TSUKUBA J. MATH.
Vol. 9 No. 1 (1985). 113–116

A CHARACTERIZATION OF ABSOLUTE NEIGHBORHOOD RETRACTS IN GENERAL SPACES

Dedicated to Professor Keiô Nagami on his 60th birthday

By

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Some characterizations of absolute neighborhood retracts were established in separable metric spaces by Hanner [4]. Hanner's characterizations were easily extended to the metric case. For this, see Hu [5] Chapter IV. In this paper, we shall extend one of Hanner's characterizations to more general spaces, especially, stratifiable spaces, spaces with a σ-almost locally finite base and paracomplexes. For the ANR theory of these spaces, we refer Cauty [2], Miwa [8] and Hyman [6], respectively.

Throughout this paper, all spaces are assumed to be paracompact normal spaces and all maps to be continuous. I and $\mathcal{S}$ denote the closed unit interval $[0,1]$ and the class of all stratifiable spaces, respectively. $ANR(Q)$ (resp. $ANE(Q)$) is the abbreviation for absolute neighborhood retract (resp. extensor) for the class Q. For these definitions, see [5].

In this paper, all theorems are proved in the class $\mathcal{S}$. But these theorems can be proved in some other classes. For instance, see Remark 2.3.

1. Preliminaries.

DEFINITION 1.1 ([3]). A space Y is *equiconnected* if there is a map $F: Y\times Y\times I\rightarrow Y$ such that $F(x, y, 0)=x$, $F(x, y, 1)=y$ and $F(x, x, t)=x$ for all $(x, y)\in Y\times Y$ and $t\in I$. The space Y is said to be *locally equiconnected* if F is defined only on $U\times I$, for some neighborhood U of the diagonal of $Y\times Y$.

DEFINITION 1.2 ([4]). Let $f, g: Y\rightarrow X$ be two maps. If X is covered by $\mathcal{U}=\{U_\alpha\}$, f and g are called *$\mathcal{U}$-near* if for each $y\in Y$ there is a $U_\alpha\in\mathcal{U}$ such that $f(y)\in U_\alpha$, $g(y)\in U_\alpha$.

DEFINITION 1.3 ([4]). Let $h_t: Y\rightarrow X$ be a homotopy. If X is covered by $\mathcal{U}=\{U_\alpha\}$, h_t is called a *$\mathcal{U}$-homotopy* if for each $y\in Y$ there is a $U_\alpha\in\mathcal{U}$ such that $h_t(y)\in U_\alpha$ for all $t\in I$.

Received March 8, 1984. Received May 8, 1984.

The following proposition is easily verified. For instance, see Cauty [2]. But for completeness, we state the proof.

PROPOSITION 1.4. *If Y is an $ANR(\mathcal{S})$, then Y is locally equiconnected.*

PROOF. Let $A=Y\times Y\times\{0,1\}\cup\Delta\times I$, where Δ is the diagonal of $Y\times Y$. We define a function $f:A\rightarrow Y$ as follows: $f(x,y,0)=x$, $f(x,y,1)=y$ and $f(x,x,t)=x$ for all $t\in I$. Then f is continuous. Since Y is an $ANR(\mathcal{S})$ by [1] Corollary 6.3, there is a neighborhood U of Δ in $Y\times Y$ and a map $F:U\times I\rightarrow Y$ such that $F|A=f$. Therefore Y is locally equiconnected.

2. Main theorems.

In this section, we extend Hanner's theorems [4] Theorem 4.1 and 4.2 to stratifiable case. Each proof is simpler than Hanner's one.

THEOREM 2.1. *If Y is an $ANR(\mathcal{S})$ and $\mathcal{U}=\{U_\alpha\}$ a given open covering of Y, then there exists an open covering $\mathcal{W}$ of Y, which is refinement of $\mathcal{U}$, such that, for any two $\mathcal{W}$-near maps $f,g:X\rightarrow Y$ defined on a stratifiable space X and any $\mathcal{W}$-homotopy $j_t:A\rightarrow Y$, $(0\leqq t\leqq 1)$, defined on a closed subspace A of X with $j_0=f|A$ and $j_1=g|A$, there exists an $\mathcal{U}$-homotopy $h_t:X\rightarrow Y$, $(0\leqq t\leqq 1)$, such that $h_0=f$, $h_1=g$ and $h_t|A=j_t$ for every $t\in I$.*

PROOF. Since Y is locally equiconnected by Proposition 1.4, there exist a neighborhood U of the diagonal of $Y\times Y$ and a map $F:U\times I\rightarrow Y$ such that $F(x,y,0)=x$, $F(x,y,1)=y$ and $F(x,x,t)=x$ for all $(x,y)\in U$ and $t\in I$. For any $y\in Y$, since I is compact, there exists an open neighborhood V_y of y such that $V_y\times V_y\subset U$ and $F(V_y\times V_y\times I)\subset U_\alpha$ for some $U_\alpha\in\mathcal{U}$. Let $\mathcal{V}=\{V_y:y\in Y\}$ and $\mathcal{V}'$ be a barycentric refinement of $\mathcal{V}$; i.e., the covering $\{\mathrm{St}(y,\mathcal{V}'):y\in Y\}$ refines $\mathcal{V}$. For any $y\in Y$, there exists an open neighborhood W_y of y such that $F(W_y\times W_y\times I)\subset V'$ for some $V'\in\mathcal{V}'$. Let $\mathcal{W}=\{W_y:y\in Y\}$. Then it is obvious that $\mathcal{W}$ refines $\mathcal{V}'$ and $W_y\times W_y\subset U$ for each $y\in Y$.

Now, let $f,g:X\rightarrow Y$ be any two $\mathcal{W}$-near maps defined on a stratifiable space X and let $j_t:A\rightarrow Y$, $(0\leqq t\leqq 1)$, be given $\mathcal{W}$-homotopy defined on a closed subspace A of X with $j_0=f|A$ and $j_1=g|A$.

By using the map F, we can construct a $\mathcal{V}'$-homotopy $k_t:X\rightarrow Y$, $(0\leqq t\leqq 1)$, by taking

$$k_t(x)=F(f(x),g(x),t)\qquad\text{for } x\in X \text{ and } t\in I.$$

Since f, g are $\mathcal{W}$-near maps, it is clear that k_t is a $\mathcal{V}'$-homotopy.

In the topological product $P=X\times I$, consider the closed subspace $Q=(X\times\{0,1\})\cup A\times I$ and define a map $m:Q\to Y$ by taking

$$m(x,t)=\begin{cases} f(x) & (\text{if } x\in X \text{ and } t=0) \\ j_t(x) & (\text{if } x\in A \text{ and } t\in I) \\ g(x) & (\text{if } x\in X \text{ and } t=1)\,. \end{cases}$$

Since Y is $ANR(\mathcal{S})$, it follows that m has an extension $m':N\to Y$ over neighborhood N of Q in P. Since I is compact, there exists an open neighborhood C of A in X such that $C\times I$ is contained in N and that a homotopy $n_t:C\to Y$, $(0\leqq t\leqq 1)$, defined by

$$n_t(x)=m'(x,t)\,, \qquad (x\in C,\ t\in I)$$

is a $\mathcal{W}$-homotopy. Therefore of course n_t is a $\mathcal{V}'$-homotopy.

Since X is stratifiable, there exists an open subset B in X such that $A\subset B\subset \bar{B}\subset C$. By Urysohn's lemma, there exists a map $e:X\to I$ such that

$$e(x)=\begin{cases} 0\,, & (\text{if } x\in X-B) \\ 1\,, & (\text{if } x\in A)\,. \end{cases}$$

Define a homotopy $h_t:X\to Y$, $(0\leqq t\leqq 1)$, by taking

$$h_t(x)=\begin{cases} k_t(x) & (\text{if } x\in X-B) \\ F(k_t(x), n_t(x), e(x)) & (\text{if } x\in C)\,. \end{cases}$$

Then h_t is well-defined. Indeed, since k_t, n_t are $\mathcal{V}'$-homotopies, for each $x\in C$ there exist some $V_1'\in\mathcal{V}'$ and $V_2'\in\mathcal{V}'$ such that $k_t(x)\in V_1'$ and $n_t(x)\in V_2'$ for any $t\in I$. By the fact $k_0(x)=n_0(x)=f(x)$, $V_1'\cap V_2'\neq\emptyset$. Therefore there is a $V_y\in\mathcal{V}$ with $V_1'\cup V_2'\subset V_y$ since $\mathcal{V}'$ is a barycentric refinement of $\mathcal{V}$. Thus for any $t\in I$, $(k_t(x), n_t(x))\in V_y\times V_y\subset U$.

It can be easily verified that h_t is a $\mathcal{U}$-homotopy satisfying the required properties. This completes the proof.

The following theorem is easy to see by Theorem 2.1 and the same method of Hanner [4] Theorem 4.2 (or Hu [5] p. 114 Theorem 1.3).

THEOREM 2.2. *A necessary and sufficient condition for a stratifiable space Y to be an $ANR(\mathcal{S})$ is the existence of an open covering $\mathcal{W}$ of Y such that, for any two $\mathcal{W}$-near maps $f,g:X\to Y$ defined on a stratifiable space X and any $\mathcal{W}$-homotopy $j_t:A\to Y$, $(0\leqq t\leqq 1)$, defined on a closed subspace A of X with $j_0=f|A$ and $j_1=g|A$, there exists a homotopy $h_t:X\to Y$, $(0\leqq t\leqq 1)$, with $h_0=f$, $h_1=g$ and $h_t|A=j_t$ for every $t\in I$.*

REMARK 2.3. In this paper, we considered exclusively in the class $\mathcal{S}$. But if we reconsider the proofs of Proposition 1.4, Theorem 2.1 and 2.2, it is found that, for each class Q satisfying the following four conditions, Proposition 1.4, Theorem 2.1 and 2.2 are valid.

(1) Every $X\in Q$ is paracompact normal.

(2) If A is a closed (resp. an open) subspace of $X\in Q$, then $A\in Q$.

(3) For $X\in Q$, $X^2\in Q$.

(4) A space $X\in Q$ is an $ANR(Q)$ if and only if X is an $ANE(Q)$.

Indeed, these conditions are used in the proofs of theorems as follows: The condition (1) has been used in the proof of Theorem 2.2 ("every local $ANR(Q)$ is an $ANR(Q)$") and the proof of Theorem 2.1 ("$\mathcal{V}'$ is a barycentric refinement of $\mathcal{V}$"). The condition (2) has been used in the proof of Theorem 2.1 ("a closed subspace A of $X\in Q$ is in Q and Q is in Q") and in the proof of Theorem 2.2 ("Q is open hereditary"). The condition (3) has been used in the proof of Proposition 1.4 ("$A\in Q$") and in the proof of Theorem 2.1 ("$X\times I\in Q$"; by $(X+I)^2\in Q$ and the condition (2)). The condition (4) has been used in the proof of Proposition 1.4 ("f has an extension F") and in the proof of Theorem 2.1 ("m has an extension m'").

Of course, the class $\mathcal{S}$ satisfies these conditions, and for instance, the following classes also satisfy these conditions: Paracomplex (Hyman [6]), space with a σ-almost locally finite base (Itō and Tamano [7] and Miwa [8]) and paracompact σ-space (Okuyama [9]).

References

[1] Borges, C. J. R., On stratifiable spaces, Pacific J. Math., **17** (1966), 1-16.
[2] Cauty, R., Rétractions dans les espaces stratifiables, Bull. Soc. Math. France, **102** (1974), 129-149.
[3] Fox, R. H., On fibre spaces II, Bull. Amer. Math. Soc., **49** (1943), 733-735.
[4] Hanner, O., Some theorems on absolute neighborhood retracts, Ark. Mat., **1** (1951), 389-408.
[5] Hu, S. T., Theory of retracts, Wayne State Univ. Press, Detroit, 1965.
[6] Hyman, D. M., A category slightly larger than the metric and CW-categories, Michigan Math. J., **15** (1968), 193-214.
[7] Itō, M. and Tamano, K., Spaces whose closed images are M_1, Proc. Amer. Math. Soc., 87 (1983), 159-163.
[8] Miwa, T., Retraction of spaces with a σ-almost locally finite base, Tsukuba J. Math., 7 (1983), 317-321.
[9] Okuyama, A., A survey of the theory of σ-spaces, Gen Top. its Appl., **1** (1971), 57-63.

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