TSUKUBA J. MATH.
Vol 6 No. 2 (1982). 289–292

HAPPLEL-RINGEL'S THEOREM ON TILTED ALGEBRAS

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In [4], Happel-Ringel have generalized the earlier work of Brenner-Butler [3] and extensively developed the theory of tilting modules. They have also introduced the notion of tilted algebras.

Let A be an artin algebra and T_A a finitely generated right A-module. Recall that T_A is said to be a *tilting module* if it satisfies the following three conditions:

(1) proj dim $T_A \leqq 1$.

(2) $\mathrm{Ext}_A^1(T_A, T_A)=0$.

(3) There is an exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$ with T'_A, T''_A direct sums of direct summands of T_A.

If A is hereditary, the endomorphism algebra $B=\mathrm{End}(T_A)$ of a tilting module T_A is said to be a *tilted algebra*.

In [4, Theorem 7.2], it has been shown that an artin algebra B is a tilted algebra if there is a component of the Auslander-Reiten quiver of B which contains all indecomposable projective modules and a finite complete slice.

Recall that a set $\mathcal{U}$ of indecomposable modules in a component $\mathcal{C}$ of the Auslander-Reiten quiver of an artin algebra is said to be a *complete slice* in $\mathcal{C}$ if it satisfies the following three conditions:

(i) For any indecomposable module X in $\mathcal{C}$, $\mathcal{U}$ contains precisely one module from the orbit $\{\tau^z X \mid z \in \boldsymbol{Z}\}$ under τ, τ^{-1}.

(ii) If there is a chain $X_0 \to X_1 \to \cdots X_r$ of indecomposable modules and non-zero maps with X_0, X_r in $\mathcal{U}$, then all X_i belong to $\mathcal{U}$.

(iii) There is no oriented cycle $U_0 \to U_1 \to \cdots \to U_r \to U_0$ of irreducible maps with all U_i in $\mathcal{U}$.

The aim of this note is to show that the condition (iii) in the definition of a complete slice is essentially dispensable, that is, to prove the following

THEOREM. *Let B be a basic artin algebra. Assume that there is a component $\mathcal{C}$ of the Auslander-Reiten quiver of B which contains all indecomposable projective modules, and that there is a finite set $\mathcal{U}=\{U_1, \cdots, U_n\}$ of indecom-*

Received November 18, 1981. Revised June 7, 1982.

posable modules in $\mathcal{C}$ *which satisfies the conditions* (i), (ii) *in the definition of a complete slice. Then* B *is either a tilted algebra or a local Nakayama algebra.*

At the same time, we shall provide a short proof of [4, Theorem 7.2] using the characterization of tilting modules due to Bongartz [2, Theorem 2.1].

Throughout this note, all modules are finitely generated and most modules are right modules. For an artin algebra A over the center C, denote by D the duality $\mathrm{Hom}_C(-, I)$, where I is the injective envelope of $C/\mathrm{rad}\, C$ over C, and by τ (resp. τ^{-1}) DTr (resp. TrD). We refer to [1] DTr and Auslander-Reiten sequences, and shall freely use the results of [1].

Proof of the Theorem.

Consider, first, the case in which $\tau U_i \cong U_i$ for some i. We claim that B is a local Nakayama algebra. (More generally, in [5] it will be shown that a basic artin algebra B is a local Nakayama algebra if there is an indecomposable module X such that $\tau X \cong X$ and the component of the Auslander-Reiten quiver of B which contains X is not *stable*). If B is simple, we are done. So we assume that B is not simple. Let $0 \to U_i \to E \to U_i \to 0$ be the Auslander-Reiten sequence. By the condition (ii), all indecomposable summands of E belong to $\mathcal{U}$. Let U_j be a summand of E. Three cases are possible:

(a) U_j is projective-injective. We get $\mathrm{rad}\, U_j \cong U_i \cong U_j/\mathrm{soc}\, U_j$, hence $\mathrm{top}\,(\mathrm{rad}\, U_j) \cong \mathrm{top}\, U_j$, this means that B is a local Nakayama algebra.

(b) U_j is not projective. We get a chain of irreducible maps $U_i \cong \tau U_i \to \tau U_j \to U_i$, hence by the conditions (i), (ii) $\tau U_j \cong U_j$.

(c) U_j is not injective. By the dual argument of (b), we get $\tau^{-1} U_j \cong U_j$, hence $\tau U_j \cong U_j$.

We claim that for any indecomposable module X in $\mathcal{C}$, either $\tau X \cong X$ or X is projective-injective. Let $X \not\cong U_i$ be an indecomposable module in $\mathcal{C}$. Note that there is a sequence $U_i = X_0, X_1, \cdots, X_r = X$ of indecomposable modules in $\mathcal{C}$ such that X_j's are pairwise non-isomorphic and for each j there is an irreducible map either from X_j to X_{j+1} or from X_{j+1} to X_j. By induction on r, we show that $X \cong U_k$ for some k and either $\tau X \cong X$ or X is projective-injective. We note that this has already been shown for $r=1$. Suppose $r>1$. By induction, for each $j<r$, $X_j \cong U_{k_j}$ for some k_j and either $\tau X_j \cong X_j$ or X_j is projective-injective. We have only to show $\tau X_{r-1} \cong X_{r-1}$, then our assertion follows from the above arguments. Suppose, on the contrary, that X_{r-1} is projective-injective. Then either $X_r \cong \mathrm{rad}\, X_{r-1}$ or $X_r \cong X_{r-1}/\mathrm{soc}\, X_{r-1}$. On the other hand, $\mathrm{rad}\, X_{r-1} \cong X_{r-2} \cong$

$X_{r-1}/\mathrm{soc}\, X_{r-1}$ since X_{r-2} can not be projective-injective. Hence $X_{r-2} \cong X_r$, a contradiction. Let P be an indecomposable projective module. By the assumption on $\mathcal{C}$, P belongs to $\mathcal{C}$, thus has to be projective-injective. Therefore, we get $\mathrm{rad}\, P \cong P/\mathrm{soc}\, P$, hence $\mathrm{top}\, P \cong \mathrm{top}\,(\mathrm{rad}\, P)$, this means that B is a local Nakayama algebra.

Next, assume that $\tau U_i \not\cong U_i$ for all i. Let $U = \bigoplus_{i=1}^{n} U_i$ and $A = \mathrm{End}\,(U)$. We claim that $D(U)$ is a tilting module and A is hereditary. Then our assertion follows from the Theorem of Brener-Butler (see [3] and [4]).

LEMMA 1 ([4]). $\mathrm{Ext}_B^1(U, U) = 0$.

PROOF. Since $\mathrm{Ext}_B^1(U, U)$ is a subgroup of $D\,\mathrm{Hom}_B(U, \tau U)$, it is sufficient to show that $\mathrm{Hom}_B(U, \tau U) = 0$. Suppose, on the contrary, that $\mathrm{Hom}_B(U_i, \tau U_j) \neq 0$ for some i, j. Using the Auslander-Reiten sequence ending in U_j, we get a chain $U_i \to \tau U_j \to * \to U_j$ of indecomposable modules and non-zero maps, hence by the conditions (i), (ii) $\tau U_j \cong U_j$, which contradicts our assumption.

PROPOSITION 2. *A is hereditary.*

PROOF. Denote by $\mathrm{add}\, U$ the category consisting of direct sums of direct summands of U. Let P_A be a projective A-module and X_A a submodule of P_A. We claim that X_A is also projective. Note that P_A is of the form $\mathrm{Hom}_B(U, U')$ for some U' in $\mathrm{add}\, U$. Let $f_1, \cdots, f_r \in X_A$ be generators and put

$$f = (f_1 \cdots f_r) : \bigoplus_{i=1}^{r} U \longrightarrow U' .$$

Then $X_A \cong \mathrm{Im}\,(\mathrm{Hom}_B(U, f))$. By the condition (ii), we get a decomposition $\mathrm{Ker}\, f = K \oplus K'$ such that $K \in \mathrm{add}\, U$ and $\mathrm{Hom}_B(U, K') = 0$. Taking a push-out, we get the commutative diagrm with exact rows

$$\begin{array}{cccccccccc}
\text{(a)} & 0 & \longrightarrow & K \oplus K' & \xrightarrow{\binom{\alpha}{\beta}} & \bigoplus_{i=1}^{r} U & \xrightarrow{f} & \mathrm{Im}\, f & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle (1\ 0)} & & \downarrow & & \Vert & & \\
\text{(b)} & 0 & \longrightarrow & K & \longrightarrow & * & \longrightarrow & \mathrm{Im}\, f & \longrightarrow & 0
\end{array}$$

By the condition (ii) $\mathrm{Im}\, f \in \mathrm{add}\, U$, hence by Lemma 1 the sequence (b) splits. Therefore, α is a split monomorphism. Applying the functor $\mathrm{Hom}_B(U, -)$ on the sequence (a), we get a split exact sequence

$$0 \longrightarrow \mathrm{Hom}_B(U, K) \longrightarrow \mathrm{Hom}_B(U, \bigoplus_{i=1}^{r} U) \longrightarrow X_A \longrightarrow 0 ,$$

which completes the proof.

LEMMA 3. $\operatorname{inj} \dim U \leqq 1$.

PROOF. Suppose that U_i is not injective, and let $P_1 \to P_0 \to \tau^{-1}U_i \to 0$ be the minimal projective resolution. By the definition of τ, we get the exact sequence

$$0 \longrightarrow U_i \longrightarrow D\operatorname{Hom}_B(P_1, B) \longrightarrow D\operatorname{Hom}_B(P_0, B) \longrightarrow D\operatorname{Hom}_B(\tau^{-1}U_i, B) \longrightarrow 0.$$

Since $D\operatorname{Hom}_B(P_j, B)$ are injective, it is sufficient to show that $\operatorname{Hom}_B(\tau^{-1}U_i, B) = 0$. Suppose, on the contrary, that $\operatorname{Hom}_B(\tau^{-1}U_i, P) \neq 0$ for some indecomposable projective module P. Note that P is of the form $\tau^r U_j$ for some j and some non-negative integer r. Using the Auslander-Reiten sequences starting from U_i and $\tau^s U_j$ with $1 \leqq s \leqq r$, we get a chain $U_i \to * \to \tau^{-1}U_i \to \tau^r U_j \to \cdots \to U_j$ of indecomposable modules and non-zero maps, hence by the conditions (i), (ii) $\tau^{-1}U_i \cong U_i$, which contradicts our assumption.

Note that by the assumption on $\mathcal{C}$, n is greater than or equal to the number of indecomposable projective modules. The next proposition due to Bongartz [2, Theorem 2.1] together with Lemmas 1, 3 completes the proof of the Theorem.

PROPOSITION (Bongartz [2]). *Let A be an artin algebra with m simple modules and $T = \bigoplus_{i=1}^{n} T_i$ a module with pairwise non-isomorphic indecomposable T_i's. Assume* $\operatorname{proj} \dim T \leqq 1$ *and* $\operatorname{Ext}_A^1(T, T) = 0$. *Then $n \leqq m$, and $n = m$ if and only if T is a tilting module.*

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