

**NON-NORMAL NUMBERS TO DIFFERENT BASES
AND
THEIR HAUSDORFF DIMENSION**

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0. Introduction.

The notion of normal numbers was first introduced by Emile Borel [3] in 1909. He considered the decimal expansion of real numbers in the unit interval to the base r and assuming that every digit of their decimal expansions is independent and also takes all possible values $0, 1, \dots$ and $r-1$ with equal probability, he proved that almost all real numbers are normal to the base r in the sense of Lebesgue measure.

For a real number ω , we denote $\{\omega\}$ the fractional part of ω defined by

$$\{\omega\} = \omega - [\omega],$$

where $[\cdot]$ is the Gauss' symbol, so that $\{\omega\}$ is contained in the unit interval $I_0 = [0, 1)$ for every real number ω . We consider the decimal expansion of $\{\omega\}$ to the base r :

$$(1) \quad \{\omega\} = \sum_{n=1}^{\infty} \frac{x_n(\omega)}{r^n},$$

where $x_n(\omega)$ is the n -th digit of development of $\{\omega\}$ and takes one of the values in $R = \{0, 1, \dots, r-1\}$. For an r -adic rational number, we agree to write a terminating expansion in the form (1) in which all digits from a certain point on are 0.

Thus every real number in I_0 is uniquely expressed by (1) and an infinite sequence of integers $\{a_n\}_{n=1, 2, \dots}$ taking one of the values in R can be corresponded to a unique real number a in I_0 defined by

$$a = \sum_{n=1}^{\infty} \frac{a_n}{r^n}.$$

We call a real number ω to be simply normal to the base r if, for each j in R ,

$$(2) \quad \lim_{N \rightarrow \infty} \frac{A_N(j; \omega)}{N} = \frac{1}{r},$$

where $A_N(j; \omega)$ is the number of indices n up to N satisfying

$$x_n(\omega) = j$$

for the expansion (1) of $\{\omega\}$.

A real number ω is said to be normal to the base r if, for every positive integer k and each string

$$\Delta_k = (j_1, j_2, \dots, j_k)$$

in R^k ,

$$(3) \quad \lim_{N \rightarrow \infty} \frac{A_N(\Delta_k; \omega)}{N} = \frac{1}{r^k},$$

where $A_N(\Delta_k; \omega)$ is the number of indices n between 1 and N satisfying

$$x_n(\omega) = j_1, x_{n+1}(\omega) = j_2, \dots, x_{n+k-1}(\omega) = j_k$$

for the expansion (1) of $\{\omega\}$.

A normal number to every positive integer base greater than 1 is called to be absolutely normal. The set of all simply normal numbers and normal numbers to the base r are denoted by $S(r)$ and $B(r)$, respectively. B denotes the set of all absolutely normal numbers and

$$B = \bigcap_{r=2}^{\infty} B(r).$$

The very definition indicates that $S(r)$ is a Borel set of the type $G_{\delta\sigma\delta}$ and from another equivalent definition of normal numbers, $B(r)$ is proved to be a Borel set of the type $G_{\delta\sigma\delta}$ (K. Nagasaka [9]). $B(r)$ is of full measure but of the first category (T. Šalát [13]).

From Borel's assumptions corresponding to (2) and (3), it seems that the decimal expansion of a normal number to the base 6 may be considered as an infinite sample paths of fair dice throwings. Indeed, an interesting application of normal numbers occurs in the foundation of probability theory, namely, in von Mises' theory of collectives (von Mises [8]). Let us consider a real number in the unit interval whose decimal expansion is identical to an arithmetic progression of that of a normal number. D.D. Wall [18] proved in his theorem 7 that this considered real number is also normal, which suggests us certain collective conditions to be satisfied for normal numbers. We know moreover this kind of property (Teturo Kamae and Benjamin Weiss [5]).

As we have seen before, the decimal expansion of a normal number is a good model of random sequences. On the other hand, the set of non-normal numbers,

simply called often as a non-normal set, attracts our attention also. Those non-normal sets are generally defined by digit properties and Lebesgue null sets, therefore Hausdorff dimension is widely used in order to compare the sizes of non-normal sets.

For any set L in Euclidean n -space R^n , $\dim L$ denotes the Hausdorff dimension of L . For a linear set M , $\dim M$ is invariant under translation, therefore we assume that every linear set may be contained in the unit interval I_0 when we consider its Hausdorff dimension, that is, we consider $S(r) \cap I_0$, $B(r) \cap I_0$ and $B \cap I_0$ instead of $S(r)$, $B(r)$ and B , respectively. We also write, for two linear sets A and B in I_0 , the difference

$$A - B = A \cap (I_0 - B).$$

By applying the theorem on entropies of Markov processes (P. Billingsley [2], Theorem 14.1), we gave a proof for which the set of all non-normal numbers to the base r has Hausdorff dimension 1 and the set of all simply normal numbers but not normal numbers to the base r has also Hausdorff dimension 1 (Nagasaka [9]). In my preceding note [10], it has been demonstrated, by making use of W.A. Beyer's calculation technique for Hausdorff dimension (Beyer [1]), that $B(r) - B(s)$ is of Hausdorff dimension 1 unless $\log r / \log s$ is rational.

In the next Section, we shall give a refinement of these results, by reconsidering the results obtained by J.W.S. Cassels [4] and Wolfgang M. Schmidt [14].

In the last Section, we shall construct uncountable non-normal numbers to both bases 3 and 5 and estimate their Hausdorff dimension. Further we shall give another simple proof of A.D. Pollington's result [12], which is a final result for the Hausdorff dimension of non-normal sets.

1. The set of normal numbers to every base except powers of one number.

H. Steinhaus once raised a question in the "New Scottish Book" as to how far the property of being normal with respect to different bases is independent. This problem was cited as Problem 144 by J.W.S. Cassels [4], but we cannot find any trace of this problem in the "Scottish Book" newly edited by R. Daniel Mauldin [7].

Cassels [4] replied to this question. Let us denote $C(3)$ a modified ternary Cantor set, that is, the set of numbers in I_0 in whose expansion to the base 3 the digit 2 never occurs. To every number

$$\omega = \sum_{n=1}^{\infty} \frac{x_n(\omega)}{3^n}, \quad (x_n(\omega) = 0 \text{ or } 1)$$

in $C(3)$, corresponds the number

$$f_{3,2}(\omega) = \sum_{n=1}^{\infty} \frac{x_n(\omega)}{2^n} \in I_0.$$

Introducing a measure μ on $C(3)$ by $\mu = \mu_0 \circ f_{3,2}$, where μ_0 is the Lebesgue measure, it is proved that μ -almost all ω in $C(3)$ are normal to every base r which is not a power of 3.

From the result above and from

$$B(3) = B(3^2) = \cdots = B(3^n) = \cdots,$$

we have $\mu(C(3) \cap (B \ominus B(3))) = 1$, where $B \ominus B(3)$ is the set of all normal numbers to every base except powers of three. Then we have

$$\dim_{\mu}(C(3) \cap (B \ominus B(3))) = 1.$$

For an $\omega \in I_0$, let us define the 3-adic cylinder set containing ω of length 3^{-n} by

$$u_n(\omega) = \{\omega' \in I_0; x_k(\omega') = x_k(\omega), k=1, 2, \dots, n\},$$

where

$$\omega = \sum_{n=1}^{\infty} \frac{x_n(\omega)}{3^n}$$

and

$$\omega' = \sum_{n=1}^{\infty} \frac{x_n(\omega')}{3^n}.$$

Then,

$$\mu_0(u_n(\omega)) = 3^{-n}$$

and

$$\mu(u_n(\omega)) = \mu_0(f_{3,2}(\omega)) = 2^{-n}.$$

Thus

$$C(3) \cap (B \ominus B(3)) \subset \left\{ \omega \in I_0; \lim_{n \rightarrow \infty} \frac{\mu(u_n(\omega))}{\mu_0(u_n(\omega))} = \frac{\log 2}{\log 3} \right\}$$

From the Theorem 14.1 of Billingsley [2], we obtain

THEOREM 1. $\dim(C(3) \cap (B \ominus B(3))) = \log 2 / \log 3$.

Abandonning to fix ourselves to $C(3)$, we can prove a stronger result :

THEOREM 2. $\dim(B\ominus B(3))=1$.

REMARK. The above value of the Hausdorff dimension of the set $B\ominus B(3)$ cannot be improved. For any linear subset L ,

$$0 \leq \dim L \leq 1.$$

PROOF. The idea of the proof of Theorem 2 is the same as that in my previous note [10]. First we need next lemma.

LEMMA 1. *For a given $y \in B(r) \cap I_0$, the vector $(x, y) \in I_0 \times I_0$ is normal to the base r for almost all $x \in I_0$ in the sense of Lebesgue measure.*

COROLLARY 1. *For a given $y \in C(3) \cap (B\ominus B(3))$ and for any positive integer k , the k -tuple $(x_1, x_2, \dots, x_{k-1}, y)$ is normal to every base r except powers of 3 for almost all $(x_1, x_2, \dots, x_{k-1}) \in I_0^{k-1}$.*

This set of full measure in the Corollary 1 is denoted by P_{k-1} , and put

$$G_k = P_{k-1} \cap [T_{k-1}(B)] \times \{y\} \subset I_0^k,$$

where $y \in C(3) \cap (B\ominus B(3))$ and T_k is a transformation from I_0 to I_0^k defined as the following: For

$$\omega = \sum_{n=1}^{\infty} \frac{x_n(\omega)}{r^n} \in I_0,$$

$$T_k(\omega) = (\omega_1, \omega_2, \dots, \omega_k) \in I_0^k,$$

where $x_j(\omega_i) = x_{(j-1) \cdot k + i}(\omega)$ for every $i=1, 2, \dots, k$ and $j=1, 2, \dots$.

An extended version of a theorem of Beyer [1] is necessary to complete the proof.

THEOREM A. (Beyer) *For any subset M in I_0 ,*

$$\dim M = \dim T_k M / k,$$

where $T_k M$ is the set of $(\omega_1, \omega_2, \dots, \omega_k) \in I_0^k$ for which there exists an $\omega \in M$ such that

$$T_k \omega = (\omega_1, \omega_2, \dots, \omega_k).$$

From the Theorem 7 of Wall [18], it is easy to see that

$$T_k^{-1} G_k \subset (B\ominus B(3)).$$

By virtue of the Theorem A and from a fundamental property of Hausdorff dimension, we get

$$\begin{aligned} \dim(B\ominus B(3)) &\geq \dim \bigcup_{k=2}^{\infty} T_k^{-1}G_k = \sup_k \dim T_k^{-1}G_k \\ &= \sup_k (k-1)/k = 1. \end{aligned}$$

⟨Q.E.D.⟩

NOTE. $C(3)$ is eventually contained in $I_0 - S(3)$. From this remark we may rewrite Theorem 1 and Theorem 2 as follows:

THEOREM 1'. $\dim(C(3) \cap (B\ominus S(3))) = \log 2 / \log 3$, where $B\ominus S(3)$ is the set of all normal numbers to every base except powers of three which are neither simply normal to the base 3.

THEOREM 2'. $\dim(B\ominus S(3)) = 1$.

Independently of Cassels, Wolfgang M. Schmidt [14] answered the Steinhaus problem. For two positive integers r and s greater than 1, we write $r \sim s$, if there exist integers n and m with $r^n = s^m$. Otherwise $r \not\sim s$. For integers s and t with $1 < t < s$, we define a function $g_{s,t}$ from I_0 to I_0 as follows: Assume that $\omega \in I_0$ is developed to the base t ,

$$\omega = \sum_{n=1}^{\infty} \frac{x_n(\omega)}{t^n}.$$

Then,

$$g_{s,t}(\omega) = \sum_{n=1}^{\infty} \frac{x_n(\omega)}{s^n} \in I_0,$$

where $x_n(\omega)$ takes one of the values $0, 1, \dots$ and $t-1$. Introducing a measure $\mu_{s,t}$ on $g_{s,t}(I_0)$ by $\mu_{s,t} = \mu_0 \circ g_{s,t}^{-1}$, it is proved that, for almost all $\omega \in g_{s,t}(I_0)$ with respect to $\mu_{s,t}$, ω is normal to the base r whenever $r \sim s$, which implies that

$$\mu_{s,t}(g_{s,t}(I_0) \cap (B\ominus B(s))) = 1,$$

where $B\ominus B(s)$ is the set of all normal numbers to every base except powers of s . Cassels' result is a special case of this result of Schmidt with $t=2$, $s=3$ and $f_{3,2} = g_{3,2}^{-1}$.

By virtue of $g_{s,t}(I_0) \subset (I_0 - S(s))$, we get analogously to Theorem 1' and Theorem 2',

THEOREM 3. For integers s and t with $1 < t < s$,

$$\dim(g_{s,t}(I_0) \cap (B\ominus S(s))) = \log t / \log s,$$

where $B\ominus S(s)$ is the set of all normal numbers to every base except powers of s

which are neither simply normal to the base s .

THEOREM 4. For any integer $r > 1$,

$$\dim(B \ominus S(r)) = 1,$$

where $B \ominus S(r)$ is the set of all normal numbers to every base except powers of r which are neither simply normal to the base r .

NOTE. $S(r) \supset B(r)$ assures that we can replace $S(r)$ in above Theorems by $B(r)$.

These Theorems are refinements of my preceding result in [10].

Recently Bodo Volkmann [17] generalized Schmidt's theorem as follows: Let r and s be integers greater than 1 with $r \sim s$. We denote $M_r(\nu_0, \nu_1, \dots, \nu_{r-1})$ the set of real numbers ω in I_0 satisfying

$$\lim_{N \rightarrow \infty} \frac{A_N(j; \omega)}{N} = \nu_j, \quad j = 0, 1, \dots, r-1,$$

where

$$(4) \quad 0 \leq \nu_j \leq 1, \quad j = 0, 1, \dots, r-1,$$

$$(5) \quad \sum_{j=0}^{r-1} \nu_j = 1$$

and there exists at least one j such that

$$(6) \quad \nu_j \neq 1/r.$$

Further, introducing a probability measure μ_ν on I_0 as the product measure of $\nu = (\nu_0, \nu_1, \dots, \nu_{r-1})$, he proved that, for μ_ν -almost all ω in $M_r(\nu_0, \nu_1, \dots, \nu_{r-1})$, ω is normal to every base $s \sim r$.

Using the same theorem of Billingsley, we obtain

THEOREM 5.

$$\dim(M_r(\nu_0, \nu_1, \dots, \nu_{r-1}) \cap B) = -\frac{1}{\log r} \sum_{j=0}^{r-1} \nu_j \cdot \log \nu_j,$$

for every integer r greater than 1.

It is clear that

$$M_r(\nu_0, \nu_1, \dots, \nu_{r-1}) \cap B \subset (B \ominus S(r)).$$

Then,

$$\dim(B \ominus S(r)) \geq \sup(M_r(\nu_0, \nu_1, \dots, \nu_{r-1}) \cap B)$$

$$\begin{aligned}
&= \sup \left\{ -\frac{1}{\log r} \sum_{j=0}^{r-1} \nu_j \cdot \log \nu_j \right\} \\
&= 1,
\end{aligned}$$

where the supremum is taken over all $\nu = (\nu_0, \nu_1, \dots, \nu_{r-1})$ satisfying (4), (5) and (6). Thus we get another proof of Theorem 4.

2. The set of non-normal numbers to different bases.

Hereafter we consider the decimal expansion of a real number ω to the base r and also to the base s . In order to distinguish different bases, we agree to write

$$\omega = \omega_r = \sum_{n=1}^{\infty} \frac{x_n^r(\omega)}{r^n},$$

if we need to specify the base r of the development of ω .

M.J. Pelling [11] proposed to construct an uncountable class of reals not normal in the scales of 3 and 5.

An uncountable class is indeed indispensable. Let us take a rational number ω in I_0 , then $\omega = a/b$, where a and b are integers with $0 \leq a < b$. The fractional part of

$$\omega, r\omega, r^2\omega, r^3\omega, \dots$$

take only values in the finite set

$$0, 1/b, 2/b, \dots, (b-1)/b.$$

From the Dirichlet's pigeon-hole principle, we may conclude that the decimal expansion of ω_r is ultimately periodic with the period of length at most b to every base r . This means that this ω has not normality of order b [6]. Hence any rational number is not normal to every base, but the set of rationals is countable.

Before mentioning the answer to Pelling's problem given by Andrew Odlyzko and also by the proposer himself, we want to give our construction of uncountable non-normal numbers to the bases 3 and 5.

For every non-negative integer k , the 10^k th through $5 \cdot 10^k$ digits of ω are prescribed to be zero to the base 3. This ω is evidently not normal to the base 3. Then let us consider the decimal expansion of the same ω to the base 9, that is ω_9 , then the $10^k/2$ th through $5 \cdot 10^k/2$ digits of ω_9 are prescribed to be zero. This prescription defines only about $4 \cdot 10^k \cdot \log 3 / \log 5$ digits in ω_5 to be zero. Then

$$\liminf_{N \rightarrow \infty} \frac{A_N(0; \omega_5)}{N} \geq 4/9 = 0.44 \dots,$$

which signifies that ω_5 is neither normal to the base 5. The digits other than the above prescription of ω are able to take all possible values. Thus we construct an uncountable class of real ω 's that are neither normal to the base 3 nor to the base 5.

Odlyzko's answer also uses the prescription technique in the decimal expansion of x in I_0 . For every nonnegative integer k , the digits 10^{2k} through $3 \cdot 10^{2k}$ of x to the base 5 are β_{2k} and 10^{2k+1} through $3 \cdot 10^{2k+1}$ to the base 3 are β_{2k+1} , where (β_n) is an infinite sequence of finite strings of 0's and 1's of length $2 \cdot 10^n$. For each sequence (β_n) , this prescribed x is normal to neither of the bases and there exist uncountably many (β_n) . Thus uncountable non-normal numbers to the bases 3 and 5 are constructed.

We can estimate the Hausdorff dimension of thus constructed non-normal numbers to the bases 3 and 5 from my theorem 4 [9]. According to our construction of non-normal numbers, the limes sup of the relative frequency of the prescribed digits is equal to $8/9$, then its Hausdorff dimension is at least

$$1 - 8/9 = 1/9 = 0.11\dots$$

By using the same calculation technique for Hausdorff dimension as in the previous Section, we obtain stronger results with the aid of another Schmidt's result [15]. The set of all positive integers greater than one is divided into two disjoint classes R and S so that equivalent integers fall in the same class under the equivalent relation \sim defined in the previous Section. Schmidt proved

LEMMA 2. *There exist uncountably many numbers which are normal to every base from R and to no base from S , where R and S are two disjoint classes of integers defined above.*

Suppose that S contains 3 and 5. Then Lemma 2 assures the existence of uncountable non-normal numbers to the bases 3 and 5, which are also normal to every base from R . Theorem 6 below shows a greater value of Hausdorff dimension of this set, hence we obtain a final result instead of mentioned before. For

$$\left(\bigcap_{r \in R} B(r) - \bigcap_{s \in S} B(s)\right) \subset (I_0 - (B(3) \cap B(5))).$$

In the case of $R = \phi$, Schmidt proved that the Hausdorff dimension of uncountably many numbers in Lemma 2 is equal to 1 [16].

Now assume that $R \neq \phi$. By Lemma 2,

$$Y = \left[\bigcap_{r \in R} B(r) - \bigcap_{s \in S} B(s)\right] \neq \phi.$$

Then, we have

COROLLARY 2 OF LEMMA 1. *For a given $y \in Y$ and for every positive integer k , the k -tuple $(x_1, x_2, \dots, x_{k-1}, y)$ is normal to every base from R for almost all $(x_1, x_2, \dots, x_{k-1}) \in I_0^{k-1}$.*

Tracing the same arguments as in Theorem 2 together with Corollary 2 of Lemma 1, we get

THEOREM 6. *The Hausdorff dimension of non-normal numbers which are normal to every base from R and to no base from S is equal to 1, where $R \neq \emptyset$ and S are two disjoint classes of all positive integers greater than 1 so that equivalent integers fall into the same class.*

NOTE. This Theorem was first proved by A.D. Pollington [12], but his proof needs rather complicated estimation technique for Hausdorff dimension and also essentially Schmidt's construction. Our proof is based on only the existence result of Schmidt and on Beyer's theorem, which seems to be much simpler than that of Pollington.

References

- [1] Beyer, W. A., Hausdorff dimension of level sets of some Rademacher series. *Pacific J. Math.* **12** (1962), 35-46.
- [2] Billingsley, P., *Ergodic Theory and Information*. John Wiley and Sons, New York-London-Sydney-Toronto 1965.
- [3] Borel, E., Les probabilités dénombrables et leurs applications arithmétiques. *Rend. Circ. Mat. Palermo* **27** (1909), 247-271.
- [4] Cassels, J. W. S., On a problem of Steinhaus about normal numbers. *Colloq. Math.* **7** (1959), 95-101.
- [5] Kamae, T. and Weiss, B., Normal numbers and selection rules. *Israel J. Math.* **21** (1975), 101-110.
- [6] Long, C. T., On real numbers having normality of order k . *Pacific J. Math.* **18** (1966), 155-160.
- [7] Mauldin, D., *The Scottish Book*. Birkhäuser, Boston-Basel-Stuttgart 1981.
- [8] Von Mises, R., *Mathematical Theory of Probability and Statistics*. Academic Press, New York-London 1964.
- [9] Nagasaka, K., On Hausdorff dimension of non-normal sets. *Ann. Inst. Stat. Math.* **23** (1971), 515-521.
- [10] Nagasaka, K., La dimension de Hausdorff de certains ensembles dans $[0,1]$. *Proc. Japan Acad.* **54**, Ser. A (1978), 109-112.
- [11] Pelling, M. J., Nonnormal numbers. *Amer. Math. Monthly* **87** (1980), 141-142.
- [12] Pollington, A. D., The Hausdorff dimension of a set of normal numbers. *Pacific J. Math.* **95** (1981), 193-204.
- [13] Šalát, T., A remark on normal numbers. *Rev. Roumaine Math. Pures Appl.* **11** (1966), 53-56.
- [14] Schmidt, W. M., On normal numbers. *Pacific J. Math.* **10** (1960), 661-672.
- [15] Schmidt, W. M., Über die Normalität von Zahlen zu verschiedenen Basen. *Acta Arith.* **7** (1961-62), 299-309.

- [16] Schmidt, W. M., On badly approximable numbers and certain games. *Trans. Amer. Math. Soc.* **123** (1966), 178-199.
- [17] Volkmann, B., On the Cassels-Schmidt theorem, I. *Bull. Sc. Math.* **108** (1984), 321-336.
- [18] Wall, D. D., Normal numbers, Thesis. Univ. of California 1949.

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