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A BOOLEAN POWER AND A DIRECT PRODUCT OF ABELIAN GROUPS

By

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A group means an abelian group in this paper. A Boolean power and a direct product of groups consist of all global sections of groups in some Boolean extensions $V^{(B)}$. We shall study about a homomorphism h whose domain is a group consisting of all the global sections of a group in $V^{(B)}$. We investigate two cases: one of them is that the range of h is a slender group, which is related to a torsion-free group, and the other is that the range of h is an infinite direct sum, which is related to a torsion group. We extend a few theorems which have been obtained in [4] and [5]. As in [5], we not only extend theorems, but improve them and give a good standing point of view.

We refer the reader to [9] or [1], for a Boolean extension $V^{(B)}$. We shall use notations and terminologies in [5], [6] and [7]. Throughout this paper, $\boldsymbol{B}$ is a complete Boolean algebra and $\mathcal{F}$ is the set of all countably complete maximal filters on $\boldsymbol{B}$. We do not mention these any more. $\check{x}$ is the element of $V^{(B)}$ such that dom $\check{x}=\{\check{y}\,;\,y\in x\}$ and range $x\subseteq\{\mathbf{1}\}$. As noted in [5], "$\hat{x}$" in [1] means our "$\check{x}$". $\hat{x}=\{y\,;\,[\![y\in x]\!]=\mathbf{1}$ and $y\in V^{(B)}\}$ for $x\in V^{(B)}$, where $V^{(B)}$ is separated. For $b\in\boldsymbol{B}$ and a group A in $V^{(B)}$, i.e. $[\![A$ is a group$]\!]=\mathbf{1}$, $\hat{A}^b$ is the subgroup of $\hat{A}$ such that $x\in\hat{A}^b$ iff $x\in\hat{A}$ and $-b\leqq[\![x=0]\!]$, where 0 is the unit of A. By this notation, $\hat{A}=\hat{A}^1$. For $x\in\hat{A}$, x^b is the element of $\hat{A}^b$ such that $b\leqq[\![x=x^b]\!]$.

1. A general setting about a complete Boolean algebra

Let $\Phi(b)$ be a property of $b\in\boldsymbol{B}$ which satisfies the following conditions:

(1) if $\{b_n\,;\,n\in N\}$ is a pairwise disjoint subset of $\boldsymbol{B}$, there exists k such that $\Phi(\bigvee_{n\geqq k}b_n)$ and $\Phi(b_n)$ hold for each $n\geqq k$;

(2) if $b\wedge c=\mathbf{0}$, $\Phi(b)$ and $\Phi(c)$ hold, then $\Phi(b\vee c)$ holds.

Let S be the subset of $\boldsymbol{B}$ such that $b\in S$ iff $\Phi(b)$ does not hold and $c\wedge c'=\mathbf{0}$ implies $\Phi(c)$ or $\Phi(c')$ for any $c, c'\leqq b$.

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LEMMA 1. *Let* F^b *be the subset of* $\boldsymbol{B}$ *defined by:* $c\in F^b$ *iff* $\Phi(b\wedge c)$ *does not hold. Then,* $F^b\in\mathfrak{F}$ *for every* $b\in S$.

PROOF. We prove only the countable completeness. Let $b_n\in F^b$ for $n\in N$. Let $c_1=\mathbf{0}$ and $c_{n+1}=\bigwedge_{k=1}^{n} b_k-b_{n+1}$. Then, $b_1=\bigvee_{n\in N} c_n\vee\bigwedge_{n\in N} b_n$. By the condition (1) and (2) of Φ and the property of S, $\Phi(b\wedge\bigvee_{n\in N} c_n)$ and so $\Phi(b\wedge\bigwedge_{n\in N} b_n)$ does not hold.

LEMMA 2. *Let* M *be a maximal pairwise disjoint subfamily of* S. *Then,* M *is finite and* $\Phi(c)$ *holds for any* c *such that* $c\wedge\bigvee M=\mathbf{0}$.

PROOF. By the condition of Φ, M is finite. Suppose that there exists c such that $\Phi(c)$ does not hold and $c\wedge\bigvee M=\mathbf{0}$. By the maximality of M, there is no element of S below c. So, there are b_0, $c_0\leqq c$ such that $b_0\wedge c_0=\mathbf{0}$ and $\Phi(b_0)$ nor $\Phi(c_0)$ does not hold. Then, take b_1, $c_1\leqq c_0$ with the same property of b_0 and c_0. In such a way, we obtain a pairwise disjoint family $\{b_n;\ n\in N\}$ such that $\Phi(b_n)$ does not hold for any $n\in N$, which is a contradiction.

2. Hom($\hat{A}$, G)

Let F be a maximal filter on $\boldsymbol{B}$. For a group A in $V^{(\boldsymbol{B})}$, $\hat{A}/F$ is the quotient of $\hat{A}$ by the equivalence relation $\sim_F$ such that $x\sim_F y$ iff $[\![x=y]\!]\in F$. In the case $A=\check{X}$, $\hat{A}$ is known as a Boolean power $X^{(\boldsymbol{B})}$ and $\hat{A}/F$ is a Boolean ultrapower $X^{(\boldsymbol{B})}/F$. (Ref. [8]) In the case that $\boldsymbol{B}=\boldsymbol{P}(I)$ and $\hat{A}=\prod_{i\in I} A_i$, where A is defined by a natural way, $\hat{A}/F$ is known as an ultraproduct $\prod_{i\in I} A_i/F$. (Ref. [2]) However, the following fact is enough to read the main part of this paper. Let K be the subgroup of $\hat{A}$ defined by: $x\in K\leftrightarrow[\![x=0]\!]\in F$. Then, $\hat{A}/F\cong\hat{A}/K$, where the right part is the quotient group.

THEOREM 1. *Let* A *be a group in* $V^{(\boldsymbol{B})}$ *and* G *a slender group. Then,* $\mathrm{Hom}(\hat{A}, G)\cong\bigoplus_{F\in\mathfrak{F}}\mathrm{Hom}(\hat{A}/F, G)$ *holds.*

PROOF. Let h be a homomorphism from $\hat{A}$ to G and $\Phi(b)$ the property "$h''\hat{A}^b=0$". Let $\{b_n;\ n\in N\}$ be a pairwise disjoint subset of $\boldsymbol{B}$ and $x_n\in\hat{A}^{b_n}$ for each $n\in N$. Think of the homomorphism $g: \boldsymbol{Z}^N\to\hat{A}$ such that $g(\sum_{n\in N} a_n\boldsymbol{e}_n)=\sum_{n\in N} a_n x_n$, where $x=\sum_{n\in N} a_n x_n$ is the element of $\hat{A}^b$ such that $b=\bigvee_{n\in N} b_n$ and $b_n\leqq[\![x=a_n x_n]\!]$ for each $n\in N$, and apply the slenderness of G to $h\cdot g$, then $h\cdot g(\boldsymbol{e}_n)=0$ and so $h(x_n)=0$ for almost all n. Hence, there exists k such that $\Phi(b_n)$ for any $n\geqq k$ and $h(\sum_{n\geqq k} x_n)=0$, by Specker's theorem. (Ref. Prop. 1 of [5] or

Lem. 94.1 of [7])

Therefore, Φ satisfies the conditions (1) and (2) of § 1. Hence, Lem. 1 and Lem. 2 hold for this Φ. Now, let $M=\{b_1 \cdots b_n\}$ and $b_0=\mathbf{1}-\bigvee M$. Let $h_i: \hat{A}/F^{b_i}\to G$ be defined by: $h_i([x]_i)=h(x^{b_i})$, where $[x]_i$ is the equivalence class containing x with respect to F^{b_i}, for each $1\leqq i\leqq n$. Since $[\![x=0]\!]\in F^{b_i}$ implies $h(x^{b_i})=0$ for $x\in\hat{A}^{-[x=0]}$, h_i is well-defined for $1\leqq i\leqq n$. For $x\in\hat{A}$, $h(x)=h(\sum_{i=0}^{m} x^{b_i})=\sum_{i=0}^{m} h(x^{b_i})=\sum_{i=1}^{m} h(x^{b_i})=\sum_{i=1}^{m} h_i([x]_i)$. The linear independence of $\{\mathrm{Hom}(\hat{A}/F, G);\ F\in\mathcal{F}\}$ is clear. Now, the proof is completed.

In view of the paragraph preceding Th. 1, Th. 1 includes Th. 2 of [5] and Th. 94.4 of [7]. We express these as corollaries.

COROLLARY 1. *Let A be a group and G a slender group. Then,* $\mathrm{Hom}(A^{(\boldsymbol{B})}, G)\cong\bigoplus_{F\in\mathcal{F}}\mathrm{Hom}(A^{(\boldsymbol{B})}/F, G)$.

COROLLARY 2. *Let A_i be a group for each $i\in I$ and G a slender group. Then,* $\mathrm{Hom}(\prod_{i\in I} A_i, G)\cong\bigoplus_{F\in\mathcal{F}}\mathrm{Hom}(\prod_{i\in I} A_i/F, G)$.

If the cardinality of A is less than the least measurable cardinal M_c or $\boldsymbol{B}$ satisfies $M_c-c.c.$, $A^{(\boldsymbol{B})}/F\cong A$ holds, so Cor. 1 is an extended form of Th. 2 of [5]. If the cardinality of I is less than M_c, then every $F\in\mathcal{F}$ is principal. Therefore, $\mathrm{Hom}(\prod_{i\in I} A_i, G)\cong\bigoplus_{i\in I}\mathrm{Hom}(A_i, G)$, which is a famous theorem. If the cardinalities of the A_i are bounded below M_c, then $\prod_{i\in I} A_i/F\cong A_i$ for some i, which was used in the proof of Cor. 2 of [5].

By Cor. 2, we can calculate a dual group of $\prod_{\lambda_1}\bigoplus_{\lambda_2}\cdots\prod_{\lambda_{2n-1}}\boldsymbol{Z}$. Now, we shall do it in a simple case. Let $j_F: V\to M_F$ be the elementary embedding, where F is a countably complete maximal filter on $\boldsymbol{P}(\lambda)$ and M_F is the transitive model which is isomorphic to V^λ/F. (Ref. [10]) Let $\boldsymbol{B}=\boldsymbol{P}(\lambda_1)$, then

$$\begin{aligned}\mathrm{Hom}(\prod_{\lambda_1}\bigoplus_{\lambda_2}\boldsymbol{Z}, \boldsymbol{Z}) &\cong\bigoplus_{F\in\mathcal{F}}\mathrm{Hom}(\prod_{\lambda_1}(\bigoplus_{\lambda_2}\boldsymbol{Z})/F, \boldsymbol{Z})\\ &\cong\bigoplus_{F\in\mathcal{F}}\mathrm{Hom}(\bigoplus_{j_F(\lambda_2)}\boldsymbol{Z}, \boldsymbol{Z})\\ &\cong\bigoplus_{F\in\mathcal{F}}\prod_{j_F(\lambda_2)}\boldsymbol{Z}.\end{aligned}$$

In the calculation, we have used the absoluteness of direct sums. Unfortunately, direct products are not absolute among transitive models. So, for the calculation of $\mathrm{Hom}(\prod_{\lambda_1}\bigoplus_{\lambda_2}\prod_{\lambda_3}\boldsymbol{Z}, \boldsymbol{Z})$, we must prepare a proposition which is obtained by modifying Cor. 2. That can be done, if we notice the fact that only the countably completeness of $\boldsymbol{B}$, not the full completeness, has been used in the proof of Th. 1.

In this paper, we deal with the case that $\boldsymbol{B}$ is a complete Boolean algebra. Therefore, unless $\boldsymbol{B}$ is very large, every element of $\mathcal{F}$ is principal. Concerning a Boolean power, a countably complete Boolean algebra can give us interesting groups, for there can be a non-principal c.c. max-filter on a non-complete but countably complete and small Boolean algebra.

3. A homomorphism into an infinite sum

In this section, we shall extend some results of [4]. We do not prove the next lemma, because the proof is in [3] and [4], and the essential idea of it will be developed in the proof of Lem. 5. For $X \subseteq I$, we identify $\prod_{i\in X} A_i$ with the subgroup of $\prod_{i\in I} A_i$ such that $x \in \prod_{i\in X} A_i$ iff $x \in \prod_{i\in I} A_i$ and $x(i)=0$ for each $i \notin X$. Similarly, we do $\bigoplus_{i\in X} A_i$ with the subgroup of $\bigoplus_{i\in I} A_i$.

LEMMA 3. (Chase [3]) *Let $h: \prod_{i\in N} A_i \to \bigoplus_{j\in J} G_j$ $(=G)$ be a homomorphism. Then, there exist an integer $n>0$ and finite subsets $F \subseteq N$ and $J' \subseteq J$ such that*

$$h''n \prod_{i\in N-F} A_i \subseteq \bigoplus_{j\in J'} G_j + \bigcap_{n\in N} nG .$$

THEOREM 2. *Let A be a group in $V^{(B)}$ and $h: \hat{A} \to \bigoplus_{j\in J} G_j$ $(=G)$ a homomorphism. Then, there exist $F_1, \cdots, F_m \in \mathcal{F}$, an integer $n^*>0$ and a finite subset J^* of J that satisfy the following condition: Let K be the subgroup of $\hat{A}$ such that $x \in K$ iff $[\![x=0]\!] \in F_i$ for each $1 \leqq i \leqq m$, then $h''n^*K \subseteq \bigoplus_{j\in J^*} G_j + \bigcap_{n\in N} nG$.*(*)

Let $\Phi(b)$ be the property "There exist an integer $n>0$ and a finite subset J' of J such that $h''n\hat{A}^b \subseteq \bigoplus_{j\in J'} G_j + \bigcap_{n\in N} nG$."

LEMMA 4. *This Φ satisfies the conditions* (1) *and* (2) *in* § 1.

PROOF. Let $b = \bigvee_{n\in N} b_n$, for a pairwise disjoint family $\{b_n ; n \in N\}$. Then, $\hat{A}^b \cong \prod_{n\in N} \hat{A}^{b_n}$. $b \leqq c$ and $\Phi(c)$ imply $\Phi(b)$. Hence, Φ satisfies the condition (1), by virtue of Lem. 3. Φ satisfies the condition (2) clearly.

LEMMA 5. *There exist an integer $n^*>0$ and a finite subset J^* of J such that, for any b which satisfies $\Phi(b)$, $h''n^*\hat{A}^b \subseteq \bigoplus_{j\in J^*} G_j + \bigcap_{n\in N} nG$.*

(*) Here we admit $m=0$ and in such a case $K=\hat{A}$.

PROOF. Suppose the negation of the conclusion. Let $\pi_j: \bigoplus_{j\in J} G_j \to G_j$ be the projection for $j\in J$. We construct $b_k\in \boldsymbol{B}$, $a_k\in \hat{A}$, $n_k\in N$, $j_k\in J$ and a finite subset J_k of J satisfying the following conditions:

(1) $\langle b_k;\, k\in N\rangle$ are pairwise disjoint and $\Phi(b_k)$ for $k\in N$;

(2) $a_k\in n_{k-1}!\,\hat{A}^{b_k}$ and $\pi_{j_k}h(a_k)\notin n_k!\,G_{j_k}$ and $\pi_{j_i}h(a_k)=0$ for each $i<k$;

(3) $h''n_{k-1}!\,\hat{A}^b\subseteq \bigoplus_{j\in J_{k-1}} G_j+\bigcap_{n\in N} nG$, where $b=\bigvee_{i=1}^{k-1} b_i$;

(4) $j_k\in J_k$ and $j_k\notin J_i$ for $i<k$;

(5) $\langle n_k;\, k\in N\rangle$ and $\langle J_k;\, k\in N\rangle$ are increasing.

Suppose that we have already defined b_i, a_i, n_i, j_i and J_i for $i\leqq k$ satisfying the above conditions. By the hypothesis, there exists b_{k+1} such that $b_{k+1}\wedge \bigvee_{i=1}^{k} b_i=\mathbf{0}$, $\Phi(b_{k+1})$ and $h''n_k!\,\hat{A}^{b_{k+1}}\nsubseteq \bigoplus_{j\in J_k} G_j+\bigcap_{n\in N} nG$. So, there exists $a_{k+1}\in n_k!\,\hat{A}^{b_{k+1}}$ such that $h(a_{k+1})\notin \bigoplus_{j\in J_k} G_j+\bigcap_{n\in N} nG$. Hence, there are $j_{k+1}\notin J_k$ and $n>n_k$ such that $\pi_{j_{k+1}}h(a_{k+1})\notin n!\,G_{j_{k+1}}$. Let $J'=J_k\cup\{j;\, \pi_j h(a_{k+1})\neq 0\}$. By the property of b_{k+1}, there exist n_{k+1} and a finite subset J_{k+1} such that $n<n_{k+1}$ and $J'\subseteq J_{k+1}$ and $h''n_{k+1}!\,\hat{A}^{b_{k+1}}\subseteq \bigoplus_{j\in J_{k+1}} G_j+\bigcap_{n\in N} nG$. $\sum_{k\in N} a_k$ exists in $\hat{A}$ and so let it be a. Then, $a-\sum_{i=1}^{k} a_i\in n_k!\,A$ and $\pi_{j_k}h(a_k)\notin n_k!\,G_{j_k}$ and $\pi_{j_k}h(a_i)=0$ for each $i<k$. Hence, $\pi_{j_k}h(a)=\pi_{j_k}h(a-\sum_{i=1}^{k} a_i)+\pi_{j_k}h(a_k)\neq 0$ for each k. Since $k\neq k'$ implies $j_k\neq j_{k'}$, it is a contradiction.

PROOF OF TH. 2. By Lem. 1, Lem. 2 and Lem. 4, M is finite and so let $M=\{b_1, \cdots, b_m\}$ and $b_0=\mathbf{1}-\bigvee M$. Let $F_i=F^{b_i}$ for $1\leqq i\leqq m$. Now, the theorem is clear by Lem. 5 and the fact that $x\in K$ implies $x\in \hat{A}^b$ for some b which satisfies $\Phi(b)$.

For a Group A, $\bar{A}$ denotes the corresponding Hausdorff group $A/\bigcap_{n\in N} nA$.

LEMMA 6. *For a group A in $V^{(B)}$, $\hat{\bar{A}}\cong\hat{\bar{A}}$.*

PROOF. By the absoluteness of N, $\bigcap_{n\in N} n\hat{A}\cong \bigcap_{n\in N} \widehat{nA}$. Hence, $\bar{\hat{A}}\cong \hat{A}/\bigcap_{n\in N} n\hat{A}\cong \hat{A}/\bigcap_{n\in N} \widehat{nA}\cong\hat{\bar{A}}$.

Let F be a maximal filter on $\boldsymbol{B}$ and $K_F^{\hat{A}}$ the subgroup of $\hat{A}$ such that $x\in K_F^{\hat{A}}$ iff $[x=0]\in F$.

LEMMA 7. *$nx\in K_F^{\hat{A}}$ implies $nx\in nK_F^{\hat{A}}$, where n is an integer.*

PROOF. Let $b=[nx=0]$. Let x' be the element of $\hat{A}$ such that $-b\leqq[x'=x]$

and $b \leqq [\![x'=0]\!]$. Then, $x' \in K_F^{\hat{A}}$ and $nx'=nx$.

LEMMA 8. *Let* $\pi: \hat{A} \rightarrow \hat{\hat{A}}$ $(\cong \bar{A})$ *be the canonical homomorphism. Then,* $\pi'' K_F^{\hat{A}} = K_F^{\hat{\hat{A}}}$.

PROOF. $\pi'' K_F^{\hat{A}} \subseteq K_F^{\hat{\hat{A}}}$ is obvious. Let $x \in K_F^{\hat{\hat{A}}}$. Then, there exists y in $\hat{A}$ such that $\pi(y)=x$. So, there exists b such that $b \in F$ and $b \leqq [\![x=0]\!]$. Let y' be the element of $\hat{A}$ such that $-b \leqq [\![y'=y]\!]$ and $b \leqq [\![y'=0]\!]$. Then, $\pi(y')=\pi(y)$ and $y' \in K_F^{\hat{A}}$.

LEMMA 9. *Let A be a torsion group in* $V^{(B)}$, *then* $\hat{A}/F$ *is also a torsion group for* $F \in \mathcal{F}$.

PROOF. Let $a \in \hat{A}$, then $\bigvee_{n \in N} [\![na=0]\!] = [\![\exists n \in N(na=0)]\!] = \mathbf{1}$. By the countable completeness of F, $[\![na=0]\!] \in F$ for some $n \in N$. So, $\hat{A}/F$ is a torsion group.

THEOREM 3. *Let A be a torsion group in* $V^{(B)}$. *Then, for each direct sum decomposition* $\bigoplus_{j \in J} G_j$ *of* $\hat{A}$, $\bar{G}_j$ *is a torsion group for almost all* $j \in J$.

PROOF. Applying Th. 2 directly, we have $F_1, \cdots, F_m \in \mathcal{F}$, an integer n and a finite subset J' of J such that $nK \subseteq \bigoplus_{j \in J'} G_j + \bigcap_{n \in N} nG$, where K and G are the same as Th. 2. Let $\pi: G \rightarrow \bar{G}$ be the canonical homomorphism. Then, $\pi'' G_j \cong \bar{G}_j$ for each $j \in J$ and $n\pi'' K \subseteq \bigoplus_{j \in J'} \pi'' G_j$.

Let $\psi: \bar{G}$ $(=\hat{\hat{A}}) \rightarrow \bar{G}/\pi'' K$ be the canonical homomorphism. Then, the restriction ψ to $n \bigoplus_{j \in J-J'} \pi'' G_j$ is a monomorphism, by Lem. 6, 7 and 8. On the other hand, $\bar{G}/\pi'' K \cong \hat{\hat{A}}^{b_1}/F_1 \oplus \cdots \oplus \hat{\hat{A}}^{b_m}/F_m \cong \hat{\hat{A}}/F_1 \oplus \cdots \oplus \hat{\hat{A}}/F_m$, by virtue of Lem. 6, 7 and 8 and the fact: $K = \hat{A}^{b_0} \oplus K_{F_1}^{\hat{A}^{b_1}} \oplus \cdots \oplus K_{F_m}^{\hat{A}^{b_m}}$. Therefore, it is a torsion group by Lem. 9 and hence $\bigoplus_{j \in J-J'} \bar{G}_j$ is a torsion group.

Let A_i be a torsion group for each $i \in I$. In view of the first paragraph of § 2, we can take a torsion group A in $V^{(P(I))}$ such that $\hat{A} \cong \prod_{i \in I} A_i$. So, Th. 3 is an improvement of Lem. 8 of [4], even in the case of a direct product, i.e. dropping the cardinality hypothesis for I. Hence, we have Th. 9 of [4] without the cardinality hypothesis for I.

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References

[1] Bell, J. L., Boolean valued models and independence proofs in Set Theory, Clarendon Press, Oxford, (1977).
[2] Bell, J. L. and Slomson, A. B., Models and Ultraproducts, North-Holland, Amsterdam, (1969).
[3] Chase, S. U., On direct sums and products of modules, Pacific Journal of Mathematics, **12** (1962), 847-854.
[4] Dugas, M. and Zimmermann-Huisgen, B., Iterated direct sums and products of modules, in L. N. in Math. **874** Springer, (1981), 179-193.
[5] Eda, K., On a Boolean power of a torsion free abelian group, to appear in J. Algebra.
[6] Fuchs, L., Infinite abelian groups, Vol. I, Academic Press, (1970).
[7] Fuchs, L., Infinite abelian groups, Vol. II, Academic Press, (1973).
[8] Mansfield, R., The Theory of Boolean Ultrapowers, Annals of Mathematical Logic, **2** (1971), 297-323.
[9] Solovay, R. M. and Tennembaum, S., Iterated Cohen extensions and Souslin's Problem, Annals of Mathematics, **94** (1971), 201-245.
[10] Jech, T., Set Theory, Academic Press, (1978).

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