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# UNIQUENESS OF CERTAIN 3-DIMENSIONAL HOMOLOGICALLY VOLUME MINIMIZING SUBMANIFOLDS IN COMPACT SIMPLE LIE GROUPS

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## 1. Introduction.

The purpose of this paper is to prove uniqueness of certain 3-dimensional homologically volume minimizing submanifolds in compact simple Lie groups.

Let $G$ be a connected compact simple Lie group whose rank is greater than 1 and $G_1$ be an analytic subgroup of $G$ associated with the highest root of $G$. The explicit definition of $G_1$ will be found in Section 2. It is well known that the homology class $[G_1]$ represented by $G_1$ generates the real homology group $H_3(G;\boldsymbol{R})$ of $G$. Furnishing $G$ with a bi-invariant Riemannian metric $\langle\ ,\ \rangle$, we consider a volume minimizing submanifold contained in the real homology class $[G_1]$. Using the notion of calibration introduced by Harvey-Lawson [1], the second named author has proved the following theorem in his paper [5].

THEOREM 1. *If $M$ is a compact oriented 3-dimensional submanifold of $G$ contained in the real homology class $[G_1]$, then*

$$\mathrm{vol}(G_1) \leqq \mathrm{vol}(M).$$

In this paper we investigate submanifolds $M$ contained in $[G_1]$ which satisfy the equality:

$$\mathrm{vol}(G_1) = \mathrm{vol}(M)$$

and obtain the following theorem.

THEOREM 2. *Let $M$ be a compact oriented 3-dimensioual submanifold of $G$ contained in $[G_1]$. The equality*

$$\mathrm{vol}(G_1) = \mathrm{vol}(M)$$

*holds if and only if $M$ is congruent with $G_1$ in $G$. In particular, $G_1$ is a unique volume minimizing submanifold contained in $[G_1]$ up to congruence in $G$.*

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REMARK. Theorem 2 is an affirmative answer to the problem posed in [5, p. 126 Remark].

## 2. Preliminaries.

Let $\mathfrak{g}$ be the Lie algebra of $G$. Take a maximal Abelian subalgebra $\mathfrak{t}$ in $\mathfrak{g}$, then the complexification $\mathfrak{t}^{C}$ of $\mathfrak{t}$ is a Cartan subalgebra of the complexification $\mathfrak{g}^{C}$ of $\mathfrak{g}$. For each element $\alpha$ in $\mathfrak{t}$, put

$$\mathfrak{g}_{\alpha}=\{X\in\mathfrak{g}^{C}; [H, X]=\sqrt{-1}\langle\alpha, H\rangle X \text{ for each } H\in\mathfrak{t}\}.$$

An element $\alpha$ in $\mathfrak{t}-\{0\}$ is called a *root* if $\mathfrak{g}_{\alpha}\neq\{0\}$. Let $\Delta$ denote the set of all roots. We obtain a direct sum decomposition of $\mathfrak{g}^{C}$:

$$\mathfrak{g}^{C}=\mathfrak{t}^{C}+\sum_{\alpha\in\Delta}\mathfrak{g}_{\alpha}.$$

Fix a lexicographic ordering on $\mathfrak{t}$ and denote by $\Delta_{+}$ the set of all positive roots in $\Delta$.

The following lemma follows from the above direct sum decomposition of $\mathfrak{g}^{C}$. For details of the proof, see Section 3 of Chapter VI in Helgason [2].

LEMMA 3. *There exist unit vectors $E_{\alpha}$, $F_{\alpha}$ in $\mathfrak{g}$ for each $\alpha\in\Delta_{+}$ in such a way that:*

i)
$$\mathfrak{g}=\mathfrak{t}+\sum_{\alpha\in\Delta_{+}}\boldsymbol{R}E_{\alpha}+\sum_{\alpha\in\Delta_{+}}\boldsymbol{R}F_{\alpha}$$

*is an orthogonal direct sum decomposition of $\mathfrak{g}$;*

ii)
$$[H, E_{\alpha}]=\langle\alpha, H\rangle F_{\alpha}, \quad [H, F_{\alpha}]=-\langle\alpha, H\rangle E_{\alpha}, \quad [E_{\alpha}, F_{\alpha}]=\alpha$$

*for $\alpha\in\Delta_{+}$ and $H\in\mathfrak{t}$.*

Let $\delta$ be the highest root in $\Delta_{+}$ and set

$$\mathfrak{g}_{1}=\boldsymbol{R}\delta+\boldsymbol{R}E_{\delta}+\boldsymbol{R}F_{\delta}.$$

Then $\mathfrak{g}_{1}$ is a compact 3-dimensional simple Lie subalgebra of $\mathfrak{g}$. Let $G_{1}$ be the analytic subgroup of $G$ corresponding to $\mathfrak{g}_{1}$. Wolf has proved that $G_{1}$ is simply connected when $G$ is centerless in the proof of Theorem 5.4 in [6]. Therefore $G_{1}$ is simply connected, even if $G$ has a nontrivial center.

Put

$$\phi(X, Y, Z)=\frac{1}{|\delta|}\langle[X, Y], Z\rangle$$

for $X$, $Y$, and $Z$ in $\mathfrak{g}$. By regarding an element of $\mathfrak{g}$ as a left-invariant vector field on $G$, $\phi$ is a bi-invariant 3-form on $G$. In particular, $\phi$ is a closed form on $G$.

We introduce an orientation on $\mathfrak{g}_1$ such that $\{\delta, E_\delta, F_\delta\}$ is a positive basis of $\mathfrak{g}_1$.

LEMMA 4. ([5]) *For each* 3*-dimensional oriented subspace* $\xi$ *in* $\mathfrak{g}$, *the inequality*

$$\phi|_\xi \leqq \mathrm{vol}_\xi$$

*holds. The equality holds if and only if there is an element* $g$ *in* $G$ *such that*

$$\xi = \mathrm{Ad}\,(g)\mathfrak{g}_1$$

*and that* $\mathrm{Ad}\,(g): \mathfrak{g}_1 \to \xi$ *is orientation preserving.*

## 3. Proof of Theorem 2.

At first we review the proof of Theorem 1.

Let $M$ be a compact oriented 3-dimensional submanifold of $G$ contained in the real homology class $[G_1]$. Since $\phi$ is a bi-invariant form on $G$, the inequality of $\phi$ stated in Lemma 4 holds at every point in $G$. The proof of Theorem 1 is as follows:

$$\mathrm{vol}\,(G_1) = \int_{G_1} \mathrm{vol}_{G_1} = \int_{G_1} \phi = \int_M \phi \leqq \int_M \mathrm{vol}_M = \mathrm{vol}\,(M)\,.$$

The equality holds if and only if $\phi|_M = \mathrm{vol}_M$. A 3-dimensional oriented submanifold $M$ of $G$ which satisfies $\phi|_M = \mathrm{vol}_M$ is called a *$\phi$-submanifold* of $G$. So the following lemma completes the proof of Theorem 2.

LEMMA 5. *If* $M$ *is a* $\phi$*-submanifold of* $G$, *then* $M$ *is congruent with a piece of* $G_1$ *in* $G$. *Furthermore, if* $M$ *is complete, then* $M$ *is congruent with* $G_1$.

PROOF. We show that $M$ is totally geodesic in $G$. Let $x$ be any point of $M$. Since $G$ is a homogeneous Riemannian manifold, we may suppose that $x$ is the identity element $e$ of $G$. We may show that the second fundamental form $h$ of $M$ vanishes at $e$. It follows from Lemma 4 that there is an element $g$ in $G$ such that $T_e(M) = \mathrm{Ad}\,(g)\mathfrak{g}_1$. For simplicity we set $T_e(M) = T_e(G_1)$ and identify $T_e(G)$ with $\mathfrak{g}$. Then the following equations hold:

$$T_e(M) = \mathfrak{g}_1 \tag{1}$$

and

$$T_e^\perp(M) = \{H \in \mathfrak{t}\,;\, \langle \delta, H \rangle = 0\} + \sum_{\alpha \in \Delta_+ - \{\delta\}} (\boldsymbol{R}E_\alpha + \boldsymbol{R}F_\alpha)\,.$$

The curvature tensor $R$ of $G$ is given by

$$R(X, Y)Z = -\frac{1}{4}[[X, Y], Z] \tag{2}$$

for $X$, $Y$ and $Z$ in $\mathfrak{g}$. See, for example, Milnor [3]. So the assumption on $M$ and Lemma 4 imply that $R(X, Y)Z$ is contained in the tangent space of $M$ for any tangent vectors $X$, $Y$, and $Z$ of $M$, that is, $M$ is a curvature invariant submanifold of $G$. As $M$ is curvature invariant and $G$ is locally symmetric, the equation

$$h(W, R(X, Y)Z)=R(h(W, X), Y)Z+R(X, h(W, Y))Z \\ +R(X, Y)h(W, Z) \tag{3}$$

holds for tangent vectors $X$, $Y$, $Z$, and $W$ of $M$. The above equation is due to Ohnita [4]. Putting $W=X$ and $Z=Y$ in (3), we obtain

$$h(X, R(X, Y)Y)=R(h(X, X), Y)Y+R(X, h(X, Y))Y \\ +R(X, Y)h(X, Y). \tag{4}$$

From now on we shall consider $h$ at $e$. Let $X$ and $Y$ be orthonormal vectors in $\mathfrak{g}_1$. Then

$$R(X, Y)Y=\frac{1}{4}|\delta|^2 X.$$

By (2) and (4),

$$|\delta|^2 h(X, X)=-[[h(X, X), Y], Y]-[[X, h(X, Y)], Y] \\ -[[X, Y], h(X, Y)]. \tag{5}$$

Let $\mathfrak{z}$ be the centralizer of $\mathfrak{g}_1$ in $\mathfrak{g}$ and $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{z}$ in $T_e^{\perp}(M)$. Then

$$\mathfrak{z}=\{H\in\mathfrak{t}\,;\langle\delta, H\rangle=0\}+\sum_{\substack{\alpha\in\Delta_+ \\ \langle\alpha,\delta\rangle=0}}(\boldsymbol{R}E_\alpha+\boldsymbol{R}F_\alpha)$$

and

$$\mathfrak{m}=\sum_{\substack{\alpha\in\Delta_+-\{\delta\} \\ \langle\alpha,\delta\rangle\neq 0}}(\boldsymbol{R}E_\alpha+\boldsymbol{R}F_\alpha).$$

According to Wolf [6], $(\mathfrak{g}, \mathfrak{z}+\mathfrak{g}_1)$ is a compact quaternionic symmetric pair. So the right hand side of the equation (5) is contained in $\mathfrak{m}$. Therefore the image of $h$ is contained in $\mathfrak{m}$.

Since $(\mathfrak{g}, \mathfrak{z}+\mathfrak{g}_1)$ is a quaternionic symmetric pair, $\mathfrak{m}$ has a quaternionic vector space structure and $\mathfrak{g}_1$ acts on $\mathfrak{m}$ as the multiplications by purely quaternionic numbers. In particular, we obtain

$$[X, [Y, h(X, Y)]]=-[Y, [X, h(X, Y)]], \tag{6}$$

because $X$ and $Y$ are orthogonal vectors in $\mathfrak{g}_1$ and $h(X, Y)\in\mathfrak{m}$.

The equation

$$|\delta|^2 h(X, X)=-[Y, [Y, h(X, X)]]+3[Y, [X, h(X, Y)]] \tag{7}$$

follows from (5), (6), and the Jacobi identity.

On the other hand

$$[U,[U,V]]=-\frac{1}{4}|\delta|^2 V \tag{8}$$

for any unit vector $U$ in $\mathfrak{g}_1$ and any vector $V$ in $\mathfrak{m}$. Indeed, since each unit vector in $\mathfrak{g}_1$ is $G_1$-conjugate to $\delta/|\delta|$ and $\mathfrak{m}$ is $\mathrm{Ad}(G_1)$-invariant, we may suppose $U=\delta/|\delta|$. Put

$$V=\sum_{\substack{\alpha\in\Delta_+-\{\delta\}\\ \langle\alpha,\delta\rangle\neq 0}}(s_\alpha E_\alpha+t_\alpha F_\alpha).$$

By ii) of Lemma 3 and the fact that

$$\frac{2\langle\alpha,\delta\rangle}{\langle\delta,\delta\rangle}=1$$

for each $\alpha$ in $\Delta_+-\{\delta\}$ with $\langle\alpha,\delta\rangle\neq 0$ (cf. Wolf [6]), we have

$$\begin{aligned}[U,[U,V]]&=-\frac{1}{|\delta|^2}\sum_{\substack{\alpha\in\Delta_+-\{\delta\}\\ \langle\alpha,\delta\rangle\neq 0}}\langle\alpha,\delta\rangle^2(s_\alpha E_\alpha+t_\alpha F_\alpha)\\ &=-\frac{|\delta|^2}{4}\sum_{\substack{\alpha\in\Delta_+-\{\delta\}\\ \langle\alpha,\delta\rangle\neq 0}}(s_\alpha E_\alpha+t_\alpha F_\alpha)\\ &=-\frac{|\delta|^2}{4}V.\end{aligned}$$

It follows from (7) and (8) that

$$\frac{1}{4}|\delta|^2 h(X,X)=[Y,[X,h(Y,X)]] \tag{9}$$

for any orthonormal vectors $X$ and $Y$ in $\mathfrak{g}_1$. By (9) we also have

$$\frac{1}{4}|\delta|^2 h(Y,Y)=[X,[Y,h(Y,X)]]. \tag{10}$$

Thus by (6), (9) and (10) we get

$$h(X,X)=-h(Y,Y).$$

Since $\mathfrak{g}_1$ is 3-dimensional, we have

$$h(X,X)=0$$

for any unit vector $X$ in $\mathfrak{g}_1$ and $h=0$.

Hence $M$ is totally geodesic in $G$. Since $G_1$ is also totally geodesic in $G$, $M$ is a piece of $G_1$. Q.E.D.

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