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A SUBMANIFOLD WHICH CONTAINS MANY EXTRINSIC CIRCLES

By

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Dedicated to Professor Kentaro Yano in honor of
the Second Order of Merit

1. Introduction

There are many simple characterizations of a sphere in E^3, which are either elementary geometric or differential geometric. For example,
(E_1) It "looks round" from an arbitrary point.
or
(E_2) A section with an arbitrary plane is a circle.
gives an elementary geometric criterion for a surface to be a sphere. On the other hand,
or
(D_2) Every geodesic is a plane curve.
gives a differential geometric criterion for a compact surface to be a sphere.

A condition such as (D_2) is simple and logical but *not practical,* because it is not so easy for an observer in E^3 to know practically that a curve on a surface is a geodesic or not.

On the contrary, it is easy to know that a curve in E^3 is a circle or not and is contained in a surface or not.
Therefore we consider an elementary geometric condition such as

(*) A circle in E^3 of (arbitrarily) given radius can be pressed entirely on an arbitrary position of a surface.

It is easy to see that (*) *is a condition for a compact surface to be a sphere.* A condition such as (*) is *practical* in the sense that it is available in verifying the sphericity of a given physical solid. We emphasize that such a condition is quite natural because an observer is an inhabitant of an ambient space. But, (*) requires a very large quantity of information because of its condition "an arbitrary position".

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Therefore we are going to give in §4 practical criterion for a compact surface to be a sphere, which is better than (*). Moerover we will extend our situation to a general Riemannian submanifold and give characterizations of an *extrinsic sphere.*

2. Basic notions

Let M be an n-dimensional submanifold immersed in an m-dimensional Riemannian manifold $\tilde{M}$. The Riemannian connections of M and $\tilde{M}$ are denoted by ∇ and $\tilde{\nabla}$, respectively, whereas the normal connection is denoted by $\nabla^{\perp}$.
The second fundamental form σ is defined by

$$\sigma(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y,$$

where X and Y are vector fields tangent to M.
For a vector field ξ normal to M, the tensor field A_ξ of type (1,1) on M is defined by

$$\tilde{\nabla}_X \xi=-A_\xi X+\nabla_X^{\perp} \xi.$$

Then σ aud A_ξ are related by

$$<\sigma(X, Y), \xi>=<A_\xi X, Y>,$$

where $<,>$ denotes the inner product with respect to the respective Riemannian metrics.
The covariant derivative $\nabla'_X\sigma$ of σ is defined by

$$(\nabla'_X \sigma)(Y, Z)=\nabla_X^{\perp}\cdot\sigma(Y, Z)-\sigma(\nabla_X Y, Z)-\sigma(Y, \nabla_X Z).$$

The *mean curvature vector field* $\mathfrak{h}$ is defined by

$$\mathfrak{h}=\frac{1}{n}\,\text{trace}\ \sigma.$$

We say that $\mathfrak{h}$ is *parallel* if $\nabla_X^{\perp}\mathfrak{h}=0$ for all X tangent to M.
We say that M is *totally umbilic* if

$$\sigma(X, Y)=<X, Y>\mathfrak{h}$$

for all X and Y. Equivalently, M is totally umbilic if

$$A_\xi=<\xi, \mathfrak{h}>I$$

for all ξ, where I denotes the identity transformation. It is known that if $\tilde{M}$ is a space form (i.e., a Riemannian manifold of constant curvature), then a totally umbilic submanifold M of $\tilde{M}$ has parallel mean curvature vector. A submanifold M of an arbitrary Riemannian manifold $\tilde{M}$ is called an *extrinsic sphere* if it is totally umbilic and has non-zero parallel mean curvature vector.

A regular curve $\gamma=\gamma(s)$ on $\tilde{M}$ parametrized by arc length s is called a *circle* if there exist a field $Y=Y(s)$ of unit vectors along γ and a positive constant k such that

$$\begin{cases}\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}=kY\\ \tilde{\nabla}_{\dot{\gamma}}Y=-k\dot{\gamma},\end{cases}$$

where $\dot{\gamma}$ denotes the unit tangent vector of γ.
The number k (resp. $1/k$) is called the *curvature* (resp. *radius*) of γ. It is easily seen that a circle $\gamma=\gamma(s)$ of curvature k in $\tilde{M}$ satisfies

$$\tilde{\nabla}_{\dot{\gamma}}\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}+k^2\dot{\gamma}=0.$$

3. A circle contained in a submanifold

Let M be a submanifold of $\tilde{M}$. A curve $\gamma=\gamma(s)$ in $\tilde{M}$ is a circle of curvature k if it satisfies

$$\tilde{\nabla}_{\dot{\gamma}}\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}+k^2\dot{\gamma}=0.$$

Using the equation of Gauss $\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}-\nabla_{\dot{\gamma}}\dot{\gamma}=\sigma(\dot{\gamma},\dot{\gamma})$, we easily obtain the following result, which is useful throughout this paper.

LEMMA. *Let M be a submanifold of $\tilde{M}$. Then a curve a curve $\gamma=\gamma(s)$ in M is a circle of curvature k in $\tilde{M}$ if and only if it satisfies*

(3.1) $$\begin{cases}\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\dot{\gamma}+k^2\dot{\gamma}-A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma}=0\\ (\nabla'_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma})+3\sigma(\dot{\gamma},\nabla_{\dot{\gamma}}\dot{\gamma})=0.\end{cases}$$

4. A practical characterization of an umbilical surface in E^3

We give an elementary geometric criterion for umbilicity, which is much better than (*) in §1. The following criterion requires the existence of two circles through each point, whereas (*) requires the existence of infinitely many circles through each point.

THEOREM 1. *Let M be a surface in E^3. Suppose that, through each point $p\in M$, there exist two circles of E^3 such that*
(i) *they are contained in M in a neighborhood of* p,
(ii) *they are tangent to each other at* p.
Then M is locally a plane or a phere.

PROOF. Let p be an arbitrary point of M and let γ_1 and γ_2 be two circles through p which satisfy the conditions (i) and (ii).
Let X_p be the common unit tangent vector of γ_1 and γ_2 at p. Then $X: p\longrightarrow X_p$

defines a vector field on M, which may not be continuous. Let Y be a unit vector field on M which, together with X, forms a right-handed orthonormal system. Then, since $\dim M=2$, it follows from Lemma that there exist $c_1\neq c_2$ such that

$$\begin{cases}(\nabla'_X\sigma)(X, X)+3\sigma(X, c_1Y)=0\\(\nabla'_X\sigma)(X, X)+3\sigma(X, c_2Y)=0\end{cases} \tag{4.1}$$

at each point. Therefore we have

$$\sigma(X, Y)=0, \tag{4.2}$$

that is,

$$<AX, Y>=0.$$

Since $\dim M=2$, we see that AX is parallel to X. Thus X is (and hence Y is also) a principal vector at each point. Let λ and μ be principal curvatures and ξ_1 and ξ_2 the corresponding principal unit vectors so that $A\xi_1=\lambda\xi_1$ and $A\xi_2=\mu\xi_2$. Put $M_0=\{p\in M|\lambda(p)\neq\mu(p)\}$. If $M_0=\emptyset$, then M is totally umbilic. Therefore we suppose that $M_0\neq\emptyset$. Then ξ_1 and ξ_2 are C^∞ vector fields in some neighborhood of each point of M_0. Hence we may put

$$\nabla_{\xi_1}\xi_1=\alpha\xi_2 \text{ and } \nabla_{\xi_2}\xi_1=\beta\xi_2$$

so that

$$\nabla_{\xi_1}\xi_2=-\alpha\xi_1 \text{ and } \nabla_{\xi_2}\xi_2=-\beta\xi_1.$$

Put $M_{0i}=\{p\in M_0|X(p)=\xi_i(p)\}$ $(i=1,2)$. Then $M_0=M_{01}\cup M_{02}$, and it is easily seen that $M_0\subset\bar{M}_{01}\cup \operatorname{Int} M_{02}$ and $M_0\subset\bar{M}_{02}\cup \operatorname{Int} M_{01}$ and hence that M_{01} or M_{02} has interior points, or M_{01} or M_{02} is dense in M_0. We may assume without loss of generality that M_{01} has interior points or M_{01} is dense in M_0. Hence it is sufficient to consider the case where $p\in M_{01}$.

Using (4.2) we obtain

$$\begin{aligned}(\nabla'_{\xi_1}\sigma)(\xi_1, \xi_1)&=\nabla^\perp_{\xi_1}\cdot\sigma(\xi_1, \xi_1)-2\sigma(\xi_1, \nabla_{\xi_1}\xi_1)\\&=\nabla^\perp_{\xi_1}(\lambda e_3)-\sigma(\xi_1, a\xi_2)\\&=(\nabla_{\zeta_1}\lambda)e_3,\end{aligned}$$

where e_3 is a local field of unit normals of M.
On the other hand, it follows from (4.1) and (4.2) that

$$(\nabla'_X\sigma)(X, X)=0. \tag{4.3}$$

Therefore we have $\nabla_X\lambda=0$ on M_{01}. If p is an interior point of M_{01}, then

$$\nabla_{\xi_1}\lambda=0 \tag{4.4}$$

holds in some neighborhood of p. If $\bar{M}_{01}$ is dense in dense in M_0, then, by continuity, (4.4) holds on $\bar{M}_0$.

We choose an orthonormal frame field e_1, e_2 in a sufficiently small tubular neighborhood of γ_1 in such a way that

$$e_1=\dot{\gamma}_1 \ along \ \gamma_1,$$

and put

$$\nabla_{e_1} e_1=ae_2 \quad \text{and} \quad \nabla_{e_2} e_1=be_2$$

so that $\nabla_{e_1} e_2=-ae_1$ and $\nabla_{e_2} e_2=-be_1$.
Let (h_{ij}) be the matrix of A with respect to e_1 and e_2. Then it follows from Lemma that, along γ_1,

$$k^2=a^2+h_{11}^2 \tag{4.5}$$

$$\nabla_{e_1}a=h_{11}h_{12} \tag{4.6}$$

$$ah_{12}+\nabla_{e_1}h_{11}=0, \tag{4.7}$$

where k is the curvature of γ_1 as a circle in E^3.
Letting θ be the angle between ξ_1 and e_1 so that

$$\begin{cases} e_1=\xi_1 \cos\theta+\xi_2 \sin\theta \\ e_2=-\xi_1 \sin\theta+\xi_2 \cos\theta, \end{cases} \tag{4.8}$$

we get

$$\begin{cases} h_{11}=\lambda \cos^2\theta+\mu \sin^2\theta \\ h_{12}=-(\lambda-\mu)\cos\theta \sin\theta \\ h_{22}=\lambda \sin^2\theta+\mu \cos^2\theta \end{cases} \tag{4.9}$$

By differentiating (4.8), we obtain

$$a=\alpha \cos\theta+\beta \sin\theta+\nabla_{e_1}\theta, \tag{4.10}$$

which is nothing but the transformation lar for Christoffel's symbols.
Moreover, by the equation of Codazzi $(\nabla_{\xi_1} A)\xi_2-(\nabla_{\xi_2} A)\xi_1=0$, we get

$$\begin{aligned} \alpha(\lambda-\mu)&=\nabla_{\xi_2} \lambda \\ \beta(\lambda-\mu)&=\nabla_{\xi_1} \mu \end{aligned} \tag{4.11}$$

Therefore, by (4.4), (4.7), (4.8), (4.9), (4.10) and (4.11) we obtain

$$[3(\lambda-\mu)\cos\theta \cdot \nabla_{e_1}\theta-\sin^2\theta \cdot \nabla_{\xi_2} \mu] \sin\theta=0. \tag{4.12}$$

Note that $\theta=0$ at p. The point p under consideration has one of the following properties:
(A) There exists no sequence $\{p_n \in \gamma_1 | \theta(p_n) \neq 0\}$ with $p=\lim p_n$
(B) There exists a sequence $\{p_n \in \gamma_1 | \theta(p_n) \neq 0\}$ with $p=\lim p_n$.
If p is a point of kind (A), then it is clear that the integral curve of ξ_1 through

p coincides with γ_1 on the connected component of $\{q\in\gamma_1 | \theta(q)=0\}$ containing p.

If p is a point of kind (B), it follows from (4.12) that

$$3(\lambda-\mu)\cos\theta\cdot\nabla_{e_1}\theta-\sin^2\theta\cdot\nabla_{\xi_2}\mu=0 \tag{4.13}$$

holds on $\{p_n\in\gamma_1 | \theta(p_n)\neq 0\}$.
Taking the limit of (4.13), we obtain

$$\nabla_{e_1}\theta=0 \quad at \quad p, \tag{4.14}$$

since $\lambda\neq\mu$.

After applying ∇_{e_1} to (4.12) and then multiplying by $\sin\theta$, we evaluate it at p to obtain

$$\nabla_{e_1}\nabla_{e_1}\theta=0 \quad at \quad p. \tag{4.15}$$

It is clear that (4.14) and (4.15) hold even if p is a point of kind (A).

It follows from (4.10) and (4.14) that

$$\alpha(p)=a(p).$$

This, together with (4.5) and (4.9), yields

$$k^2=(\alpha(p))^2+(\lambda(p))^2, \tag{4.11}$$

which implies that *the curvature of* γ_1 *is* $\sqrt{(\alpha(p))^2+(\lambda(p))^2}$.

Applying ∇_{e_1} to (4.10) and evaluating at p, we obtain

$$\nabla_{\xi_1}\alpha=\nabla_{\xi_1}a \quad at \quad p$$

because of (4.14) and (4.15).
On the other hand, from (4.6) we get

$$\nabla_{\xi_1}a=\nabla_{e_1}a=h_{11}h_{12}=0 \quad at \quad p$$

Therefore we have

$$\nabla_{\xi_1}\alpha=0 \quad at \quad p.$$

Since p is arbitrary, we get $\nabla_{\xi_1}\alpha=0$ on M_{01}.
If p is an interior point of M_{01}, then

$$\nabla_{\xi_1}\alpha=0 \tag{4.17}$$

holds in some neighborhood of p. If M_{01} is dense in M_0, then, by continuity, (4.17) holds on M_0.
Eliminating ξ_2 from $\nabla_{\xi_1}=\alpha\xi_2$ and $\nabla_{\xi_1}\xi_2=-\alpha\xi_1$, we obtain

$$\nabla_{\xi_1}\nabla_{\xi_1}\xi_1+\alpha^2\xi_1=0.$$

Moreover, since ξ_1 is a principal vector, we get

$$A_{\sigma(\xi_1,\xi_1)}=\lambda^2\xi_1.$$

Therefore we have

(4.18) $$\nabla_{\xi_1}\nabla_{\xi_1}\xi_1+(\alpha^2+\lambda^2)\xi_1-A_{\sigma(\xi_1,\xi_1)}\xi_1=0.$$

Furthermore, since $\nabla_{\xi_1}\xi_1=\alpha\xi_2$, it follows from (4.2) and (4.3) that $(\nabla'_{\xi_1}\sigma)(\xi_1,\xi_1)+3\sigma(\xi_1,\nabla_{\xi_1}\xi_1)=0$ on M_{01}.

(4.19) $$(\nabla'_{\xi_1}\sigma)(\xi_1,\xi_1)+3\sigma(\xi_1,\nabla_{\xi_1}\xi_1)=0$$

holds in some neighborhood of p. If M_{01} is dense in M_0, then, by continuity, (4.19) holds on M_0.

By (4.18), (4.19) and Lemma, we see that *the integral curve of ξ_1 through p is a circle of curvature $\sqrt{(\alpha(p))^2+(\lambda(p))^2}$ in E^3.*

Since we can apply the same argument to γ_2, letting θ_i $(i=1,2)$ be the angle between ξ_1 and γ_i, we consider the following cases:

$(A)_1$: There exists no sequence $\{p_n\in\gamma_i|\theta_i(p_n)\neq 0\}$ with $p=\lim p_n$

$(B)_1$: There exists a sequence $\{p_n\in\gamma_i|\theta_i(p^u)\neq 0\}$ with $p=\lim p_n$.

It is clear that the case $(A)_1$ and $(A)_2$ does not occur. If $(A)_1$ and $(B)_2$, then γ_1 and γ_2 have the same curvature and the integral curve of ξ_1 through p coincides with γ_1. If $(B)_1$ and $(A)_2$, then γ_1 and γ_2 have the same curvature and the integral curve of ξ_1 through p coincides with γ_2. If $(B)_1$ and $(B)_2$, then γ_1,γ_2 and the integral curve of ξ_1 through p have the same curvature and hence, the integral curve of ξ_1 through p coincides with γ_1 or γ_2, since $\dim M=2$. This is a contradiction so that the last case does not occur.

Since we can apply the same argument to the possible two cases, we suppose that the $(B)_1$ and $(A)_2$ is the case, that is, the integral curve of ξ_1 through p coincides with γ_2. Then, from

$$\begin{aligned}<\tilde{\nabla}_{\xi_1}\xi_1,\tilde{\nabla}_{e_1}e_1> &= <\nabla_{\xi_1}\xi_1+\sigma(\xi_1,\xi_1),\nabla_{e_1}e_1+\sigma(e_1,e_1)> \\ &= <\alpha\xi_2+\lambda e_3, ae_2+h_{11}e_3>,\end{aligned}$$

we get

$$<\tilde{\nabla}_{\xi_1}\xi_1,\tilde{\nabla}_{e_1}e_1>\lambda^2+\alpha^2 \quad at \quad p,$$

since $\alpha=a$, $\lambda=h_{11}$ and $\xi_2=e_2$ at p.

Let $\widehat{\gamma_1\gamma_2}$ denote the angle between γ_1-plane and γ_2-plane. Then we have

$$\cos\widehat{\gamma_1\gamma_2}=\frac{<\tilde{\nabla}_{\xi_1}\xi_1,\tilde{\nabla}_{e_1}e_1>}{\|\tilde{\nabla}_{\xi_1}\xi_1\|\ \|\tilde{\nabla}_{e_1}e_1\|}=1.$$

Thus γ_1-plane and γ_2-plane coincide along γ_2. This implies that the set of γ_1's along γ_2 near p forms a part of a plane, which contradicts the non-umbilicity of M_0.

Therefore $M_0=\emptyset$ so that M is totally umbilic. (Q.E.D.)

REMARK. The assumption (i) in Theorem 1 is not sufficient for a surface to be totally umbilic, because an ordinary torus and a suitable part of an ellipsoid or an elliptic paraboloid satisfy the assumption (i). Moreover, it is not difficult to see that a torus actually contains *four* circles of E^3 through each point ([1]). On the contrary, it is easily seen that a surface is a plane if it contains three straight lines through each point (cf. §7). It seems to be natural to conjecture that *a simply connected complete surface in E^3 which contains two circles through each point must be totally umbilic.* This conjecture is hairbreadth, because an ellipsoid and an elliptic paraboloid actually satisfy the assumption except at only four and two points respectively.

5. An extrinsic sphere in a general Riemannian manifold

Let M be an n-dimensional submanifold in $\tilde{M}$. If $n=2$, then $(\tilde{\nabla}_{\dot{\gamma}_1}\dot{\gamma}_1)_p$ and $(\tilde{\nabla}_{\dot{\gamma}_2}\dot{\gamma}_2)_p$ are linearly dependent for two circles γ_1 and γ_2 in $\tilde{M}$ which are tangent to each other at $p\in M$ and contained in M. On the contrary, the situation is not so simple if $n>2$. Therefore we introduce the following notion.

Circles $\gamma_1, \cdots, \gamma_k$ in $\tilde{M}$ which are tangent to one another at $p\in M$ and contained in M in a neighborhood of p are said to be *r-independent* if $\dim\{\{(\tilde{\nabla}_{\dot{\gamma}_1}\dot{\gamma}_1)_p, \cdots, (\tilde{\nabla}_{\dot{\gamma}_k}\dot{\gamma}_k)_p\}\}=r$ and there exist $c_1, \cdots c_k$ with $\sum c_i\neq 0$ such that $\sum c_i(\tilde{\nabla}_{\dot{\gamma}_i}\dot{\gamma}_i)_p=0$, where $\{\{\cdots\}\}$ denotes the linear space spanned by $\cdots$.

We shall give an elementary geometric characterization of an extrinsic sphere, that is, we shall prove the following.

THEOREM 2. *Let M be an n-dimensional submanifold immersed in a Riemannian manifold $\tilde{M}$. Then M is either a totally geodesic submanifold or an extrinsic sphere if, through each point $p\in M$, there exist n^2 circles of $\tilde{M}$ such that*

(i) *they are contained in M in a neighborhood of p,*

(ii) *they are mutually tangent n by n and respective n circles are $(n-1)$-independent,*

(iii) *none of them are orthogonal to each other at p,*

(iv) *the set of all tangent vectors to them at p spans $T_p(M)$.*

PROOF Let $\gamma_{ij}(1\leqq i, j\leqq n)$ be n^2 circles satisfying the assumption and suppose that $\gamma_{i1}, \cdots, \gamma_{in}$ are tangent to one another at p. We denote by X_i the common unit tangent vector of $\gamma_{i1}, \cdots, \gamma_{in}$ at p. Then by (i) and Lemma we have

$$(5.1)\qquad \begin{cases}(\nabla'_{X_i}\sigma)(X_i, X_i)+3\sigma(X_i, (\tilde{\nabla}_{\dot{\gamma}_{in}}\dot{\gamma}_{in})_p)=0\\ \cdots\cdots\cdots\cdots\cdots\\ (\nabla'_{X_i}\sigma)(X_i, X_i)+3\sigma(X_i, (\tilde{\nabla}_{\dot{\gamma}_{in}}\dot{\gamma}_{i1})_p)=0\end{cases}$$

for $1 \leqq i \leqq n$.

On the other hand, since $\gamma_{i1}, \cdots, \gamma_{in}$ are $(n-1)$-independent by (ii), we see that $\{\{X_i\}\}^{\perp} = \{\{(\nabla_{\dot\gamma_{in}} \dot\gamma_{i1})_p, \cdots, (\nabla_{\dot\gamma_{in}} \dot\gamma_{in})_p\}\}$. Thus, from (5.1) we obtain

(5.2) $$(\nabla'_{X_i} \sigma)(X_i, X_i) = 0$$

and

(5.3) $$\sigma(X_i, \{\{X_i\}\}^{\perp}) = 0.$$

It follows from (5.3) that

$$< A_\xi X_i, \{\{X_i\}\}^{\perp} > = 0$$

for an arbitrary normal vector ξ at p.
This implies that X_i is a principal vector with respect to ξ.
Therefore we see from (iii) and (iv) that A_ξ is proportional to the identity transformation for an arbitrary normal vector ξ at p, which means that p is an umbilic point.
Since p is arbitrary, M is totally umbilic.

Moreover, (5.2), together with the umbilicity of M, allows the identity

$$\nabla^{\perp}\dot\gamma_{ij} \cdot \sigma(\dot\gamma_{ij}, \dot\gamma_{ij}) = (\nabla'\dot\gamma_{ij}\sigma)(\dot\gamma_{ij}, \dot\gamma_{ij}) + 2\sigma(\dot\gamma_{ij}, \nabla_{\dot\gamma_{ij}} \dot\gamma_{ij})$$

to boil down to

$$\nabla^{\perp}_{X_i} \mathfrak{h} = 0.$$

Thus we get $\nabla^{\perp}\mathfrak{h} = 0$ at p by (iv), and hence we have

$$\nabla^{\perp}\mathfrak{h} = 0 \quad on \quad M,$$

since p is arbitrary.

Thus M is totally umbilic with parallel mean curvature vector. More precisely, M is a totally geodesic submanifold or an extrinsic sphere according as $\mathfrak{h} = 0$ or $\mathfrak{h} \neq 0$. (Q.E.D.)

The following result gives another characterization of an extrinsic sphere.

THEOREM 3. *Let M be an n-dimensional submanifold immersed in a Riemannian manifold $\tilde M$. Then M is either a totally geodesic submanifold or an extrinsic sphere if, at each point p of M, there exist an orthonormal basis $e_1, \cdots, e_n$ of $T_p(M)$ and real numbers $\alpha_2, \cdots, \alpha_n$ $(0 < \alpha_j < \pi/2)$ with the following properties: For each pair $(X, Y) = (e_i, e_j)$ and $(X, Y) = (e_1 \cos\alpha_j + e_j \sin\alpha_j, e_1 \sin\alpha_j - e_j \cos\alpha_j)$, $1 \leqq i < j \leqq n$, there exist two circles γ_1 and γ_2 of $\tilde M$ such that*

(i) $\gamma_1(0) = \gamma_2(0) = p$

(ii) $\dot{\gamma}_1(0)=\dot{\gamma}_2(0)=X$

(iii) $(\nabla_{\dot{\gamma}_1}\dot{\gamma}_1)_p=c_1Y$ *and* $(\nabla_{\dot{\gamma}_2}\dot{\gamma}_2)_p=c_2Y$ *for some* $c_1\neq c_2$ (i.e., γ_1 *and* γ_2 *are* 1*-independent*)

(iv) γ_1 *and* γ_2 *are contained in* M *in a neighborhood of* p.

PROOF By (iv) and Lemma, we have

$$\begin{cases}(\nabla'_X\sigma)(X, X)+3\sigma(X, c_1Y)=0\\(\nabla'_X\sigma)(X, X)+3\sigma(X, c_2Y)=0.\end{cases} \tag{5.4}$$

Therefore

$$\sigma(X, Y)=0 \tag{5.5}$$

holds for all pairs $(X, Y)=(e_i, e_j)$ and $(e_1\cos\alpha_j+e_j\sin\alpha_j, e_1\sin\alpha_j-e_j\cos\alpha_j)$, $1\leqq i<j\leqq n$. Thus we obtain

$$\sigma(e_i, e_j)=0 \quad (1\leqq i<j\leqq n)$$
$$\sigma(e_1, e_1)=\cdots=\sigma(e_n, e_n),$$

which implies that p is an umbilic point.
Since p is arbitrary, M is totally umbilic.

Moreover, by (5.4) and (5.5) we get

$$(\nabla'_X\sigma)(X, X)=0$$

for $X=e_1, \cdots, e_{n-1}$ and $e_1\cos\alpha_n+e_n\sin\alpha_n$.
Applying the same argument as in the proof of Theorem 2, we obtain

$$\nabla^{\perp}\mathfrak{h}=0$$

so that we can complete the proof.

COROLLARY 1. *Let* M *be an* n*-dimensional submanifold of* E^m. *If* M *satisfies the assumption of Theorem* 2 *or Theorem* 3, *then* M *is locally* E^n *or* S^n.

COROLLARY 2. *Let* M *be a surface in* E^m. *Suppose that, through each point of* M, *there exist four circles of* E^m *such that*

(a) *they are contained in* M *in a neighborhood of* p,

(b) *they are tangent two by two at* p,

(c) *none of them are orthogonal to each other at* p.

Then M *is locally a plane or a sphere.*

6. A submanifold with parallel second fundamental form

We shall give an elementary geometric characterization for a submanifold with parallel second fundamental form. We first prove the following general result.

PROPOSITION. *Let M be a submanifold of $\tilde{M}$. Suppose, through each point of M and in each direction tangent to M, there exist r (which may depend on the point and the direction) circles of $\tilde{M}$ such that*

(a) *they are contained in M in a neighborhood of the point,*

(b) *they are $(r-1)$-independent.*

Then $(\nabla'_X\sigma)(X, X)=0$ holds for all X tangent to M.

PROOF Let p be an arbitrary point of M and $X\in T_p(M)$ be an arbitrary unit vector. Then, by assumption, there exist r circles $\gamma_1, \cdots, \gamma_r$ of M which satisfy $\gamma_i(0)=p, \dot{\gamma}_i(0)=X$ and (a) and (b). Thus, by Lemma, we have

$$\begin{cases}(\nabla'_X\sigma)(X, X)+3\sigma(X, (\nabla_{\dot{\gamma}_1}\dot{\gamma}_1)_p=0 \\ \cdots\cdots\cdots\cdots\cdots \\ (\nabla'_X\sigma)(X, X)+3\sigma(X, (\nabla_{\dot{\gamma}_r}\dot{\gamma}_r)_p)=0.\end{cases}$$

Therefore, by (b) we get $(\nabla'_X\sigma)(X, X)=0$. (Q.E.D.)

It is easily seen that if the equation of Codazzi reduces to $(\nabla'_X\sigma)(Y, Z)-(\nabla'_Y\sigma)(X, Z)=0$, then we can deduce $\nabla'\sigma=0$ from $(\nabla'_X\sigma)(X, X)=0$. For example, we have the following.

COROLLARY. *Let M be a submanifold of a space form. If M satisfies the assumption of Proposition, then the second fundamental form of M is parallel.*

It is clear that the same result holds for a Kaehler or totally real submanifold of a complex space form (i.e., a Kaehler manifold of constant holomorphic curvature).

7. A submanifold which contains many extrinsic geodesics

An extrinsic circle is not necessarily an intrinsic circle, whereas an extrinsic geodesic is an intrinsic geodesic. Therefore it is natural to expect that a submanifold with extrinsic geodesics is simpler than a submanifold with extrinsic circles. The following result gives a sufficient condition for a submanifold to be totally geodesic.

THEOREM 4. *Let M be an n-dimensional submanifold immersed in a Riemannian manifold $\tilde{M}$. Then a point $p\in M$ is a geodesic point if there exist $n(n+1)/2$ geodesics $\gamma_{ij}(1\leqq i\leqq j\leqq n)$ of $\tilde{M}$ such that*

(i) $\gamma_{ij}(0)=p$

(ii) $T_p(M)=\{\{\dot{\gamma}_{11}(0), \cdots, \dot{\gamma}_{nn}(0)\}\}$

(iii) $\dot{\gamma}_{ij}(0)\in\{\{\dot{\gamma}_{ii}(0), \dot{\gamma}_{jj}(0)\}\}$

(iv) γ_{ij} *are contained in* M *in a neighborhood of* p

PROOF By (iv) and the equation of Gauss, we have

$$\sigma(\dot{\gamma}_{ij}, \dot{\gamma}_{ij})=\tilde{\nabla}_{\dot{\gamma}_{ij}}\dot{\gamma}_{ij}-\nabla_{\dot{\gamma}_{ij}}\dot{\gamma}_{ij}=0 \ (1\leqq i\leqq j\leqq n) \text{ in a neighborhood of } p.$$

Therefore it follows from (iii) that

$$\sigma(\dot{\gamma}_{ii}(0), \dot{\gamma}_{ii}(0))=0 \ (1\leqq i\leqq n)$$
$$\sigma(a_i\dot{\gamma}_{ii}(0)+b_j\dot{\gamma}_{jj}(0), a_i\dot{\gamma}_{ii}(0)+b_j\dot{\gamma}_{jj}(0))=0 \ (1\leqq i<j\leqq n)$$

for some a_i and b_j.
Hence we have

$$\sigma(\dot{\gamma}_{ii}(0), \dot{\gamma}_{jj}(0))=0 \ (1\leqq i\leqq j\leqq n),$$

which, together with (ii), implies $\sigma=0$ at p.

COROLLARY. *Let* M *be a surface in a Riemannian manifold* $\tilde{M}$. *If, through each point* p *of* M, *there exist three geodesics of* $\tilde{M}$ *which are contained in* M *in a neighborhood of then* M *is totally geodesic.*

REMARK. The condition (iii) is automatically satisfied when $\dim M=2$, from which Corollary follows.
On the contrary, in the case $\dim M>2$, p is not necessarily a geodesic point even if there exist infinitely many geodesics of $\tilde{M}$ which satisfy (i), (ii) and (iv).

References

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