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INNER DERIVATIONS OF HIGHER ORDERS

By

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Summary. We define inner derivations of higher order of a ring R and we prove that they correspond to the inner automorphisms of a suitable ring. Moreover, we prove that any higher derivation of R is inner if and only if any usual derivation of R is inner.

I.

Let R be a ring with identity and let S be a segment of $N=\{0, 1, 2, \cdots\}$, that is, $S=N$ or $S=\{0, 1, \cdots, s\}$ for some $s\geqq 0$.

A family $d=(d_n)_{n\in S}$ of mappings $d_n: R\rightarrow R$ is called a *derivation of order* s *of* R (where $s=\sup S\leqq\infty$) if the following properties are satisfied:

(1) $d_n(a+b)=d_n(a)+d_n(b)$,

(2) $d_n(ab)=\sum_{i+j=n} d_i(a)d_j(b)$,

(3) $d_0=id_R$.

The set of derivations of order s of R, denoted by $D_s(R)$, is the group under the multiplication $*$ defined by the formula

$$(d * d')_n=\sum_{i+j=n} d_i\circ d'_j,$$

where $d, d'\in D_s(R)$ and $n\in S$ ([1], [5], [7]).

It is easy to prove the following two lemmas.

LEMMA 1.1. *Let* $a\in R$, $d_0=id_R$, *and*

$$d_n(x)=a^n x-a^{n-1}xa=a^{n-1}(ax-xa)$$

for $n\geqq 1$, $x\in R$. *Then* $d=(d_n)_{n\in S}$ *belongs to* $D_s(R)$.

LEMMA 1.2. *Let* $d\in D_s(R)$, $k\in S\setminus\{0\}$ *and let* $\delta=(\delta_n)_{n\in S}$ *be the family of mappings from* R *to* R *defined by*

$$\delta_n=\begin{cases} 0, & \text{if } k\nmid n,\\ d_r, & \text{if } n=rk.\end{cases}$$

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Then $\delta\in D_s(R)$.

The derivation d from Lemma 1.1 will be denoted by $[a, 1]$ and the derivation δ from Lemma 1.2, for $d=[a, 1]$, will be denoted by $[a, k]$. Therefore, for $a\in R$, $k\in S\setminus\{0\}$, $x\in R$, $n\in S$:

$$[a, k]_n(x)=\begin{cases} x , & \text{if } n=0, \\ 0 , & \text{if } k\nmid n, \\ a^r x-a^{r-1}xa, & \text{if } n\neq 0 \text{ and } n=kr. \end{cases}$$

Let $\boldsymbol{a}=(a_n)_{n\in S}$ be a sequence in R. Denote by $\Delta(\boldsymbol{a})$ the element in $D_s(R)$ defined by

$$\Delta(\boldsymbol{a})_n=([a_1, 1]*[a_2, 2]*\cdots*[a_n, n])_n .$$

For example

$$\Delta(\boldsymbol{a})_1(x)=a_1x-xa_1$$

$$\Delta(\boldsymbol{a})_2(x)=a_1^2x-a_1xa_1+a_2x-xa_2$$

$$\Delta(\boldsymbol{a})_3(x)=a_1^3x-a_1^2xa_1+a_1a_2x+xa_2a_1-a_1xa_2-a_2xa_1+a_3x-xa_3$$

$$\Delta(\boldsymbol{a})_4(x)=a_1^4x-a_1^3xa_1+a_2^2x-a_2xa_2+a_1^2a_2x-a_1^2xa_2-a_1a_2xa_1$$
$$+a_1xa_2a_1+a_1a_3x-a_1xa_3-a_3xa_1+xa_3a_1+a_4x-xa_4 .$$

DEFINITION 1.3. Let $d\in D_s(R)$. If there exists a sequence $\boldsymbol{a}=(a_n)_{n\in S}$ of elements of R such that $d=\Delta(\boldsymbol{a})$ then d is called an *inner derivation of order* s *of* R.

II.

Denote by T the additive group of the product of $s+1$ copies of R. The element $(a_n)_{n\in S}$ will be always denoted by $\boldsymbol{a}$. We define a multiplication on T as follows:

$$\boldsymbol{ab}=\boldsymbol{c}, \quad \text{where} \quad c_n=\sum_{i+j=n} a_ib_j .$$

T is a ring with identity $(1, 0, 0, \cdots)$ ([7], [8]). Notice that an element $\boldsymbol{a}$ is invertible in T iff a_0 is invertible in R.

For any $k\in S$, let π_k denote the k-th projection from T to R. If $a\in R$ then $j_k(a)$, $p_k(a)$ and $q_k(a)$ (where $k\in S$, $l\in S\setminus\{0\}$) denote the elements of T defined by the following conditions:

$$\pi_nj_k(a)=\begin{cases} 0, & \text{for } n\neq k, \\ a, & \text{for } n=k, \end{cases} \qquad \pi_np_l(a)=\begin{cases} 0 , & \text{if } l\nmid n, \\ a^r, & \text{if } n=rl, \end{cases}$$

$$\pi_n q_l(a)=\begin{cases} 1, & \text{for } n=0, \\ 0, & \text{for } n\geqq 1,\ n\neq l, \\ a, & \text{for } n=l. \end{cases}$$

Let T_k (for $k\in S\setminus\{0\}$) denote the set of elements $\boldsymbol{a}$ in T such that $a_0=1$ and $a_i=0$ for $i=1, 2, \cdots, k$, and let T_0 be the set of elements $\boldsymbol{a}$ in T such that $a_0=1$.

Observe that $q_k(a)=1_T+j_k(a)$, and every element in T_k is of the form $1+j_{n+1}(1)\boldsymbol{a}$, for some $\boldsymbol{a}\in T$.

It is easy to verify the following

LEMMA 2.1. *Let $k\in S$, $a\in R$.*

(1) *If $\boldsymbol{a}$, $\boldsymbol{b}\in T_k$ then $\boldsymbol{ab}$, $\boldsymbol{a}^{-1}\in T_k$.*

(2) $p_k(a)^{-1}=q_k(-a)$.

(3) *If $\boldsymbol{b}\in T_0$ then $\boldsymbol{b}p_k(-b_k)=\boldsymbol{a}$, where $a_n=b_n$ for $n=0, 1, \cdots, k-1$, and $a_k=0$.*

Now we prove two lemmas.

LEMMA 2.2. *Let $\boldsymbol{b}\in T_0$. Then there exists an element $\boldsymbol{a}$ in T_0 such that $\boldsymbol{b}p_1(a_1)p_2(a_2)\cdots p_k(a_k)\in T_k$, for any $k\in S\setminus\{0\}$.*

PROOF. Let $a_1=-b_1$. Then, by Lemma 2.1(3), we have $\boldsymbol{b}p_1(a_1)\in T_1$. Suppose that elements $a_1, \cdots, a_n$ satisfy the condition

$$\boldsymbol{v}^{(k)}=\boldsymbol{b}p_1(a_1)\cdots p_k(a_k)\in T_k$$

for $k=1, 2, \cdots, n$.

Put $a_{n+1}=-\pi_{n+1}(\boldsymbol{v}^{(n)})$. Then

$$\begin{aligned}\boldsymbol{v}^{(n+1)}&=\boldsymbol{v}^{(n)}p_{n+1}(a_{n+1})\\ &=\boldsymbol{b}p_1(a_1)\cdots p_{n+1}(a_{n+1})\in T_{n+1}\end{aligned}$$

by Lemma 2.1(3).

LEMMA 2.3. *Let $\boldsymbol{a}\in T_0$ Then there exists $\boldsymbol{b}\in T_0$ such that*

$$p_1(a_1)p_2(a_2)\cdots p_k(a_k)\boldsymbol{b}\in T_k$$

for any $k\in S\setminus\{0\}$.

PROOF. Put $b_0=1$ and $b_n=\pi_n(\boldsymbol{u}^{(n)})$, for $n\geqq 1$, where $\boldsymbol{u}^{(n)}=q_n(-a_n)\cdots q_1(-a_1)$.

Then $b_n=\pi_n(\boldsymbol{u}^{(k)})$ for any $n\in S\setminus\{0\}$ and $k\geqq n$. In fact, if $k\geqq n$ then

$$\pi_n(\boldsymbol{u}^{(k+1)})=\pi_n(\boldsymbol{u}^{(k)}+j_{k+1}(-a_{k+1})\boldsymbol{u}^{(k)})$$
$$=\pi_n(\boldsymbol{u}^{(k)})+\pi_n(j_{k+1}(-a_{k+1})\boldsymbol{u}^{(k)})$$
$$=\pi_n(\boldsymbol{u}^{(k)})\,.$$

Therefore, if $\boldsymbol{b}=(b_n)_{n\in S}$ then $\pi_i(\boldsymbol{b}-\boldsymbol{u}^{(k)})=0$ for $i=0, 1, \cdots, k$. So $\boldsymbol{b}=\boldsymbol{u}^{(k)}+j_{k+1}(1)\boldsymbol{v}^{(k)}$, for some $\boldsymbol{v}^{(k)}\in T$, and, by Lemma 2.1, we have

$$p_1(a_1)p_2(a_2)\cdots p_k(a_k)\boldsymbol{b}=p_1(a_1)\cdots p_k(a_k)(q_k(-a_k)\cdots q_1(-a_1)+j_{k+1}(1)\boldsymbol{v}^{(k)})$$
$$=1_T+j_{n+1}(1)\boldsymbol{c}\,,$$

for some $\boldsymbol{c}\in T$. This completes the proof.

III.

If $d\in D_s(R)$ then $\exp(d)$ will denote the ring automorphism of T defined as follows:

$$\exp(d)(\boldsymbol{a})=\boldsymbol{b},\quad \text{where}\quad b_n=\sum_{i+j=n} d_i(a_j) \qquad ([5], [7], [8]).$$

In [7] Ribenboim showed that the mapping exp is a group isomorphism from $D_s(R)$ to the group $B_s(R)$ of such automorphisms $h: T\to T$ that $h(j_1(1))=j_1(1)$, $\pi_0 h j_0=id_R$. If $h\in B_s(R)$ then the derivation $d=(d_n)_{n\in S}$, where $d_n(x)=\pi_n h j_0(x)$ for $x\in R$, satisfies the condition $h=\exp(d)$ ([7]).

For any $\boldsymbol{a}\in T_0$ denote by $\langle\boldsymbol{a}\rangle$ the inner automorphism of T defined by $\langle\boldsymbol{a}\rangle(\boldsymbol{x})=\boldsymbol{a}^{-1}\boldsymbol{x}\boldsymbol{a}$. Observe that $\langle\boldsymbol{a}\rangle$ belongs to $B_s(R)$.

LEMMA 3.1. (1) *If* $a\in R$, $k\in S\setminus\{0\}$ *then* $\exp([a, k])=\langle q_k(-a)\rangle$.

(2) *Let* $\boldsymbol{a}\in T_k$. *If* $d=(d_n)_{n\in S}$ *is an element of* $D_s(R)$ *such that* $\exp(d)=\langle\boldsymbol{a}\rangle$, *then* $d_1=d_2=\cdots d_k=0$.

PROOF. (1) If $d\in D_s(R)$ satisfies $\exp(d)=\langle q_k(-a)\rangle$ then

$$d_n(x)=\pi_n\langle q_k(-a)\rangle j_0(x)$$
$$=\pi_n q_k(-a)^{-1}j_0(x)q_k(-a)$$
$$=\pi_n p_k(a)j_0(x)(1_T+j_k(-a))\,,\qquad \text{for}\quad n\in S.$$

Hence $d_n(x)=0$ if $k\nmid n$, and $d_n(x)=a^r x-a^{r-1}xa$ if $n=kr$. Therefore $d=[a, k]$.

(2) It follows from Lemma 2.1(1) since $d_n=\pi_n\langle\boldsymbol{a}\rangle j_0$.

Now we are ready to prove the following

THEOREM 3.2. *Let* $d\in D_s(R)$. *Then* d *is inner iff there exists* $\boldsymbol{b}\in T_0$ *such* *that* $\exp(d)=\langle \boldsymbol{b}\rangle$.

PROOF. Let $d=\Delta(\boldsymbol{a})$, where $\boldsymbol{a}\in T_0$, and let $\boldsymbol{b}$ be as in Lemma 2.3. Moreover, let $\delta=(\delta_n)_{n\in S}$ be the unique derivation satisfying $\exp(\delta)=\langle \boldsymbol{b}\rangle$. We show that $\delta=d$.

Let $n\in S\setminus\{0\}$. It follows from Lemmas 2.3, 2.1 that

$$\boldsymbol{b}=q_n(-a_n)\cdots q_1(-a_1)\boldsymbol{v}^{(n)},$$

where $\boldsymbol{v}^{(n)}$ is an element of T_n.

Therefore, if $F=\exp^{-1}$ then

$$\delta=F\langle \boldsymbol{b}\rangle=F\langle \boldsymbol{v}^{(n)}\rangle * F\langle q_1(-a_1)\rangle * \cdots * F\langle q_n(-a_n)\rangle\,,$$

and, by Lemma 3.1,

$$\delta_n=([a_1, 1] * \cdots * [a_n, n])_n=d_n\,.$$

Conversely, let $\boldsymbol{b}\in T_0$, $d=\exp^{-1}(\langle \boldsymbol{b}\rangle)$ and let $\boldsymbol{a}$ be such as in Lemma 2.2. We show that $d=\Delta(\boldsymbol{a})$.

Let $n\in S\setminus\{0\}$. It follows from Lemmas 2.2, 2.1 that

$$\boldsymbol{b}=\boldsymbol{v}^{(n)}q_n(-a_n)\cdots q_1(-a_1)\,,$$

where $\boldsymbol{v}^{(n)}\in T_n$, and hence

$$d=F\langle \boldsymbol{b}\rangle=F\langle q_1(-a_1)\rangle * \cdots * F\langle q_n(-a_n)\rangle * F\langle \boldsymbol{v}^{(n)}\rangle\,,$$

where $F=\exp^{-1}$.

Therefore, by Lemma 3.1, we have

$$d_n=([a_1, 1] * \cdots * [a_n, n])_n \qquad \text{i.e.}\quad d=\Delta(\boldsymbol{a}).$$

COROLLARY 3.3. *The set of inner derivations of order s of R is a normal subgroup of $D_s(R)$.*

IV.

Recall that the *usual* (*classical*) *derivation* of R is the additive mapping $\delta: R\to R$ such that $\delta(ab)=\delta(a)b+a\delta(b)$, for all elements $a, b\in R$. The set of usual derivations of R corresponds bijectively, in the natural way, to the set $D_1(R)$. Evidently a usual derivation is innner iff there exists an element $a\in R$ such that $\delta(x)=ax-xa$ for any $x\in R$.

It is easy to see that

LEMMA 4.1. *Let $d, d'\in D_s(R)$. If $d_i=d'_i$ for $i=0, 1, \cdots, n<s$ then $d_{n+1}-d'_{n+1}$*

is a usual derivation.

Now we can prove

THEOREM 4.2. *If every classical derivation of R is inner then so is every derivation of order s of R.*

PROOF. Let $d \in D_s(R)$. We must construct an element $\boldsymbol{a} \in T$ such that $d = \Delta(\boldsymbol{a})$.

Since d_1 is a classical derivation then there exists $a_1 \in R$ such that $d_1(x) = a_1 x - x a_1$, for any $x \in R$. So we have $d_1 = [a_1, 1]$.

Let $d_2' = [a_1, 1]_2$. Then $(1_R, d_1, d_2')$ and $(1_R, d_1, d_2)$ are derivations of order 2 and hence, by Lemma 4.1, there exists $a_2 \in R$ such that $d_2(x) = d_2'(x) + a_2 x - x a_2$ for any $x \in R$. Therefore,

$$d_2 = d_2' + [a_2, 2]_2 = [a_1, 1]_2 + [a_2, 2]_2$$
$$= ([a_1, 1] * [a_2, 2])_2 .$$

Next let $d_3' = ([a_1, 1] * [a_2, 2])_3$ Since $(1_R, d_1, d_2, d_3')$, $(1_R, d_1, d_2, d_3)$ are derivations of order 3 then, by Lemma 4.1, $d_3(x) = d_3'(x) + a_3 x - x a_3$ for some $a_3 \in R$. So we have

$$d_3 = d_3' + [a_3, 3]_3$$
$$= ([a_1, 1] * [a_2, 2])_3 + [a_3, 3]_3$$
$$= ([a_1, 1] * [a_2, 2] * [a_3, 3])_3$$

and so on.

The assumption of the above theorem is satisfied for a large class of rings (see for example [3], [4], [2]).

V.

We end this paper with the following three remarks.

REMARK 5.1. Let $a \in R$. If $d = [a, 1]^{-1}$ then $d_n(x) = x a^n - a x a^{n-1}$, for $n \geqq 1$, $x \in R$.

REMARK 5.2. Let $a \in R$. Let $d = (d_n)_{n \in S}$ be the family of mappings from R to R defined by

$$d_0(x) = x$$
$$d_1(x) = a x - x a$$

$$d_n(x)=a^n x+x(-a)^n+2\sum_{k=1}^{n-1} a^{n-k}x(-a)^k, \qquad \text{for} \quad n\geqq 2.$$

Then $d\notin D_s(R)$ (in general) but $\delta=(2d_n)_{n\in S}$ is an inner derivation of order s of R. Namely, $\delta=[a, 1]*[-a, 1]^{-1}$.

REMARK 5.3. Let $d\in D_s(R)$. Suppose that there exists an element $a\in R$ such that $d_n=a^{n-1}d_1$ for any $n\in S\setminus\{0\}$. If the set $d_1(R)$ contains a regular element then $d=[a, 1]$.

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