TSUKUBA J. MATH.
No. 7 No. 2 (1983), 265—279

ON A PROBLEM OF MAHLER FOR TRANSCENDENCY OF FUNCTION VALUES II

By

Kumiko NISHIOKA

1. Introduction.

In what follows, Ω is a $n\times n$ matrix whose entries are non-negative integers, and Ω satisfies:

(0) The characteristic polynomial of Ω is irreducible over $\boldsymbol{Q}$, the field of rational numbers, and Ω has the eigenvalues $\rho_1, \rho_2, \cdots, \rho_n$ such that $\rho_1>1$ and $\rho_1>|\rho_2|\geqq\cdots\geqq|\rho_n|$.

Let (A_{ij}) be the classical adjoint (the transpose of the matrix of cofactors) of matrix $\Omega-\rho_1 E$, where E is the $n\times n$ identity matrix. For a non-negative integer k, we put $\Omega^k=(\omega_{ij}^{(k)})$, and for a n-tuple of independent variables $z=(z_1, \cdots, z_n)$, we define

$$T^k z=(z_1^{(k)}, \cdots, z_n^{(k)}),\ z_i^{(k)}=\prod_{j=1}^n z_j^{\omega_{ij}^{(k)}}.$$

Let F be a finite algebraic number field and $f(z)=\sum_{h_i\geqq 0} a_{h_1\cdots h_n} z_1^{h_1}\cdots z_n^{h_n}$ be a power series with coefficients in F. By $\bar{\boldsymbol{Q}}$ we denote the algebraic closure of $\boldsymbol{Q}$ in $\boldsymbol{C}$, the field of complex numbers. Mahler [4] proved:

THEOREM (Mahler). *Let $f(z)$ be not algebraic over $\bar{\boldsymbol{Q}}(z_1, \cdots, z_n)$ and satisfy the functional equation*

$$f(Tz)=\sum_{i=0}^m a_i(z)f(z)^i/\sum_{i=0}^m b_i(z)f(z)^i,$$

where the coefficients $a_i(z)$ and $b_i(z)$ are polynomials with algebraic coefficients and $m<\rho_1$. $\Delta(z)$ denotes the resultant of $\sum_{i=0}^m a_i(z)u^i$ and $\sum_{i=0}^m b_i(z)u^i$ as polynomials in u. If $\alpha=(\alpha_1, \cdots, \alpha_n)\in\bar{\boldsymbol{Q}}^n$ satisfies that $\alpha_1\cdots\alpha_n\neq 0$, the real part of $\sum_{j=1}^n |A_{1j}|\log\alpha_j$ is negative, $f(z)$ converges at $z=\alpha$ and $\Delta(T^k\alpha)\neq 0$ for all $k\geqq 0$, then $f(\alpha)$ is transcendental.

For example, $f(z)=\sum_{h=0}^\infty z^{2^h}$ satisfies the functional equation $f(z^2)=f(z)-z$. Then for an algebraic number such that $0<|\alpha|<1$, $f(\alpha)$ is transcendental. Refer to Loxton and van der Poorten [2], [3] for other examples. Mahler [5], [6]

Received February 7, 1983

treated matrices of the form ρE and algebraic independency of values of several functions satisfying a certain type of functional equation. In [7], Mahler gave a summary of his earlier work and proposed three problems connected with it. Two of the three problems have been studied by Kubota, Loxton, van der Poorten and Masser. The present investigation is concerned with the remaining problem:

RROBLEM. Assume that $f(z)$ satisfies an algebraic functional equation of the form

$$P(z, f(z), f(Tz))=0,$$

where $P(z, u, v)\neq 0$ is a polynomial in $u, v, z_1, \cdots, z_n$ with algebraic coefficients. To investigate the transcendency of function values $f(\alpha)$ where α is an algebraic point satisfying suitable further restrictions.

Our earlier paper [9] considered this problem in the case $n=1$. Now we consider the general case $n\geqq 1$, and treat more generalized power series and transformations. In [9], the coefficients of power series must satisfy some conditions but in this paper we shall show that the conditions are deduced from the functional equation.

2. Preliminaries, theorems and lemmas.

As usual, if α is an algebraic number, we denote by $\overline{|\alpha|}$ the maximum of absolute values of the conjugates of α and by $d(\alpha)$ the least positive integer such that $d(\alpha)\alpha$ is an algebraic integer, and we set $\mathrm{size}(\alpha)=\max\{\log\overline{|\alpha|}, \log d(\alpha)\}$. Assume that Ω satisfies (0), t is a positive integer, and $\rho_1/t>1$. For a n-tuple of independent variables $z=(z_1, \cdots, z_n)$ and a non-negative integer k, we put

$$T^k z=(z_1^{(k)}, \cdots, z_n^{(k)}), \quad z_i^{(k)}=\prod_{j=1}^n z_j^{\omega_{ij}^{(k)}/t^k}.$$

Let $f(z)=\sum a_{h_1\cdots h_n} z_1^{h_1}\cdots z_n^{h_n}$ be a formal power series with powers being non-negative rational numbers and coefficients in F. We may assume $a_{0\cdots 0}=0$ without loss of generality. We consider the following four properties on $f(z)$.

(1) There are constants $c_1>0$ and $0\leqq\eta<1/n$ such that for any $h>0$ there exists a positive integer $\delta_h\leqq c_1 h^{\eta}$ with $\delta_h h_i\in \boldsymbol{Z}$ $(1\leqq i\leqq n)$ if $h_i\leqq h$ $(1\leqq i\leqq n)$ and $a_{h_1\cdots h_n}\neq 0$.

By the property (1), for a non-negative number h, the cardinality of terms $a_{h_1\cdots h_n} z_1^{h_1}\cdots z_n^{h_n}$ with $a_{h_1\cdots h_n}\neq 0$ and $h_i\leqq h$ $(1\leqq i\leqq n)$ is not greater than $(c_1 h^{1+\eta}+1)^n$.

(2) $f(z)$ is not algebraic over $\overline{\boldsymbol{Q}}(z_1, \cdots, z_n)$.

(3) $f(z)$ satisfies an algebraic functional equation of the form:

(2.1) $$Q_0(z, f(z))f(Tz)^l + Q_1(z, f(z))f(Tz)^{l-1} + \cdots + Q_l(z, f(z)) = 0,$$

where $Q_i(z, u) \in \overline{\boldsymbol{Q}}[z_1, \cdots, z_n, u]$ and $Q_0(z, f(z)) \neq 0$.

Since we may assume that $Q_0(z, u), \cdots, Q_l(z, u)$ have no common divisor as polynomials in u, there are elements $g_0(z, u), \cdots, g_l(z, u)$ of $\overline{\boldsymbol{Q}}[z_1, \cdots, z_n, u]$ such that

$$g(z) \text{ (say)} = \sum_{i=0}^{l} g_i(z, u) Q_i(z, u)$$

is independent of u and not zero. We set

$$m = \max_{0 \leqq i \leqq l} \deg_u Q_i(z, u).$$

(4) If d_h is the least positive integer such that $d_h a_{h_1 \cdots h_n}$ is an algebraic integer for all $(h_1, \cdots, h_n)$ with $h_i \leqq h$ $(1 \leqq i \leqq n)$, then there are constants c_2 and $L \geqq 1$ such that

$$\log \left|\overline{a_{h_1 \cdots h_n}}\right| \leqq c_2 (\max\{h_1, \cdots, h_n\})^L, \quad \log d_h \leqq c_2 h^L.$$

Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \boldsymbol{C}^n$, $\alpha_1 \cdots \alpha_n \neq 0$, and fix a branch of $\log \alpha_i$ $(1 \leqq i \leqq n)$. For a non-negative integer k, we put

$$\log \alpha_i^{(k)} = \sum_{j=1}^{n} (o_{ij}^{(k)} / t^k) \log \alpha_j.$$

For a power series $f(z)$ with the property (1), we define

$$f(T^k \alpha) = \sum a_{h_1 \cdots h_n} e^{h_1 \log \alpha_1^{(k)} + \cdots + h_n \log \alpha_n^{(k)}},$$

if it absolutely converges.

THEOREM 1. *Let $f(z)$ have the properties (1)~(4). Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \overline{\boldsymbol{Q}}^n$, $\alpha_1 \cdots \alpha_n \neq 0$, and suppose the real part of $\sum_{j=1}^{n} |A_{1j}| \log \alpha_j$ is negative. Assume that $f(T^k \alpha)$ is defined and $g(T^k \alpha) \neq 0$ for any non-negative integer k. By n_0, we denote the rank of the multiplicative group generated by $\alpha_1, \cdots, \alpha_n$. If*

(2.2) $$(\rho_1/t) \times \min\{(\rho_1/t)^{(1-n\eta)/(L+n(1+\eta)-1)}, (\rho_1/|\rho_2|)^{(1-n\eta)/n(1+\eta)}\}$$
$$> (t^{n_0} l)^{n+1} \times \max\{\rho_1/t, m\},$$

then $f(\alpha)$ is transcendental.

THEOREM 2. *If $f(z)$ satisfies (1) and (3), then $f(z)$ satisfies (4) with $L = \max\{1 + 2\eta + \varepsilon, (2 + 3\eta)(n-1)\}$, where ε is any positive number.*

In the previous paper [9], we considered the transformation $Tz = t_p z^p + \cdots + t_{p+N} z^{p+N}$, where $t_p, \cdots, t_{p+N}$ are algebraic numbers, p and N are integers with

$p \geqq 2$ and $N \geqq 0$, and $t_p t_{p+N} \neq 0$. In this case, we also have the following theorem.

THEOREM 3. *Let* $f(z)=\sum_{h=0}^{\infty} a_h z^h$ *be a formal power series with powers being non-negative integers, the coefficients in* F *and* $a_0=0$. *If* $f(z)$ *satisfies* (3), *then* $f(z)$ *satisfies* (4) *with* $L=1+\varepsilon$, *where* ε *is an arbitrary positive number.*

REMARK. Mahler considered the case $t=1$, $\eta=0$ and $l=1$, and the condition $m<\rho_1$ is needed. In this case, by Theorem 1 and Theorem 2, we only need

$$m<\rho_1 \times \min\{\rho_1^{1/(L+n-1)}, (\rho_1/|\rho_2|)^{1/n}\}.$$

Note that the part of the minimum in the above inequality is greater than 1.

EXAMPLE 1. Let a be an integer greater than 1, $\Omega=\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$, and $t=1$. Ω satisfies (0) and $\rho_1>a$, $|\rho_2|<1$. The power series

$$f(z_1, z_2)=\prod_{k=0}^{\infty}(1-z_1^{(k)} z_2^{(k)})^{l^k} \qquad (l \geqq 1)$$

satisfies the functional equation: $f(Tz)^l(1-z_1 z_2)=f(z_1, z_2)$. If $a_1=a_2=1$ and $a_k=a a_{k-1}+a_{k-2}$, then $z_1^{(k)} z_2^{(k)}=z_1^{a_{k+1}} z_2^{a_k}$. It is shown that $f(z_1, z_2)$ is transcendental over $\overline{\boldsymbol{Q}}(z_1, z_2)$, and $f(z_1, z_2)$ satisfies (4) with $L=1$. By Theorem 1, if $l^6 \leqq a$, $\alpha_i \in \overline{\boldsymbol{Q}}$ and $0<|\alpha_i|<1$, then $f(\alpha_1, \alpha_2)$ is transcendental.

EXAMPLE 2. Let p be a positive integer, $t \geqq 2$, p and t be coprime and $p/t>1$. Assume that

$$A(z, X)=a_0(z)+a_1(z)X+\cdots+a_l(z)X^l \in \overline{\boldsymbol{Q}}[z, x]$$

satisfies $a_0(0)=0$, $a_0(z) \neq 0$, $a_1(0) \neq 0$, and the coefficients of $a_i(z)$ $(0 \leqq i \leqq l)$ are positive. Put

$$w_0(z)=0, \quad w_n(z)=A(z, w_{n-1}(z^{p/t})) \qquad (n \geqq 1).$$

Then $\operatorname{ord}_z(w_{n+1}(z)-w_n(z)) \geqq \operatorname{ord}_z(w_n(z^{p/t})-w_{n-1}(z^{p/t}))=(p/t)\operatorname{ord}_z(w_n(z)-w_{n-1}(z))$. Since $\operatorname{ord}_z(w_1(z)-w_0(z))>0$, there exists a formal power series $f(z)=\sum_{h=0}^{\infty} a_h z^h$ with the powers being non-negative rational numbers such that $\lim_{n\to\infty} w_n(z)=f(z)$. We have $f(z)=A(z, f(z^{p/t}))$. If $a_h \neq 0$, then $h=n_0+n_1(p/t)+\cdots+n_i(p/t)^i$ for some non-negative integers $n_0, \cdots, n_i$. Therefore $f(z)$ satisfies (1) with $\eta=\log t/(\log p-\log t)$. If $\operatorname{ord}_z a_0(z)=i_0$, then $a_{i_0(p/t)^i} \neq 0$ for any $i \geqq 0$. Hence $f(z)$ is not a Puiseux series, and $f(z)$ is not algebraic over $\overline{\boldsymbol{Q}}(z)$. There is a constant $c>1$ computable from the coefficients of $A(z, X)$ such that

$$\overline{|a_h|} \leqq c^h, \quad d_h \leqq c^h \qquad (h>0).$$

By Theorem 1, if $(p/t)^{(1-\eta)/(1+\eta)}>(tl)^2$, $\alpha \in \overline{\boldsymbol{Q}}$, $0<|\alpha|<1/c$ and $a_l(\alpha^{(p/t)^k}) \neq 0$ for any $k\geqq 0$, then $f(\alpha)$ is transcendental. Especially, if $A(z, X)=z+X$ and $p>t^5$, then $f(z)=\sum_{h=0}^{\infty} z^{(p/t)^h}$ and $f(\alpha)$ is transcendental for any algebraic number such that $0<|\alpha|<1$. In the case $n>1$, we can also construct examples similarly.

We need some lemmas for the proof of theorems. Mahler [4] proved that if Ω satisfies (0), then $A_{11}, \cdots, A_{n1}$ and also $A_{11}, \cdots, A_{1n}$ are linearly independent over $\boldsymbol{Q}$, having the same sign, and $\Omega^k=\sum_{i=1}^n \rho_i{}^k \Gamma_i$ $(k\geqq 0)$ where Γ_i is independent of k, the entries of Γ_1 are positive, and $\Gamma_1=A_1(A_{i1}A_{1j})$ for some nonzero number A_1, these lead the following two lemmas.

LEMMA 1. *Let* $\alpha=(\alpha_1, \cdots, \alpha_n)\in \boldsymbol{C}^n$, $\alpha_1\cdots\alpha_n\neq 0$, *and* $\Lambda=\sum_{j=1}^n |A_{1j}|\log \alpha_j$. *We denote the real part of* Λ *by* $\mathrm{Re}\,\Lambda$. *Then for any* $h_1, \cdots, h_n\in \boldsymbol{Q}$,

$$\log|(\alpha_1^{(k)})^{h_1}\cdots(\alpha_n^{(k)})^{h_n}|$$
$$=(\rho_1/t)^k|A_1|(\mathrm{Re}\,\Lambda)\textstyle\sum_{i=1}^n h_i|A_{i1}|+\phi(h_1, \cdots, h_n, k),$$

where $|\phi(h_1, \cdots, h_n, k)|\leqq c_3(\sum_{i=1}^n|h_i|)(|\rho_2|/t)^k$ *and* c_3 *depends only on* Ω *and* α.

LEMMA 2. *Let* $\alpha=(\alpha_1, \cdots, \alpha_n)\in\overline{\boldsymbol{Q}}^n$, $\alpha_1\cdots\alpha_n\neq 0$. *Then there is a constant* c_4 *depending only on* Ω *and* α *such that* $\log\overline{|\alpha_i^{(k)}|}\leqq c_4(\rho_1/t)^k$ $(1\leqq i\leqq n,\ k\geqq 0)$, *and there is a positive integer* d *depending only on* Ω *and* α *such that* $d^{[(\rho_1/t)^k]}\alpha_i^{(k)}$ $(1\leqq i\leqq n,\ k\geqq 0)$ *are algebraic integers.*

LEMMA 3. *Let* $f(z)=\sum a_{h_1\cdots h_n}z_1{}^{h_1}\cdots z_n{}^{h_n}$ *have the property* (1). *Assume that* $a_{h_1\cdots h_n}\neq 0$, $a_{h_1'\cdots h_n'}\neq 0$, $(h_1, \cdots, h_n)\neq(h_1', \cdots, h_n')$ *and* $h_i, h_i'\leqq h$ $(1\leqq i\leqq n)$. *Then there is a positive constant* c_5 *depending only on* Ω *and* c_1 *such that*

$$|\textstyle\sum_{i=1}^n(h_i-h_i')|A_{i1}||\geqq c_5h^{-n(1+\eta)+1}.$$

PROOF. Since $A_{11}, \cdots, A_{n1}$ are linearly independent real numbers over $\boldsymbol{Q}$, $|A_{11}|, \cdots, |A_{n1}|$ are also linearly independent over $\boldsymbol{Q}$. Since ρ_1 is an algebraic integer, A_{i1} $(1\leqq i\leqq n)$ are algebraic integers and $[\boldsymbol{Q}(A_{11}, \cdots, A_{n1}):\boldsymbol{Q}]\leqq[\boldsymbol{Q}(\rho_1):\boldsymbol{Q}]$ $=n$. By the property (1), there is a positive integer $\delta_h\leqq c_1h^\eta$ such that $\delta_h(h_i-h_i')\in\boldsymbol{Z}$ $(1\leqq i\leqq n)$. Therefore

$$N_{\boldsymbol{Q}(\rho_i)/\boldsymbol{Q}}\delta_h(\textstyle\sum_{i=1}^n(h_i-h_i')|A_{i1}|)\geqq 1$$

so we have Lemma 3.

LEMMA 4. *Suppose that* $B_0\neq 0$, $B_1, \cdots, B_l$ *are algebraic numbers and* $B_0\beta^l+B_1\beta^{l-1}+\cdots+B_l=0$. *Then*

$$\overline{|B_0\beta|}<\overline{|B_0|}+\overline{|B_1|}+\cdots+\overline{|B_l|}.$$

Futher, if D *is a positive integer such that* $DB_0, DB_1, \cdots, DB_l$ *are algebraic* *integers, then so is* $DB_0\beta$.

3. Proof of Theorem 1.

Let the power series $f(z)$ and the number α satisfy all the requirement of Theorem 1 and suppose, in addition, that $f(\alpha)$ is algebraic. Under these assumptions, we shall derive a contradiction, which proves the theorem. We set

$$\Lambda=\sum_{j=1}^{n}|A_{1j}|\log\alpha_j, \text{ and } M=\max\{\rho_1/t,\ m\}.$$

In the following, $c_8, c_9, \cdots$ denote constants greater than 1 depending on Ω, f, α and the functional equation in (3), whose coefficients we may assume algebraic integers.

For any $r\geqq 0$, $f(T^r\alpha)$ and $f(T^{r+1}\alpha)$ are defined. Then by the property (3),

$$Q_0(T^r\alpha,\ f(T^r\alpha))f(T^{r+1}\alpha)^l+Q_1(T^r\alpha,\ f(T^r\alpha))f(T^{r+1}\alpha)^{l-1} + \cdots +Q_l(T^r\alpha,\ f(T^r\alpha))=0\,. \tag{3.1}$$

Since $g(T^r\alpha)\neq 0$ by hypothesis, at least one of

$$Q_0(T^r\alpha,\ f(T^r\alpha)),\ \cdots,\ Q_{l-1}(T^r\alpha,\ f(T^r\alpha))$$

is nonzero. We set

$$j_r=\min\{j : Q_j(T^r\alpha,\ f(T^r\alpha))\neq 0\},$$

and define Y_r $(r\geqq 0)$ inductively, as follows:

$$Y_0=1\,,\quad Y_r=Q_{j_{r-1}}(T^{r-1}\alpha,\ f(T^{r-1}\alpha))Y_{r-1}{}^m \qquad (r\geqq 1).$$

Thus $Y_r\neq 0$ for all $r\geqq 0$. The next lemma gives estimates for these quantities.

LEMMA 5. *For* $r\geqq 1$,

$$[F(\alpha_1,\ \cdots,\ \alpha_n,\ \cdots,\ \alpha_1^{(r)},\ \cdots,\ \alpha_n^{(r)},\ f(\alpha),\ \cdots,\ f(T^r\alpha) : \boldsymbol{Q}]\leqq c_9(lt^{n_0})^r,$$

and

$$\mathrm{size}(Y_r)\,,\quad \mathrm{size}(Y_rf(T^r\alpha))\leqq c_{10}rM^r.$$

PROOF. The first part of the lemma follows by induction using (3.1). Let $\deg_{z_i}Q_j(z,\ u)$ be not greater than s, the hause of $f(\alpha)$ and the coefficients of $Q_j(z,\ u)$ be not greater than c_8 with $c_4{}^{ns}\leqq c_8$, and D be a common multiple of $d(f(\alpha))$ and d^{ns} (c_4 and d are in Lemma 2). Then for any integer $r\geqq 1$, we can prove

$$(*)\qquad \begin{cases} \overline{|Y_r|},\ \overline{|Y_rf(T^r\alpha)|}\leqq\{(l+1)(s+1)^n(m+1)c_8\}^{1+M+\cdots+M^{r-1}}c_8{}^{rM^{r-1}+M^r}, \\ D^{[rM^{r-1}+M^r]}Y_r \text{ and } D^{[rM^{r-1}+M^r]}Y_rf(T^r\alpha) \text{ are algebraic integers,} \end{cases}$$

by induction. If $r=1$, then $Y_1=Q_{j_0}(\alpha, f(\alpha))$,

$$\begin{cases} Y_1 f(T\alpha)^{l-j_0}+\cdots+Q_l(\alpha, f(\alpha))=0\,, \\ \overline{|Q_j(\alpha, f(\alpha))|}\leqq(s+1)^n(m+1)c_8\times c_8^{1+m} \quad (0\leqq j\leqq l), \\ D^{1+m}Q_j(\alpha, f(\alpha)) \quad (0\leqq j\leqq l) \text{ are algebraic integers.}\end{cases}$$

This implies (*) by Lemma 4. If $r>1$, then

$$Y_r f(T^r\alpha)^{l-j_{r-1}}+\cdots+Y_{r-1}{}^m Q_l(T^{r-1}\alpha, f(T^{r-1}\alpha))=0\,.$$

By the induction hypothesis and Lemma 2, we obtain

$$\begin{cases} \overline{|Y_{r-1}{}^m Q_j(T^{r-1}\alpha, f(T^{r-1}\alpha))|} \\ \leqq(s+1)^n(m+1)c_8 c_8{}^{(\rho_1/t)^{r-1}} \\ \quad\times\{((l+1)(s+1)^n(m+1)c_8)^{1+M+\cdots+M^{r-2}}c_8{}^{(r-1)M^{r-2}+M^{r-1}}\}^m, \\ D^{[(\rho_1/t)^{r-1}]}(D^{[(r-1)M^{r-2}+M^{r-1}]})^m Y_{r-1}{}^m Q_j(T^{r-1}\alpha, f(T^{r-1}\alpha)) \\ \quad (0\leqq j\leqq l) \text{ are algebraic integers.}\end{cases}$$

This implies (*) by Lemma 4.

By (2.2), we have

$$\begin{aligned}(3.2)\qquad &\min\{(\rho_1/t)^{1/(L+n(1+\eta)-1)}, (\rho_1/|\rho_2|)^{1/n(1+\eta)}\} \\ &>t^{n_0}l\{(t/\rho_1)(t^{n_0}l)^{n(1+\eta)}M\}^{\eta/(1+\eta)(1-n\eta)} \\ &\quad\times\{(t/\rho_1)(t^{n_0}l)^{n(1+\eta)}M\}^{1/(1+\eta)(1-n\eta)}\,.\end{aligned}$$

Then there exists q_2 such that

$$(3.3)\qquad q_2>\{(t/\rho_1)(t^{n_0}l)^{n(1+\eta)}M\}^{1/(1+\eta)(1-n\eta)} \qquad (\geqq 1)$$

and

$$(3.4)\qquad \min\{(\rho_1/t)^{1/(L+n(1+\eta)-1)}, (\rho_1/|\rho_2|)^{1/n(1+\eta)}\}>t^{n_0}lq_2{}^{1+\eta}.$$

By (3.3),

$$(3.5)\qquad q_2>(t/\rho_1)t^{n_0}lM(t^{n_0}lq_2{}^{\eta})^{n(1+\eta)-1}.$$

By (3.4) and (3.5), there is q_1 such that

$$\begin{cases} (3.6)\qquad q_1>t^{n_0}lq_2{}^{\eta}, \\ (3.7)\qquad q_2>(t/\rho_1)t^{n_0}lMq_1{}^{n(1+\eta)-1}, \\ (3.8)\qquad \min\{(\rho_1/t)^{1/(L+n(1+\eta)-1)}, (\rho_1/|\rho_2|)^{1/n(1+\eta)}\}>q_1q_2\,.\end{cases}$$

By (3.7), $t^{n_0}l<q_1q_2$ so that by (3.8),

$$(3.9)\qquad t^{n_0}l(q_1q_2)^L<(\rho_1/t)q_1q_2\,.$$

The next lemma is one in relation to the construction of the auxiliary function.

LEMMA 6. *Let k be a positive integer and set $\gamma_1=2(c_1+1)^n q_1^{n(1+\eta)k}$ and $\gamma_2=q_2^{(1+\eta)k}$. Then there are $[\gamma_1]+1$ polynomials $P_j(z)=\sum_{0\le h_i\le[\gamma_2]} b_{h_1\cdots h_n}^{(j)} z_1^{h_1}\cdots z_n^{h_n}$ with degrees at most $[\gamma_2]$ whose coefficients are algebraic integers in F with sizes at most $c_{11}k(q_1q_2)^{Lk}$, such that the power series*

$$E_k(z)=\sum_{j=0}^{[\gamma_1]} P_j(z)f(z)^j=\sum b_{h_1\cdots h_n} z_1^{h_1}\cdots z_n^{h_n}$$

is not zero, but all the coefficients $b_{h_1\cdots h_n}$ with $h_i<(q_1q_2)^k$ $(1\le i\le n)$ vanish. Further,

$$\mathrm{size}(b_{h_1\cdots h_n})\le c_{12}k(\max\{h_1, \cdots, h_n\})^L,$$

and

$$\log|b_{h_1\cdots h_n}|\le c_{13}k(q_1q_2)^{Lk}+c_{13}\max\{h_1, \cdots, h_n\}.$$

PROOF. Set $f(z)^j=\sum a_{h_1\cdots h_n}^{(j)} z_1^{h_1}\cdots z_n^{h_n}$, for $j\ge 0$. By the properties (1) and (4), we have for $j\ge 1$,

$$\overline{|a_{h\cdots h_n}^{(j)}|}\le(c_1(\max\{h_1, \cdots, h_n\})^{1+\eta}+1)^{nj}\times e^{c_2(n\times\max\{h_1,\cdots,h_n\})^L}$$
$$\le(\max\{h_1, \cdots, h_n\})^{n(1+\eta)j}c_{16}{}^{j+(\max\{h_1,\cdots h_n\})^L}.$$

By the assumption that $f(\alpha)$ converges, $\log|a_{h_1\cdots h_n}|\le c_{17}\max\{h_1, \cdots, h_n\}$. Then for $j\ge 1$,

$$|a_{h\cdots h_n}^{(j)}|\le(\max\{h_1, \cdots, h_n\})^{n(1+\eta)j}c_{18}{}^{j+\max\{h_1,\cdots,h_n\}}.$$

The polynomials $P_j(z)$ have $([\gamma_1]+1)([\gamma_2]+1)^n$ coefficients $b_{h_1\cdots h_n}^{(j)}$ in all. We can achieve the property required of the auxiliary power series $E_k(z)$ by choosing $b_{h_1\cdots h_n}^{(j)}$ so as to satisfy the linear equations

$$\sum a_{h'_1-h_1,\cdots,h'_n-h_n}^{(j)} b_{h_1\cdots h_n}^{(j)}=0 \qquad (0\le h'_i<(q_1q_2)^k), \tag{3.10}$$

where the sum is taken over all $h_1, \cdots, h_n, j$ satisfying $0\le j\le[\gamma_1]$ and $0\le h_i\le \min\{[\gamma_2], h'_i\}$ $(1\le i\le n)$. For any $b_{h_1\cdots h_n}^{(j)}$, $E_k(z)$ has the property (1) with the same c_1 and η for $f(z)$. Therefore the number of linear equations is not greater than $(c_1(q_1q_2)^{(1+\eta)k}+1)^n$. The integer $D=(\prod_{r=1}^{[\gamma_1]} d_{(q_1q_2)^{k/r}})^n$ will serve as a common denominator for all the $a_{h_1\cdots h_n}^{(j)}$ appering in those equations. The property (4) gives $\log D\le c_{19}k(q_1q_2)^{Lk}$. By a standard version of Siegel's lemma, as given, for example, in Lang [1], page 4, the equations (3.10) have a non-trivial solution in which the $b_{h_1\cdots h_n}^{(j)}$ are algebraic integers in F and

$$\mathrm{size}(b_{h_1\cdots h_n}^{(j)})\le c_{20}kq_1^{n(1+\eta)k}+c_{21}k(q_1q_2)^{Lk}$$
$$\le c_{11}k(q_1q_2)^{Lk} \qquad (0\le j\le[\gamma_1],\ 0\le h_i\le[\gamma_2])$$

by (3.7). By the construction of $E_k(z)$, (3.11) $$b_{h'_1\cdots h'_n}=\sum a^{(j)}_{h'_1-h_1,\cdots,h'_n-h_n}b^{(j)}_{h_1\cdots h_n}$$

where the sum is taken over all $h_1, \cdots, h_n$, j satisfying $0\leqq j\leqq[\gamma_1]$ and $0\leqq h_i\leqq \min\{[\gamma_2], h'_i\}$ $(1\leqq i\leqq n)$. In estimating $b_{h'_1\cdots h'_n}$ we can suppose that $\max\{h'_1, \cdots, h'_n\} \geqq(q_1q_2)^k$, since otherwise $b_{h'_1\cdots h'_n}=0$. We have

$$\begin{aligned}\log|\overline{b_{h'_1\cdots h'_n}}|&\leqq\log([\gamma_1]+1)([\gamma_2]+1)^n+[\gamma_1]n(1+\eta)\log\max\{h'_1, \cdots, h'_n\}\\&\quad+([\gamma_1]+(\max\{h'_1, \cdots, h'_n\})^L)\log c_{16}+c_{11}k(q_1q_2)^{Lk}\\&\leqq c_{22}k(\max\{h'_1, \cdots, h'_n\})^L.\end{aligned}$$

When $h'=\max\{h'_1, \cdots, h'_n\}$, the integer $D_{h'}=(\prod_{r=1}^{[\gamma_1]}d_{h'/r})^n$ will serve as a common denominator for all the $a^{(j)}_{h'_1-h_1,\cdots,h'_n-h_n}$ appearing in (3.11), so that

$$\log d(b_{h'_1\cdots h'_n})\leqq\log D_{h'}\leqq c_{23}kh'^L.$$

Finally, again using (3.11),

$$\begin{aligned}\log|b_{h'_1\cdots h'_n}|&\leqq\log([\gamma_1]+1)([\gamma_2]+1)^n+n(1+\eta)[\gamma_1]\log\max\{h'_1, \cdots, h'_n\}\\&\quad+([\gamma_1]+\max\{h'_1, \cdots, h'_n\})\log c_{18}+c_{11}k(q_1q_2)^{Lk}\\&\leqq c_{13}k(q_1q_2)^{Lk}+c_{14}\max\{h'_1, \cdots, h'_n\}.\end{aligned}$$

This completes the proof of lemma.

Let $E_k(z)$ be the function constructed in Lemma 6. We set

$$H=\min\{\textstyle\sum_{i=1}^n h_i|A_{i1}| : b_{h_1\cdots h_n}\neq0\},$$

and $H=\sum_{i=1}^n H_i|A_{i1}|$. Let K be the integer such that $(q_1q_2)^K\leqq\max\{H_1, \cdots, H_n\} <(q_1q_2)^{K+1}$. By Lemma 6, we have $\max\{H_1, \cdots, H_n\}\geqq(q_1q_2)^k$, so that $K\geqq k$.

LEMMA 7. *For $k\geqq1$, we have*

$$[\boldsymbol{Q}(Y_K{}^{[\gamma_1]}E_k(T^K\alpha)) : \boldsymbol{Q}]\leqq c_{24}(t^{n_0}l)^K,$$

$$\mathrm{size}(Y_K{}^{[\gamma_1]}E_k(T^K\alpha))\leqq c_{25}K(q_1q_2)^{LK}+c_{26}((\rho_1/t)q_2{}^{1+\eta})^K+c_{27}K(q_1{}^{n(1+\eta)}M)^K.$$

PROOF. The first assertion follows at once from Lemma 5. For the second, we use the representation

$$Y_K{}^{[\gamma_1]}E_k(T^K\alpha)=\textstyle\sum_{j=0}^{[\gamma_1]}P_j(T^K\alpha)(Y_Kf(T^K\alpha))^jY_K{}^{[\gamma_1]-j}.$$

From Lemma 2, 5 and the estimate for the size of coefficients of the polynomials $P_j(z)$ in Lemma 6, we find

$$\begin{aligned}\mathrm{size}(Y_K{}^{[\gamma_1]}E_k(T^K\alpha))&\leqq\log([\gamma_1]+1)([\gamma_2]+1)^n+c_{11}k(q_1q_2)^{Lk}\\&\quad+c_{28}n[\gamma_2](\rho_1/t)^K+[\gamma_1]c_{10}KM^K.\end{aligned}$$

This yields the assertion of the lemma.

LEMMA 8. *If k is sufficiently large, then $Y_K^{[\gamma_1]}E_k(T^K\alpha)$ is not zero and*

$$\log|Y_K^{[\gamma_1]}E_k(T^K\alpha)|\leqq(\mathrm{Re}\,\Lambda/2)|A_1|(\min_{1\leq i\leq n}|A_{i1}|)(\rho_1/t)^K(q_1q_2)^K.$$

PROOF. We can write

$$E_k(z)=b_{H_1\cdots H_n}z_1^{H_1}\cdots z_n^{H_n}\{1+\sum(b_{h_1\cdots h_n}/b_{H_1\cdots H_n})z_1^{h_1-H_1}\cdots z_n^{h_n-H_n}\},$$

where the sum is taken over all $(h_1, \cdots, h_n)$ such that $\sum_{i=1}^n H_i|A_{i1}|<\sum_{i=1}^n h_i|A_{i1}|$. By using the fundamental inequality of transcendence theory (If β is a nonzero algebraic number, then $\log|\beta|\geqq-2[\boldsymbol{Q}(\beta):\boldsymbol{Q}]\mathrm{size}(\beta)$.), and Lemma 6,

$$\begin{aligned}(3.12)\qquad &\log|b_{h_1\cdots h_n}/b_{H_1\cdots H_n}|\\ &\leqq c_{13}k(q_1q_2)^{Lk}+c_{13}\max\{h_1, \cdots, h_n\}+c_{29}k(\max\{H_1, \cdots, H_n\})^L\\ &\leqq c_{30}K(q_1q_2)^{LK}+c_{31}\textstyle\sum_{i=1}^n h_i|A_{i1}|.\end{aligned}$$

For any nonnegative integer y, we set

$$B_y=\sum(b_{h_1\cdots h_n}/b_{H_1\cdots H_n})e^{(h_1-H_1)\log\alpha_1^{(k)}+\cdots+(h_n-H_n)\log\alpha_n^{(k)}},$$

where the sum is taken over all $(h_1, \cdots, h_n)$ satisfying

$$(**)\qquad \sum_{i=1}^n H_i|A_{i1}|+y+1\geqq\sum_{i=1}^n h_i|A_{i1}|>\sum_{i=1}^n H_i|A_{i1}|+y.$$

Then

$$(3.13)\qquad E_k(T^K\alpha)=b_{H_1\cdots H_n}e^{H_1\log\alpha_1^{(k)}+\cdots+H_n\log\alpha_n^{(k)}}(1+\textstyle\sum_{y=0}^{\infty}B_y).$$

Using the fact that $E_k(z)$ has the property (1) with the same c_1 and η for $f(z)$, Lemma 1 and (3.12), we have

$$\begin{aligned}\log|B_y|\leqq&\log c_{32}((q_1q_2)^{K+1}+y+1)^{n(1+\eta)}\\ &+c_{30}K(q_1q_2)^{LK}+c_{31}\max\{\textstyle\sum_{i=1}^n h_i|A_{i1}|\}\\ &+(\rho_1/t)^K|A_1|(\mathrm{Re}\,\Lambda)\min\{\textstyle\sum_{i=1}^n(h_i-H_i)|A_{i1}|\}\\ &+c_3\max\{\textstyle\sum_{i=1}^n|h_i-H_i|\}(|\rho_2|/t)^K,\end{aligned}$$

where max and min are taken over all $(h_1, \cdots, h_n)$ satisfying (**). If $y\geqq1$, then $\min\{\sum_{i=1}^n(h_i-H_i)|A_{i1}|\}>y\geqq1$. By (3.8), if k is sufficiently large, then

$$(3.14)\qquad \log|B_y|\leqq(\mathrm{Re}\,\Lambda/2)|A_1|(\rho_1/t)^K y$$

for any $y\geqq1$. If $y=0$, then by Lemma 3, for any $(h_1, \cdots, h_n)$ satisfying (**),

$$\sum_{i=1}^n(h_i-H_i)|A_{i1}|\geqq c_{33}^{-1}(q_1q_1)^{(-n(1+\eta)+1)K}.$$

By this inequality and (3.8),

(3.15) $$\log|B_0|\leqq(\mathrm{Re}\,\Lambda/2)|A_1|(\rho_1/t)^K c_{33}^{-1}(q_1q_2)^{(-n(1+\eta)+1)K},$$

if k is sufficiently large. By (3.14) and (3.15), we have

$$|\textstyle\sum_{y=0}^{\infty}B_y|<1,$$

if k is sufficiently large. Therefore, by (3.13), $E_k(T^K\alpha)$ is not zero, if k is sufficiently large. By Lemma 1, 5, 6, (3.7) and (3.8), we have

$$\begin{aligned}&\log|Y_K^{[\gamma_1]}E_k(T^K\alpha)|\\ &\leqq 2(c_1+1)^n q_1^{n(1+\eta)k}c_{10}KM^K+c_{13}k(q_1q_2)^{Lk}+c_{13}\max\{H_1,\cdots,H_n\}\\ &\quad+(\rho_1/t)^K|A_1|(\mathrm{Re}\,\Lambda)\textstyle\sum_{i=1}^n H_i|A_{i1}|+c_3(\sum_{i=1}^n H_i)(|\rho_2|/t)^K+\log 2.\\ &\leqq(\mathrm{Re}\,\Lambda/2)|A_1|(\min_{1\leqq i\leqq n}|A_{i1}|)(\rho_1/t)^K(q_1q_2)^K,\end{aligned}$$

if k is sufficiently large.

To complete the proof of Theorem 1, we apply the fundamental inequality of transcendence theory to the number $Y_K^{[\gamma_1]}E_k(T^K\alpha)$. By Lemma 7 and Lemma 8, we obtain

$$\begin{aligned}&(\mathrm{Re}\,\Lambda/2)|A_1|(\min_{1\leqq i\leqq n}|A_{i1}|)(\rho_1/t)^K(q_1q_2)^K\\ &\geqq -2c_{24}(t^{n_0}l)^K\{c_{25}K(q_1q_2)^{LK}+c_{26}((\rho_1/t)q_2^{1+\eta})^K+c_{27}K(q_1^{n(1+\eta)}M)^K\},\end{aligned}$$

providing k is sufficiently large. Since $\mathrm{Re}\,\Lambda<0$ and $K\geqq k$, this contradicts (3.6), (3.7) and (3.9).

4. Proof of Theorem 2.

At the first, we prove the theorem in the case where $f(z)$ is a power series with powers being non-negative integers and $t=1$. Adopting Ω^r for Ω, if necessary, we may assume that the entries of Ω are greater than 1. Let

$$S=\{(\lambda_1,\cdots,\lambda_n):0\leqq\lambda_i\in\boldsymbol{Z}\},$$

and we define $(\lambda_1,\cdots,\lambda_n)<(\lambda_1',\cdots,\lambda_n')$ if and only if $\lambda_1+\cdots+\lambda_n<\lambda_1'+\cdots+\lambda_n'$ or $\lambda_1+\cdots+\lambda_n=\lambda_1'+\cdots+\lambda_n'$ and $\lambda_1=\lambda_1',\cdots,\lambda_i=\lambda_i',\lambda_{i+1}<\lambda_{i+1}'$. Then S is a totally ordered set. For $\lambda=(\lambda_1,\cdots,\lambda_n)\in S$ and $z=(z_1,\cdots,z_n)$, we set $|\lambda|=\lambda_1+\cdots+\lambda_n$, $\lambda!=\lambda_1!\cdots\lambda_n!$, $z^\lambda=z_1^{\lambda_1}\cdots z_n^{\lambda_n}$, $\dfrac{\partial^\lambda}{\partial z^\lambda}=\dfrac{\partial^{\lambda_1}}{\partial z_1^{\lambda_1}}\cdots\dfrac{\partial^{\lambda_n}}{\partial z_n^{\lambda_n}}$. Then $f(z)=\sum_{\lambda\in S}a_\lambda z^\lambda$. By the property (3), there is a polynomial $P(z,u,v)\in\overline{\boldsymbol{Q}}[z_1,\cdots,z_n,u,v]$ with coefficients being algebraic integers such that

$$P(z,f(z),f(Tz))=0,$$

$$P_u(z,f(z),f(Tz))=(\mathrm{put})\textstyle\sum_{\lambda\in S}A_\lambda z^\lambda\neq 0,$$

where $P_u(z, u, v)$ is the partial derivative of $P(z, u, v)$ in u. We denote by λ_0, the least λ with $A_\lambda \neq 0$, and put $|\lambda_0| = m_0$. Let $\{y_\lambda\}_{\lambda \in S}$ be variables, and we set

$$f(y, z) = \sum_{\lambda \in S} y_\lambda z^\lambda,$$

$$P_u(z, f(y, z), f(y, Tz)) = \sum_{\lambda \in S} A_\lambda(y) z^\lambda,$$

$$P(z, f(y, z), f(y, Tz)) = \sum_{\lambda \in S} B_\lambda(y) z^\lambda.$$

If $\lambda \in \boldsymbol{Z}^n$ and $\notin S$, then we put $A_\lambda(y) = B_\lambda(y) = 0$. Hence $A_\lambda(y)$ and $B_\lambda(y)$ are polynomials in $\{y_\lambda\}_{\lambda \in S}$ with coefficients being algebraic integers. Substituting a_λ to y_λ in $A_\mu(y)$ and $B_\mu(y)$, we obtain A_μ and 0 respectively.

LEMMA 9. *Let $\nu, \mu \in S$, and $2|\nu| > |\mu|$. Then $\deg_{y_\nu} B_\mu(y) \leqq 1$ and the coefficient of y_ν in $B_\mu(y)$ is $A_{\mu-\nu}(y)$.*

PROOF. Since $2|\nu| > |\mu|$, $\deg_{y_\nu} B_\mu \leqq 1$. The coefficient of y_ν in $B_\mu(y)$ is equal to

$$(4.1) \qquad \frac{\partial}{\partial y_\nu} \cdot \frac{1}{\mu!} \cdot \frac{\partial^\mu}{\partial z^\mu} P(z, f(y, z), f(y, Tz)) \Big|_{z=0}$$

$$= \frac{\partial}{\partial y_\nu} \cdot \frac{1}{\mu!} \cdot \frac{\partial^\mu}{\partial z^\mu} P(z, \sum_{\lambda \leq \mu} y_\lambda z^\lambda, \sum_{\lambda \Omega \leq \mu} y_\lambda z^{\lambda \Omega}) \Big|_{z=0}$$

Since $2|\nu| > |\mu|$, y_ν does not appear in $\sum_{\lambda \Omega \leq \mu} y_\lambda z^{\lambda \Omega}$ and therefore (4.1) is equal to

$$\frac{1}{\mu!} \cdot \frac{\partial^\mu}{\partial z^\mu} \{P_u(z, \sum_{\lambda \leq \mu} y_\lambda z^\lambda, \sum_{\lambda \Omega \leq \mu} y_\lambda z^{\lambda \Omega}) z^\nu\} \Big|_{z=0}$$

$$= \frac{1}{\mu!} \cdot \frac{\partial^\mu}{\partial z^\mu} \{P_u(z, f(y, z), f(y, Tz)) z^\nu\} \Big|_{z=0}$$

$$= A_{\mu-\nu}(y).$$

Let $P(z, u, v) = \sum_{i \in S} \sum_{j=0}^{J} \sum_{k=0}^{K} b_{ijk} z^i u^j v^k$, where b_{ijk} are algebraic integers. We set

$$M = \max\{\sum_{i \in S} \sum_{j=0}^{J} \sum_{k=0}^{K} |\overline{b_{ijk}}|, 1\}.$$

Let a constant $c_{40} \geqq 1$ and a positive integer D satisfy

$$(4.2) \qquad \begin{cases} |\overline{A_\lambda}| (|\lambda| = m_0), \ |\overline{a_\lambda}| (|\lambda| \leqq m_0), \ |\overline{1/A_{\lambda_0}}| \leqq c_{40}, \\ DA_\lambda (|\lambda| = m_0), \ Da_\lambda (|\lambda| \leqq m_0), \ D(1/A_{\lambda_0}) \text{ are algebraic integers.} \end{cases}$$

Then we can prove (4.3) and (4.4) in the case $n=1$ and $n>1$ respectively. for any $\mu \in S$ with $|\mu| = m_0 + m$ $(m \geqq 1)$ by induction in m.

$$(4.3) \qquad \begin{cases} |\overline{a_\mu}| \leqq \{M c_{40} (m_0+1)^{J+K}\}^{2m-1} (m!)^{J+K} c_{40}{}^{m(2m_0+1)}, \\ D^{2m-1} D^{m(2m_0-1)} a_\mu \text{ is an algebraic integer.} \end{cases}$$

$$(4.4)\quad \begin{cases} \overline{|a_\mu|} \leqq (4c_{40}{}^2)^{(m_0+2)^{n-1}m^{2(n-1)}}\{M(m_0+1)^{n(J+K)}\}^{2m-1}(m!)^{n(J+K)}c_{40}{}^{m(2m_0+1)}, \\ D^{2(m_0+2)^{n-1}m^{2(n-1)}}D^{m(2m_0+1)}a_\mu \text{ is analgebraic integer.} \end{cases}$$

In the case where $f(z)$ is a power series with powers being non-negative integers, (4.3) and (4.4) lead the theorem. Since the proof of (4.3) is easier than the proof of (4.4), we only prove (4.4). We give a number to each element of $\{\mu: |\mu|=m+m_0\}$ as: $\mu_1<\mu_2<\cdots<\mu_{l(m)}$. Note that

$$l(m)\leqq(m+m_0+1)^{n-1}\leqq(m_0+2)^{n-1}m^{n-1}.$$

By Lemma 9, we have

$$(4.5)\qquad B_{\lambda_0+\mu_i}(y)=\textstyle\sum_{2m_0+m\geqq|\lambda|\geqq m_0+m}A_{\lambda_0+\mu_i-\lambda}(y)y_\lambda+C_{\lambda_0+\mu_i}(y),$$

where $C_{\lambda_0+\mu_i}(y)$ is the coefficient of $z^{\lambda_0+\mu_i}$ in

$$\textstyle\sum_{i\in S}\sum_{j=0}^{J}\sum_{k=0}^{K}b_{ijk}z^i(\sum_{|\lambda|<m_0+m}y_\lambda z^\lambda)^j(\sum_{|\lambda|<m_0+m}y_\lambda z^{\lambda\Omega})^k.$$

If $\lambda>\mu_i$, then $\lambda_0+\mu_i-\lambda\notin S$ or $\lambda_0+\mu_i-\lambda<\lambda_0$. In any case, $A_{\lambda_0+\mu_i-\lambda}=0$. Substituting a_λ to y_λ in (4.4), we have

$$A_{\lambda_0+\mu_i-\mu_1}a_{\mu_1}+A_{\lambda_0+\mu_i-\mu_2}a_{\mu_2}+\cdots+A_{\lambda_0}a_{\mu_i}=-C_{\lambda_0+\mu_i},$$

where $C_{\lambda_0+\mu_i}$ denotes the values of $C_{\lambda_0+\lambda_i}(y)$ at $y_\lambda=a_\lambda$ $(\lambda\in S)$. Therefore

$$(4.6)\qquad \begin{pmatrix} A_{\lambda_0} & & \\ & A_{\lambda_0} & 0 \\ & & \ddots \\ & * & & A_{\lambda_0}\end{pmatrix}\begin{pmatrix}a_{\mu_1}\\ a_{\mu_2}\\ \vdots \\ a_{\mu_{l(m)}}\end{pmatrix}=\begin{pmatrix}-C_{\lambda_0+\mu_1}\\ -C_{\lambda_0+\mu_2}\\ \vdots\\ -C_{\lambda_0+\mu_{l(m)}}\end{pmatrix},$$

where the entries in $*$ consist of A_λ $(|\lambda|=m_0)$. Hence by (4.2) we have

$$\begin{aligned}(4.7)\qquad \overline{|a_{\mu_i}|} &\leqq c_{40}{}^{l(m)}l(m)2^{l(m)}c_{40}{}^{l(m)}M(m_0+m)^{n(J+K)} \\ &\quad\times\max\overline{|a_{\lambda_1}\cdots a_{\lambda_r}a_{\nu_1}\cdots a_{\nu_s}|}, \\ &\leqq \max(4c_{40}{}^2)^{(m_0+2)^{n-1}m^{n-1}}M(m_0+1)^{n(J+K)}m^{n(J+K)} \\ &\quad\times\overline{|a_{\lambda_1}\cdots a_{\lambda_n}a_{\nu_1}\cdots a_{\nu_s}|},\end{aligned}$$

where max is taken over all $(\lambda_1,\cdots,\lambda_r,\nu_1,\cdots,\nu_s)$ such that

$$(4.8)\qquad \begin{cases}\lambda_i,\ \nu_j\in S,\ |\lambda_i|=m_0+m_i \text{ with } 1\leqq m_i<m,\ |\nu_j|\leqq m_0, \\ (m_0+m_1)+\cdots+(m_0+m_r)+|\nu_1|+\cdots+|\nu_s|\leqq m+2m_0.\end{cases}$$

If d is a common multiple of all $d(a_{\lambda_1}\cdots a_{\lambda_r}a_{\nu_1}\cdots a_{\nu_s})$ with $(\lambda_1,\cdots,\lambda_r,\nu_1,\cdots,\nu_s)$ satisfying (4.8), then $D^{2l(m)}d\,a_{\mu_i}$ is an algebraic integer. If $m=1$, then $r=0$, and

$$\overline{|a_{\nu_1}\cdots a_{\nu_s}|}\leqq c_{40}{}^{2m_0+1},$$

$D^{2m_0+1}a_{\nu_1}\cdots a_{\nu_s}$ is an algebraic integer.

This implies (4.4). If $m>1$, then by the induction hypothesis,

$$\overline{|a_{\lambda_1}\cdots a_{\lambda_r}a_{\nu_1}\cdots a_{\nu_s}|}\leqq(4c_{40}^2)^{(m_0+2)^{n-1}(m_1^{2(n-1)}+\cdots+m_r^{2(n-1)})}$$
$$\times\{M(m_0+1)^{n(J+K)}\}^{2(m_1+\cdots+m_r)-r}(m_1!\cdots m_r!)^{n(J+K)}$$
$$\times c_{40}^{(m_1+\cdots+m_r)(2m_0+1)}c_{40}^{m+2m_0-(m_0+m_1)-\cdots-(m_0+m_r)}.$$

If $r=0$, then $\overline{|a_{\nu_1}\cdots a_{\nu_s}|}\leqq c_{40}^{m+2m_0}$ and

$$(4c_{40}^2)^{(m_0+2)^{n-1}m^{n-1}}M(m_0+1)^{n(J+K)}m^{n(J+K)}\overline{|a_{\nu_1}\cdots a_{\nu_s}|}$$
$$\leqq(4c_{40}^2)^{(m_0+2)^{n-1}m^{2(n-1)}}\{M(m_0+1)^{n(J+K)}\}^{2m-1}(m!)^{n(J+K)}c_{40}^{m(2m_0+1)}.$$

If $r=1$, then

$$(4c_{40}^2)^{(m_0+2)^{n-1}m^{n-1}}M(m_0+1)^{n(J+K)}m^{n(J+K)}\overline{|a_{\lambda_1}a_{\nu_1}\cdots a_{\nu_s}|}$$
$$\leqq(4c_{40}^2)^{(m_0+2)^{n-1}(m^{n-1}+(m-1)^{2(n-1)})}\{M(m_0+1)^{n(J+K)}\}^{2(m-1)}$$
$$\times(m\times(m-1)!)^{n(J+K)}c_{40}^{m(2m_0+1)}$$
$$\leqq(4c_{40}^2)^{(m_0+2)^{n-1}m^{2(n-1)}}\{M(m_0+1)^{n(J+K)}\}^{2m-1}(m!)^{n(J+K)}c_{40}^{m(2m_0+1)},$$

by the inequality $m^{n-1}+(m-1)^{2(n-1)}\leqq(m+(m-1)^2)^{n-1}\leqq m^{2(n-1)}$. If $r\geqq2$, then $m_1+\cdots+m_r\leqq m$ by (4.8), and

$$(4c_{40}^2)^{(m_0+2)^{n-1}m^{n-1}}\{M(m_0+1)^{n(J+K)}\}m^{n(J+K)}\overline{|a_{\lambda_1}\cdots a_{\lambda_r}a_{\nu_1}\cdots a_{\nu_s}|}$$
$$\leqq(4c_{40}^2)^{(m_0+2)^{n-1}(m_1^{2(n-1)}+\cdots+m_r^{2(n-1)}+m^{n-1})}$$
$$\times\{M(m_0+1)^{n(J+K)}\}^{2m-1}(m\times m_1!\cdots m_r!)^{n(J+K)}c_{40}^{m(2m_0+1)}$$
$$\leqq(4c_{40}^2)^{(m_0+2)^{n-1}m^{2(n-1)}}\{M(m_0+1)^{n(J+K)}\}^{2m-1}(m!)^{n(J+K)}c_{40}^{m(2m_0+1)},$$

by the inequalities $m\times m_1!\cdots m_r!\leqq m!$ and

$$m_1^{2(n-1)}+\cdots+m_r^{2(n-1)}+m^{r-1}$$
$$\leqq(m_1^2+\cdots+m_r^2+m)^{r-1}\leqq m^{2(r-1)}.$$

The denominator of a_{μ_i} is also estimated in the same way. These imply (4.4).

In general case, there is a polynomial $P(z, u, v)$, $\overline{Q}[z_1, \cdots, z_n, u, v]$ such that

$$P_u(z, f(z), f(Tz))=\sum A_{h_1\cdots h_n}z_1^{h_1}\cdots z_n^{h_n}\neq0.$$

m_0' is the least number in all $h_1+\cdots+h_n$ with $A_{h_1\cdots h_n}\neq0$. Let $a_{\lambda_1\cdots\lambda_n}\neq0$ and $\lambda_i\leqq h$ $(1\leqq i\leqq n)$. Substituting $z_i^{t\delta_{nh+m_0'}}$ to z_i, we can treat $\{a_{h_1\cdots h_n}: h_1+\cdots+h_n\leqq nh\}$ in the same way as above with $m_0=t\delta_{nh+m_0'}m_0'$ $(\leqq c_{41}h^{\eta})$ and $m_0+m=(\lambda_1+\cdots+\lambda_n)t\delta_{nh+m_0'}$ $(\leqq c_{42}h^{1+\eta})$. Thus we have the theorem.

Similarly we can prove Theorem 3, and we omit the proof.

References

[1] Lang, S., Introduction to Transcendental Numbers. Addison-Wesley, 1966.

[2] Loxton, J. H. and van der Poorten, A. J., Transcendence and algebraic independence by a method of Mahler. Transcendence Theory—Advances and Applications ed. Baker, A. and Masser, D. W., Academic Press (1977), 211-226.

[3] Loxton, J. H. and van der Poorten, A. J., Arithmetic properties of certain functions in several variables. Journal of number theory **9** (1977), 87-106.

[4] Mahler, K., Arithmetische Eigenschaften der Lösugen einer Klasse von Funktionalgleichungen. Math. Ann. **101** (1929), 342-366.

[5] Mahler, K., Über das Verschwinden von Potenzreihen mehrerer Veränderlichen in speziellen Punktfolgen. Math. Ann. **103** (1930), 573-587.

[6] Mahler, K., Arithmetische Eigenschaften einer Klass transzendental-transzendenter funktionen. Math. Z. **32** (1930), 545-585.

[7] Mahler, K., Remarks on a paper by W. Shwarz. Journal of number theory **1** (1969), 512-521.

[8] Masser, D. W., A vanishing theorem for power series. Invent. math. **67** (1982), 275-296.

[9] Nishioka, K., On a problem of Mahler for transcendency of function values. J. Austral. Math. Soc. (Series A) **33** (1982), 386-393.

Department of Mathematics
Nara Women's University
Kita-Uoya Nishimachi, Nara
630 Japan