X_2 \arrow[r] & 0
\end{tikzcd}$$

with b_1, $b_2 \in k$. But it is easy to see that this sequence splits. It's a contradiction, therefore ${}_AX$ is indecomposable.

If $\tau_R V \simeq V$, V belongs to some homogeneous tube $\mathcal{C}$ [4]. Next we will state the construction of the homogeneous tube $\mathcal{C}$. Here we denote $V(s)$ the module in $\mathcal{C}$ which has the quasi-length s [5].

THEOREM 2.4. *Let $V=(1, X, \phi)$ be a non-projective indecomposable R-module. And assume $\tau_R V \simeq V$. Then V is quasi-simple, and $V(s)=(s, X^s, \Phi(s))$, where $\Phi(s)=$*

$$\begin{bmatrix}\phi & \psi & & 0\\ & \phi & \ddots & \\ & & \ddots & \psi\\ 0 & & & \phi\end{bmatrix}$$

with ψ being an arbitrary linear map which is linearly independent of ϕ. Further the Auslander-Reiten sequence which has the end-term $V(s)$ has the following form:

$$\begin{tikzcd}
 & M^{s+1} \arrow[d, "\Phi(s+1)"] \arrow[rdd, "-J(s)"] & \\
 & X^{s+1} \arrow[rdd, "-J(s)"] & \\
M^s \arrow[d, "\Phi(s)"] \arrow[ruu, "I(s)"] \arrow[rdd, "J(s-1)"'] & & M^s \arrow[d, "\Phi(s)"] \\
X^s \arrow[ruu, "I(s)"] \arrow[rdd, "J(s-1)"'] & & X^s \\
 & M^{s-1} \arrow[d, "\Phi(s-1)"] \arrow[ruu, "I(s-1)"'] & \\
 & X^{s-1} \arrow[ruu, "I(s-1)"'] &
\end{tikzcd}$$

where $I(s)=(E(s)/0)$, $J(s)=(0\,|\,E(s))$ with $E(s)$ the unit matrix of degree s.

PROOF. It is easy to see that V is quasi-simple. We prove the rest parts by the induction on s. First, by Lemma 2.1, the Auslander-Reiten sequence which has the end-term V has the following form:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\begin{bmatrix}1\\0\end{bmatrix}"] \arrow[d, "\phi"] & M\oplus M \arrow[r, "(0\ 1)"] \arrow[d, "\begin{bmatrix}\phi & \psi\\ 0 & \phi\end{bmatrix}"] & M \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & X \arrow[r, "\begin{bmatrix}1\\0\end{bmatrix}"'] & X\oplus X \arrow[r, "(0\ 1)"'] & X \arrow[r] & 0
\end{tikzcd}$$

with some ψ. If ψ is linearly dependent of ϕ, this sequence splits. Consequently it must be linearly independent of ϕ. Here it is easy to see that the arbitrary ψ which is linearly independent of ϕ makes the isomorphic extension. Second, assume that the form of $V(s)$ and the form of the Auslander-Reiten sequence which has the end-term $V(s-1)$ are checked. Then the Auslander-Reiten sequence which has the end-term $V(s)$ is decided except θ as the following form. But routine calculations show that we can take $\theta=0$.

$$\begin{tikzcd}
 & M^{s+1} \arrow[d, "\begin{bmatrix}\phi & \psi & 0 & \cdots & 0 & \theta\\ & \phi & \ddots & & & 0\\ & & & \ddots & & \vdots\\ & & & & & 0\\ & & & \ddots & & \psi\\ 0 & & & & & \phi\end{bmatrix}"] \arrow[ddr, "-J(s)"] & \\
 & X^{s+1} \arrow[ddr, "-J(s)"'] & \\
M^{s} \arrow[d, "\Phi(s)"] \arrow[uur, "I(s)"] \arrow[ddr, "J(s-1)"'] & & M^{s} \arrow[d, "\Phi(s)"] \\
X^{s} \arrow[uur, "I(s)"] \arrow[ddr, "J(s-1)"'] & & X^{s} \\
 & M^{s-1} \arrow[d, "\Phi(s-1)"] \arrow[uur, "I(s-1)"] & \\
 & X^{s-1} \arrow[uur, "I(s-1)"'] &
\end{tikzcd}$$

Recently Ringel considered the stable separating tubular families, and he made $\mathbf{P}_1k$-family of stable tubes [6]. In connection with it, we show the following.

PROPOSITION 2.5. *Let* $V=(1, X, \phi)$ *be a non-projective indecomposable R-module. Assume* $\tau_R V \simeq V$, ϕ *a monomorphism,* $\mathrm{End}_A(X)=k$, *and* k *an infinite field. Then we can make*

$|k|$-*family of homogeneous tubes.* (| | *means the cardinal number.*)

PROOF. We write the canonical extension

$$0 \longrightarrow M \xrightarrow{\phi} X \xrightarrow{\pi} \mathrm{Cok}\,\phi \longrightarrow 0$$

and let

$$0 \longrightarrow X \xrightarrow{\lambda} E \xrightarrow{\mu} \mathrm{Cok}\,\phi \longrightarrow 0$$

be the Auslander-Reiten sequence. Since π is not a splitable epimorphism, there exists λ' such that $\pi=\mu\lambda'$. If necessary, adding some $a\lambda$ $(a\in k)$ to λ', we can take λ' as a monomorphism. Further, since λ' is not a splitable monomorphism, there exists ζ such that $\lambda'=\zeta\lambda$. We can also take ζ as an automorphism. Now, using λ' above, we can make the following commutative diagram with exact rows and columns, with some $\phi'\in\mathrm{Hom}_A\,(M, X)$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M \arrow[r, "\phi"] \arrow[d, "\phi'"'] & X \arrow[r, "\pi"] \arrow[d, "\lambda'"] & \mathrm{Cok}\,\phi \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "\lambda"'] \arrow[d] & E \arrow[r, "\mu"'] \arrow[d] & \mathrm{Cok}\,\phi \arrow[r] & 0 \\
 & \mathrm{Cok}\,\phi' \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{Cok}\,\lambda' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Notice $\mathrm{Cok}\,\phi\simeq\mathrm{Cok}\,\lambda'$ from the commutative diagram below:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\lambda"] \arrow[d, equal] & E \arrow[r, "\mu"] \arrow[d, "\zeta"', "\wr"] & \mathrm{Cok}\,\phi \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & X \arrow[r, "\lambda'"'] & E \arrow[r, "\mu'"'] & \mathrm{Cok}\,\lambda' \arrow[r] & 0
\end{tikzcd}$$

where each row is exact. Set $V'=(1, X, \phi')$, then by Corollary 2.3, $\tau_R V'\simeq V'$. It is easy to see $V\not\simeq V'$. In this way we can construct $|k|$-number of τ_R-invariant modules.

EXAMPLE. *We give an example where* $\mathrm{gl.dim}_A\,A=\infty$ *and there exists a left A-module M such that $R(A, M)$ has homogeneous tubes.*

Let

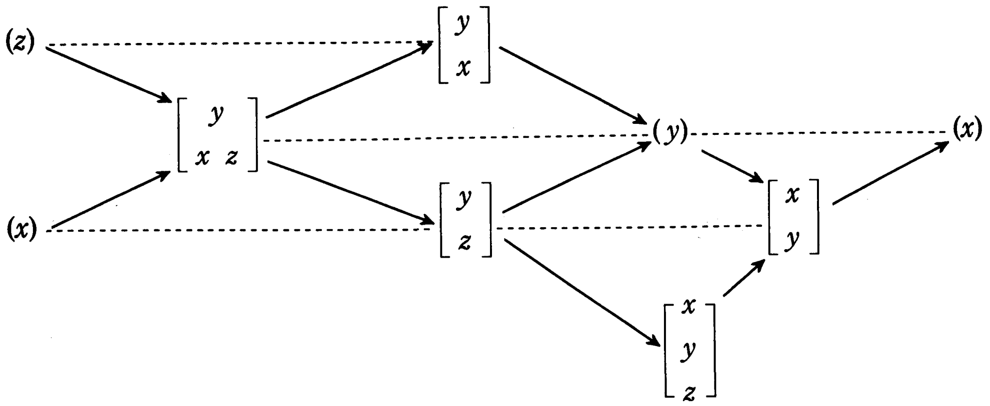
$$A=\left\{\begin{bmatrix} z & 0 & 0 & \alpha & \beta \\ 0 & x & \gamma & 0 & 0 \\ 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & y & \delta \\ 0 & 0 & 0 & 0 & x \end{bmatrix} \in M_5(k)\right\}.$$

In other words, A is defined by the following quiver with relations:

$$\begin{tikzcd} x \arrow[r, shift left, "\delta"] & y \arrow[l, shift left, "\gamma"] \arrow[r, "\alpha"] & z, \end{tikzcd}$$

with $\gamma\delta=\delta\gamma=0$ $(\beta=\alpha\delta)$.

A is representation-finite, and has the following Auslander-Reiten quiver:



Here, for example, $\begin{pmatrix} & y & \\ x & & z \end{pmatrix}$ means the indecomposable A-module N such that top $N\simeq S_y$ and soc $N\simeq S_x\oplus S_z$, where S_- means the simple A-module corresponding to the idempotent $-$. Let $M=(x)\oplus(z)$. Then R-modules $V=\left(1, \begin{pmatrix} & y & \\ x & & z \end{pmatrix}, \phi\right)$, where ϕ are inclusions in the sense of Proposition 2.5, are τ_R-invariant.

REMARK. (Ringel [6]) *Under the additional assumption that* $\mathrm{End}_A(X)=k$, *the homogeneous tube in* mod R *constructed in Theorem 2.4. is an abelian category which is serial, and is closed under extensions in* mod R.

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