ORDER OF THE STANDARD ISOMETRIC MINIMAL IMMERSIONS OF CROSS AS HELICAL GEODESIC IMMERSIONS

By

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0. Introduction.

Let $\phi: M \to \tilde{M}$ be an isometric immersion of a Riemannian manifold M into another Riemannian manifold $\tilde{M}$. Let γ be an arbitrary geodesic in M parametrized by arc-length. If the curve $\phi \circ \gamma$ in $\tilde{M}$ is of rank d and has constant curvatures $\kappa_1, \cdots, \kappa_{d-1}$ which are independent of the choise of the geodesic γ, then $\phi: M \to \tilde{M}$ is called a *helical geodesic immersion of order d* [7].

The standard isometric minimal immersions of compact rank one symmetric spaces (CROSS) into spheres, which we will define in §. 3, are examples of helical geodesic immersions. In [10] Tsukada calculated the order of the standard isometric minimal immersions of CROSS as helical geodesic immersions except the Cayley projective plane. In this paper we calculate the order of the standard isometric minimal immersions of CROSS in a different manner from the Tsukada's one. Namely we prove the following

THEOREM. *The k-th standard isometric minimal immersions* ϕ_k *of* CROSS *into spheres are helical geodesic immersions. And the order of the immersions are given as follows;*

M	order of ϕ_k
S^n , $n \geq 2$	k
CP^n , $n \geq 2$	$2k$
QP^n , $n \geq 2$	$2k$
$\mathrm{Cay}\, P^2$	$2k$

From Theorems of the author [5], [6], the order of the standard isometric minimal immersions of CROSS coincide with their degree.

1. Order of helices in a sphere.

Let $c : I \to M$ be a curve parametrized by arc-lengrh in a Riemannian manifold M. We assume that $\dot{c}(t)$, $\nabla_t \dot{c}(t)$, $\cdots$, $\nabla^{(d-1)}{}_t \dot{c}(t)$ are linearly independent on I and $\dot{c}(t)$, $\nabla_t \dot{c}(t)$, $\cdots$, $\nabla^{(d)}{}_t \dot{c}(t)$ are linearly dependent on I. Then we have the usual Frenet-Serret's formula;

$$\nabla_t \dot{c}(t) = \kappa_1(t) Y_2(t)$$

$$\nabla_t Y_2(t) = -\kappa_1(t) \dot{c}(t) + \kappa_2(t) Y_3(t),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\nabla_t Y_{d-1}(t) = -\kappa_{d-2}(t) Y_{d-2}(t) + \kappa_{d-1}(t) Y_d(t),$$

$$\nabla_t Y_d(t) = -\kappa_{d-1}(t) Y_{d-1}(t),$$

where κ_i are positive functions on I, Y_i are orthonormal vector fields along the curve c and ∇ is the Levi-Civita connection of M. A curve c is called a *helix of order d*, if κ_i are constants.

Let S_1^N be the unit sphere in R^{N+1} centered at the origin. Then $f_0(t) = c(t) \cdot c(0)$, $f_1(t) = c(t) \cdot \dot{c}(0)$, $\cdots$, $f_d(t) = c(t) \cdot Y_d(0)$ satisfies the following differntial equation ([2], p. 177).

(1.1)
$$\begin{pmatrix} f_0' \\ f_1' \\ \cdot \\ \cdot \\ \cdot \\ f_{d-1}' \\ f_d' \end{pmatrix} = \begin{pmatrix} 0 & -1 & & & & \\ 1 & 0 & -\kappa_1 & & & \\ & \kappa_1 & 0 & -\kappa_2 & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & 0 & -\kappa_{d-1} \\ & & & & \kappa_{d-1} & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \cdot \\ \cdot \\ f_{d-1} \\ f_d \end{pmatrix}$$

By a suitable coordinate transformation, a helix in the unit sphere S_1^N is an orbit of the one parameter subgroup in $SO(N+1)$ generated by the following vector ([2], p. 177).

(1.2)
$$\begin{pmatrix} 0 & -1 & & & & & & \\ 1 & 0 & -\kappa_1 & & & & & \\ & \kappa_1 & 0 & -\kappa_2 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & 0 & -\kappa_{d-1} & & \\ & & & & \kappa_{d-1} & 0 & 0 & \\ & & & & & 0 & \cdot & \cdot \\ & & & & & & \cdot & 0 & 0 \\ & & & & & & & 0 & 0 \end{pmatrix}$$

Then the helix is contained in the totally geodesic submanifold $S_1^d = \{(x_0, x_1, \cdots, x_d, 0, \cdots, 0) \mid \sum_{i=0}^{d}(x_i)^2 = 1\}$ and is not contained in any totally geodesic submanifold of S_1^d.

The order of a helix in the unit sphere is characterized by the following

LEMMA 1.1. *Let* $c : I \to S_1^N$ *be a helix of order* d. *Then there exists a polynomial* P *of degree* $d+1$ *such that*

(1.3) $P(d/dt)(f_0) = 0$, $t \in I$,

where $f_0(t) = c(t) \cdot c(0)$. *For any polynomial of which degree is less that* $d+1$ (1.3) *does not hold.*

PROOF. Let $P(t)$ be the charasteristic polynomial of (1.2). Then $f_0(t)$ satisfies (1.3) ([3], p. 161).

We may assume that $N = d$, i.e., there does not exists any totally geodesic submanifold which contains the helix c.

By the differential equation (1.1) we get the following

(1.4) $(d/dt)^n f_0 = a_{n,n} f_n + a_{n,n-1} f_{n-1} + \cdots + a_{n,0} f_0$, $1 \leq n \leq d$,

where $a_{n,i}$, $0 \leq i \leq n$, are constants and $a_{n,n} \neq 0$, $1 \leq n \leq d$. Let $P = \sum_{j=0}^{m} b_j x^{m-j}$, $1 \leq m \leq d$. Now we assume that (1.3) holds for P. Then by (1.3) and (1.4) we get

(1.5) $0 = c_m f_m + c_{m-1} f_{m-1} + \cdots + c_0 f_0$

where $c_m = a_{m,m}$, $c_{m-1}, \cdots, c_0$ are constants. (1.5) implies that $c(t)$ is contained in the linear subspace which is perpendicular to a non-zero vector $(c_0, c_1, \cdots, c_m, 0, \cdots, 0)$, which is a contradiction. So we get the Lemma.   Q. E. D.

REMARK 1.2. (i) The above Lemma is also true, if the parameter t is proportional to the arc-length.
(ii) The polynomial with the smallest degree for which (1.3) holds is unique up to constant factor.

2. Jaccbi polynomials and their finite cosine expansion.

Let p, q be real numbers which are greater than -1. Then for each non-negative integer k Jacobi polynomial $P_k^{(p,q)}(t)$ is defined by

$$P_k^{(p,q)}(t) = ((p+1)_k / k)_2 F_1(-k, \; k+p+q+1; \; p+1; \; (1-t)/2), \qquad t \in R,$$

where $_2F_1$ is the hypergeometric function of Gauss and $(p+1)_k = (p+1) \cdot (p+2) \cdots (p+k)$. The Jacobi polynomials are characterized by the following

LEMMA 2.1.   *The Jacobi polynomial $P_k^{(p,q)}(t)$ is the unique analytic solution of the following differential equation with $y(1)=1$,*

$$(1-t^2)d^2y/dt^2-((p+q+2)t+(p-q))dy/dt+k(p+q+k+1)y=0.$$

When $p=q=n/2-1$ (n; positive integer), the Jacobi polynomial $P_k^{(p,q)}(\cos \theta)$ has finite cosine expansion.

LEMMA 2.2.   ([8], p. 94).

$$P_k^{(n/2-1,\,n/2-1)}(\cos \theta)$$

$$=a_0\cos k\theta+a_1\cos (k-2)\theta+ \cdots +$$

$$\begin{cases} a_{[k/2]}\cos \theta , & \text{if } k \text{ is an odd integer}, \\ a_{[k/2]} & , \text{if } k \text{ is an even integer}, \end{cases}$$

where a_j, $0\leq j\leq[k/2]$, are positive constants.

We can derive the finite cosine expansion for some Jacobi polynomials by the following

LEMMA 2.3.   ([8], p. 250).   *Let p, q, r be real numbers such that $r\geq p\geq 0$, $q>-1$.   Then*

$$P_k^{(r,q)}(t)= \sum_{j=0}^{k} a_{k,j}P_j^{(p,q)}(t)$$

where $a_{k,j}$ are positive constants.

3. Proof of the Theorem.

Let (G, K) be an irreducible symmetric pair of compact type.   Let $\mathfrak{g}$ [resp. $\mathfrak{k}$] be the Lie algebra of G [resp. K], and $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form B of $\mathfrak{g}$.   Let g_0 be the G-invariant Riemannian metric on $M=G/K$, which is a G-invariant extension of $-B|_{\mathfrak{p}\times\mathfrak{p}}$.   Then (M, g_0) is a Riemannian symmetric space.

Let V be a real irreducible representation of G with a G-invariant inner product $\langle \, , \, \rangle$, which contains a K-fixed unit vector v_0 (such a representation is called a *class one representation* of the pair).   Then

(3.1) $\phi : M \longrightarrow V; gK \longrightarrow g\cdot v_0$

realizes an isometric immersion of (M, c^2g_0) for some constant c into the unit sphere in V centered at the origin [11].

Define a linear map $A: V\rightarrow C^\infty(M)$ by

$$A(v)(gK) = \langle v, \, g \cdot v_0 \rangle, \qquad g \in G.$$

Then $A(v)$ is an eigen-function of the Laplace-Beltrami operator $\varDelta$ of (M, g_0) [11]. Actually A is a G-homomorphism. Since $\varDelta \circ g = g \circ \varDelta$, $\varDelta$ is a scalar operator on $A(V)$, i.e., $A(V)$ is contained in some eigin-space of $\varDelta$, say the k-th eigen-space V^k. If M is of rank one $A(V) = V^k$ [1]. So we denote by ϕ_k the immersion defined by (3.1) when $A(V) = V^k$, and call the k-th *standard isometric minimal immersion* of M.

Hereafter we assume that M is of rank one. The v_0 is unique up to sign for each class one representation [1]. It is easily seen that ϕ_k is a helical geodesic immersion.

Since v_0 is a K-fixed vector $h = A(v_0)$ satisfies

$$(3.2) \qquad\qquad h(k \cdot p) = h(p), \qquad k \in K, \qquad p \in M.$$

The function in V^k which satisfies (3.2) is called a *zonal spherical function* of V^k, which is unique up to sign (since v_0 is so).

Now we give a explicit expression of $h = A(v_0)$ using Jacobi polynomials.

Assume that a function $h \in C^\infty(M)$ satisfies (3.2), then thre exists a function h_0 defined on $[0, \pi]$, where π is the diameter of (M, g_0), such that

$$(3.3) \qquad\qquad h(p) = h_0(d(p))$$

where $d(p)$ is the distance of p from the origin eK on M. For such functions there exists a differential operator $\partial(\varDelta)$ on $(0, \pi)$ such that

$$(3.4) \qquad (\varDelta h)(p) = (\partial(\varDelta) h_0)(d(p)), \qquad d(p) \in (0, \pi).$$

From [4, p. 270] or [9, p. 203], $\partial(\varDelta)$ is calculated as follows

LEMMA 3.1.

$$\partial(\varDelta) = \begin{cases} -d^2/d\theta^2 & , \ \ if \ M = S^1, \\ -d^2/d\theta^2 - (n-1) \cot\theta \, d/d\theta & , \ \ if \ M = S^n, \ n \geq 2 \\ -d^2/d\theta^2 - ((n-1) \cot\theta/2 + \cot\theta)d/d\theta & , \ \ if \ M = CP^n, \ n \geq 2, \\ -d^2/d\theta^2 - (2(n-1) \cot\theta/2 + 3\cot\theta)d/d\theta & , \ \ if \ M = QP^n, \ n \geq 2, \\ -d^2/d\theta^2 - (4\cot\theta/2 + 7\cot\theta)d/d\theta & , \ \ if \ M = \mathrm{Cay}\,P^2. \end{cases}$$

After a change of variable $t = \cos\theta$, $\partial(\varDelta)$ is given as follows.

LEMMA 3.2.

$$\partial(\Delta)=\begin{cases} -(1-t^2)d^2/dt^2+ntd/dt & , \ \ if \ M=S^n, \ n\geqq1, \\ -(1-t^2)d^2/dt^2+((n+1)t+(n-1))d/dt & , \ \ if \ M=CP^n, \ n\geqq2, \\ -(1-t^2)d^2dt^2+(2(n+1)t+2(n-1))d/dt, & if \ M=QP^n, \ n\geqq2, \\ -(1-t^2)d^2/dt^2+(12t+4)d/dt & , \ \ if \ M=\mathrm{Cay}\,P^2. \end{cases}$$

By a formula of Freudenthal [9] we can calculate the eigenvalue of Δ.

LEMMA 3.3. *The k-th eigen-value of Δ is*

$$k(k+n-1) \ , \ \ if \ M=S^n, \ n\geqq1$$

$$k(k+n) \ \ \ \ , \ \ if \ M=CP^n, \ n\geqq2$$

$$k(k+2n+1), \ \ if \ M=QP^n, \ n\geqq2$$

$$k(k+11) \ \ \ \ , \ \ if \ M=\mathrm{Cay}\,P^2.$$

Let k be an eigen-function of Δ corresponding to the k-th eigen-value which satisfies (3.2). Then h_0 (defined by (3.3)) is an eigen-function of Δ corresponding to the same eigen-value. So from Lemma 2.1, Lemma 3.2 and Lemma 3.3 we get the following

LEMMA 3.4. *The zonal spherical function h of (M, g_0) which is contained in V^k is $h(p)=h_0(d(p))$, where h_0 is*

$$h_0(\theta)=\begin{cases} P_k^{(n/2-1,n/2-1)}(\cos\theta), & if \ M=S^n, \ n\geqq1, \\ P_k^{(n-1,0)}(\cos\theta) & , \ \ if \ M=CP^n, \ n\geqq2, \\ P_k^{(2n-1,1)}(\cos\theta) & , \ \ if \ M=QP^n, \ n\geqq2, \\ P_k^{(7,3)}(\cos\theta) & , \ \ if \ M=\mathrm{Cay}\,P^2. \end{cases}$$

Let $\gamma_X(t)$ be a geodesic in M issuing from the origin eK in the unit direction $X\in T_{eK}M=\mathfrak{p}$. Since M is of rank one (3.2) implies that $A(v_0)(\gamma_X(t))=\gamma_X(t)\cdot\gamma_X(0)$ is independent of the choise of X. From the uniqueness of the zonal spherical function we get $A(v_0)(\gamma_X(t))=kh_0(t)$, $t\in(0,\pi)$, for some non-zero constant k. So we have only to calculate the smallest degree of the polynomial such that (1.3) holds.

CASE 1. $M=S^n$ $(n\geqq2)$. Since h_0 has finite cosine expansion by Lemma 2.2, h_0 satisfies (1.3) for

$$P(t)=(t^2+k^2)\cdot(t^2+(k-2)^2)\cdots\cdots\begin{cases}(t^2+1),\ \text{if}\ k\ \text{is an odd integer,}\\(t^2+4)\cdot t,\ \text{if}\ k\ \text{is an even integer.}\end{cases}$$

It is easily verified that (1.3) does not hold for any polynomial of which degree is less that $k+1$. So the Theorem is proved.

CASE 2. $M=CP^n$ $(n\geqq2)$, $M=QP^n$ $(n\geqq2)$ or $M=\mathrm{Cay}\,P^2$. Since h_0 has finite expansion by Jacobi polynomials from Lemma 2.3, h_0 has finite cosine expansion

$$h_0(\theta)=\sum_{j=0}^{k} a_j\cos j\theta,$$

where a_j are positive constants. So h_0 satisfies (1.3) for

$$P(t)=\Big(\prod_{j=0}^{k}(t^2+j^2)\Big)\cdot t.$$

Also in this case it is easily verified that (1.3) does not hold for any polynomial of which degree is less than $2k+1$. So the Theorem is proved.

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