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COUNTABLE-POINTS COMPACTIFICATIONS OF PRODUCT SPACES

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§1. Introduction.

Throughout this paper all spaces are assumed to be completely regular and T_1. A compactification αX of a space X is said to be a countable-points compactification (abbreviated to CCF) if the cardinality of the remainder $\alpha X-X$ is at most countable. Every locally compact space or, by L. Zippin [7], every rim-compact, Čech-complete separable metrizable space has a CCF. Thus, the problem to characterize those spaces which have a CCF was raised by K. Morita in [4], and for the case of metric spaces it was solved by the author [5].

In the present paper we consider the above problem on product spaces. Indeed, even for the case of a separable metrizable space X with a CCF and a compact space Y the product space $X\times Y$ does not have a CCF in general. More precisely, we shall establish the following theorems.

THEOREM 1. *Let X be a space having a CCF, and Y a zero-dimensional compact metrizable space. Then $X\times Y$ has also a CCF.*

THEOREM 2. *Let X be a paracompact space and Y a compact space. Then $X\times Y$ has a CCF iff X is locally compact or X has a CCF and Y is zero-dimensional and metrizable.*

Theorems 1 and 2 will yield further the following theorem which characterizes a product space of paracompact spaces to have a CCF.

THEOREM 3. *Let X and Y be paracompact spaces. Then $X\times Y$ has a CCF iff one of the following three conditions is satisfied:*

(a) *X and Y are both locally compact;*

(b) *one of X and Y is zero-dimensional, locally compact, separable metrizable, and the other has a CCF.*

(c) *X and Y are both zero-dimensional, Čech-complete separable metrizable.*

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§ 2. Preliminaries and proofs of Theorems 1 and 2.

A space is said to be rim-compact if it has a base consisting of open sets with compact boundaries. A zero-dimensional space means a space with a base consisting of open-and-closed sets.

If a space X has a CCF, then every closed set or every open set of X has also a CCF, and X is known to be rim-compact and Čech-complete. While Čech-completeness is (countably) productive, rim-compactness is not. Hence the following lemma, proved in [6], will be useful.

LEMMA 2.1. *A product space $X\times Y$ is rim-compact iff one of the three conditions below is satisfied*:

(i) *X and Y are both locally compact*;

(ii) *one of X and Y is locally compact and zero-dimensional and the other is rim-compact*;

(iii) *X and Y are both zero-dimensional.*

For a space X $R(X)$ denotes the set of all points at which X is not locally compact. The following lemma was proved in [5].

LEMMA 2.2. *If a paracompact space X has a CCF, then $R(X)$ is Lindelöf.*

With this lemma we easily have

LEMMA 2.3. *Let $X=\vee X_\alpha$ be the topological sum of paracompact spaces, each of which has a CCF. Then X has a CCF iff all but a countable number of X_α's are locally compact.*

PROOF OF THEOREM 1. Assume that X has a CCF and Y is zero-dimensional, compact metrizable. Then there is a sequence $\{\mathcal{V}_n \mid n\in N\}$ of finite disjoint open covers of Y such that $\mathcal{V}_{n+1}$ refines $\mathcal{V}_n$ for $n\in N$ and $\{St(y, \mathcal{V}_n) \mid n\in N\}$ is a local base at each point y of Y, where N=the set of all natural numbers. Let αX be a CCF of X, and $\alpha X-X=\{p_n \mid n\in N\}$. Let us put

$$\mathcal{D}=\{\{(x, y)\} \mid (x, y)\in X\times Y\}\cup\{\{p_n\}\times V \mid V\in\mathcal{V}_n,\ n\in N\}.$$

Then $\mathcal{D}$ is an upper semi-continuous decomposition of $\alpha X\times Y$. To see this, we show that for any open set G of $\alpha X\times Y$ the set $H=\cup\{D\in\mathcal{D} \mid D\subset G\}$ is open in $\alpha X\times Y$. Let $(x, y)\in H$. Then we can choose an open set U of αX and $V\in\mathcal{V}_n$ for some n so that $(x, y)\in U\times V\subset G$, according as $(x, y)\in X\times Y$ or $(x, y)\in\{p_n\}\times V$, $V\in\mathcal{V}_n$. Hence, in any case we have

$$(x, y) \in (U - \{p_1, \cdots, p_{n-1}\}) \times V \subset H,$$

that is, H is open in $\alpha X \times Y$. Let Z be the decomposition space of $\alpha X \times Y$ with respect to $\mathscr{D}$. Then the above shows that Z is a CCF of $X \times Y$ as required. This proves the theorem.

A space X is said to be locally countably compact if every point of X has its neighborhood whose closure is countably compact. Since for paracompact spaces, more generally for iso-compact spaces in the sense of Bacon [3], every countably compact closed subset is compact, Theorem 2 follows from Theorem 1 and the next theorem.

THEOREM 2.4. *Let X be a space which is not locally countably compact and Y a compact space. If $X \times Y$ has a CCF, then Y is zero-dimensional and metrizable.*

PROOF. Suppose that $X \times Y$ has a CCF. Then $X \times Y$ is rim-compact and Čech-complete. By assumptions on X and Y and Lemma 2.1 Y is zero-dimensional. Let x_0 be the point at which X is not locally countably compact. Since X is Čech-complete, by [2] X is a space of point countable type. Hence, there exists a compact subset K of X that contains x_0 and admits a countable neighborhood base $\{U_n \mid n \in N\}$ with $U_{n+1} \subset U_n$ for $n \in N$. Since $\mathrm{Cl}\, U_n$ is not countably compact, we can select $n_1 < \cdots < n_i < \cdots$ of natural numbers and a countably infinite discrete closed set $\{x_{ik} \mid k \in N\}$ for each $i \in N$ such that $x_{ik} \in U_{n_i} - U_{n_{i+1}}$ for $i, k \in N$. Let us put

$$X_0 = \{x_{ik} \mid i, k \in N\} \cup K.$$

Then X_0 is a σ-compact closed subset of X, and is not locally compact. Let us rewrite X_0 as $\{x_n \mid n \in N\} \cup K$. Since $X_0 \times Y$ is closed in $X \times Y$, $X_0 \times Y$ has a CCF $\alpha(X_0 \times Y)$. Let $\alpha(X_0 \times Y) - X_0 \times Y = \{q_i \mid i \in N\}$. Note that each q_i is G_δ in $\alpha(X_0 \times Y)$ since $X_0 \times Y$ is σ-compact. Therefore, q_i has a countable neighborhood base $\{V_{ik} \mid k \in N\}$ of open sets in $\alpha(X_0 \times Y)$ since $\alpha(X_0 \times Y)$ is compact Hausdorff. Finally let us put

$$\mathscr{B} = \{p_Y((X_0 \times Y) \cap V_{ik}) \mid i, k \in N\},$$

where p_Y is the projection from $X_0 \times Y$ to Y. Now we show that $\mathscr{B}$ is a base of Y. Let y be any point of Y and G an open set of Y with $y \in G$. Let W be an open set of $\alpha(X_0 \times Y)$ such that $X_0 \times G = W \cap (X_0 \times Y)$. Since X_0 is not locally compact, the set

$$(W \cap \mathrm{Cl}_{\alpha(X_0 \times Y)}(X_0 \times \{y\})) - X_0 \times Y$$

is non-empty, and contains some q_i. Then for some k we have $V_{ik} \subset W$ and

$V_{ik}\cap(X_0\times\{y\})\neq\emptyset$. Hence it follows that

$$y\in p_Y((X_0\times Y)\cap V_{ik})\subset G,$$

which shows that $\mathcal{B}$ is a base of Y. Thus, Y is metrizable and the proof is completed.

§ 3. Proof of Theorem 3.

With the aid of Theorem 2.4 as well as the method of its proof, we shall obtain further the following result.

THEOREM 3.1. *Let X be a space which is not locally countably compact, and Y a paracompact space. If $X\times Y$ has a CCF, then Y is zero-dimensional, Čech-complete separable metrizable.*

PROOF. Suppose that $X\times Y$ has a CCF. Then Y is zero-dimensional and Čech-complete. Let $X_0=\{x_n\mid n\in N\}\cup K$ be as constructed in the proof of Theorem 2.4. Then $X_0\times Y$ has a CCF $\alpha(X_0\times Y)$. Let $\alpha(X_0\times Y)-X_0\times Y=\{q_i\mid i\in N\}$.

CLAIM 1. *$R(Y)$ is separable metrizable.*

PROOF OF CLAIM 1. Since X_0 is σ-compact, $X_0\times Y$ is paracompact, and so, by Lemma 2.2 $R(X_0\times Y)=\mathrm{Cl}_{\alpha(X_0\times Y)}(\{q_i\mid i\in N\})-\{q_i\mid i\in N\}$ is Lindelöf. Therefore, the countable space $\{q_i\mid i\in N\}$ is a space of countable type [2], and hence, it is metrizable. Then we see that each q_i has a countable neighborhood base $\{V_{ik}\mid k\in N\}$ of open sets in $\mathrm{Cl}_{\alpha(X_0\times Y)}(\{q_i\mid i\in N\})$. Now, using $R(X_0\times Y)=(R(X_0)\times Y)\cup(X_0\times R(Y))$ it can be verified by the same way as in the proof of Theorem 2.4 that the collection

$$\{p_Y((X_0\times R(Y))\cap V_{ik})\mid i,\ k\in N\}$$

is a base of the subspace $R(Y)$. Hence, $R(Y)$ is separable metrizable.

CLAIM 2. *$Y-R(Y)$ is an F_σ subset of Y.*

PROOF OF CLAIM 2. Let us put and rewrite

$$\begin{aligned}D&=\{q_i\mid q_i\notin\mathrm{Cl}_{\alpha(X_0\times Y)}(X_0\times R(Y)),\ i\in N\}\\&=\{q_{i_k}\mid k\in N\}.\end{aligned}$$

Then for each $q_{i_k}\in D$ there is an open F_σ subset W_k of $\alpha(X_0\times Y)$ such that $q_{i_k}\in W_k$, $W_k\cap(X_0\times R(Y))=\emptyset$. Let us set

$$G_{nk}=p_Y((\{x_n\}\times Y)\cap W_k) \quad \text{for} \quad n, k\in N,$$

and

$$\mathcal{G}=\{G_{nk}\mid n, k\in N\}.$$

Then it should be noted that each G_{nk} is an open F_σ subset of Y and $G_{nk}\cap R(Y)=\emptyset$. We show that $\mathcal{G}$ covers $Y-R(Y)$. Let $y\in Y-R(Y)$. Since $R(Y)$ is closed, there are open sets U, V of Y such that $y\in U$, $R(Y)\subset V$ and $U\cap V=\emptyset$. Take open sets G, H of $\alpha(X_0\times Y)$ such that $X_0\times U=G\cap(X_0\times Y)$ and $X_0\times V=H\cap(X_0\times Y)$. Then as is seen before, $G\cap \mathrm{Cl}_{\alpha(X_0\times Y)}(X_0\times\{y\})-X_0\times Y$ contains some q_i. Since $G\cap H=\emptyset$ and $X_0\times R(Y)\subset H$, we have $q_i\notin \mathrm{Cl}_{\alpha(X_0\times Y)}(X_0\times R(Y))$. Hence $q_i=q_{i_k}\in D$ for some $k\in N$, and since $q_{i_k}\in \mathrm{Cl}_{\alpha(X_0\times Y)}(X_0\times\{y\})$ and $K\times\{y\}$ is compact, we have $(x_n, y)\in W_k$ for some $n\in N$. Hence $y\in G_{nk}\in\mathcal{G}$, and the claim follows.

Since $R(X_0\times Y)$ is Lindelöf, so is Y. Hence by Claim 2 the locally compact subspace $Y-R(Y)$ is also Lindelöf, and so it is σ-compact. Therefore for a compact subset C_i, $i\in N$ we have $Y=R(Y)\cup\cup\{C_i\mid i\in N\}$. Since each C_i is metrizable by Theorem 2.4, in view of Claim 1 Y is a union of a countable number of separable metrizable subspaces. Since Y is Čech-complete, it is separable metrizable by [1]. This completes the proof of the theorem.

Let us now prove Theorem 3.

PROOF OF THEOREM 3. Since the "only if" part directly follows from Theorem 3.1, we shall prove the "if" part. Assume (b) in the theorem and that X has a CCF and Y is zero-dimensional, locally compact separable metrizable. Then $Y\cong$(is homeomorphic to)$\vee C_i$, the topological sum of a countable number of zero-dimensional compact metrizable subspaces C_i, $i\in N$. Since $X\times Y\cong\vee(X\times C_i)$, and each $X\times C_i$ has a CCF by Theorem 1, $X\times Y$ has also a CCF by Lemma 2.3. Assume (c). Then $X\times Y$ is zero-dimensional Čech-complete separable metrizable. Thus, by [7] $X\times Y$ has a CCF. This proves the theorem.

REMARK. In view of Theorem 3, it should be noted that there exists a paracompact space X such that X has a CCF but $R(X)$ is not metrizable. Indeed, let $X=\beta R-N$, where βR=the Stone-Čech compactification of the real line R. Then X has a CCF and $R(X)=\beta N-N$ [8, Example 3], but $X=(\beta R-R)\cup(R-N)$, which is σ-compact.

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