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A DIFFERENTIAL GEOMETRIC CHARACTERIZATION OF HOMOGENEOUS SELF-DUAL CONES

(Dedicated to Professor K. Murata on his sixtieth birthday)

By

Hirohiko SHIMA

In this note we give a differential geometric characterization of self-dual cones among affine homogeneous convex domains not containing any full straight line. Let Ω be an affine homogeneous convex domain in an n-dimensional real vector space V^n. Then Ω admits an invariant volume element

$$v=\phi dx^1\wedge\cdots\wedge dx^n \tag{1}$$

and the canonical bilinear form defined by

$$g=\sum_{i,j}\frac{\partial^2\log\phi}{\partial x^i\partial x^j}dx^i dx^j \tag{2}$$

is positive definite and so gives an invariant Riemannian metric on Ω, where $\{x^1,\cdots,x^n\}$ is an affine coordinate system on V^n [5]. In an affine coordinate system $\{x^1,\cdots,x^n\}$ the components of the Riemannian connection Γ and the Riemannian curvature tensor R for g are expressed as follows

$$\Gamma^i{}_{jk}=\frac{1}{2}\sum_p g^{ip}\frac{\partial^3\log\phi}{\partial x^j\partial x^k\partial x^p}, \tag{3}$$

$$R^i{}_{jkl}=\sum_p(\Gamma^i{}_{pk}\Gamma^p{}_{jl}-\Gamma^i{}_{pl}\Gamma^p{}_{jk}), \tag{4}$$

where $g_{ij}=\frac{\partial^2\log\phi}{\partial x^i\partial x^j}$ and $\sum_p g^{ip}g_{pj}=\delta^i{}_j$ (Kronecker's delta). Since $\frac{1}{2}\sum_p g^{ip}\frac{\partial^3\log\phi}{\partial x^j\partial x^k\partial x^p}$ defines a tensor field on Ω, we denote this tensor field by the same letter Γ.

An open convex set Ω in V^n is called a cone with vertex o if $o+\lambda(x-o)\in\Omega$ for all $x\in\Omega$ and $\lambda>0$. An open convex cone Ω with vertex o is said to be a self-dual cone if V^n admits an inner product $\langle\ ,\ \rangle$ such that

(i) $\langle x-o, y-o\rangle>0$ for all $x, y\in\Omega$;

(ii) if $x\in V^n$ is a vector such that $\langle x-o, y-o\rangle\geqq 0$ for all $y\in\bar{\Omega}$ then $x\in\bar{\Omega}$, where $\bar{\Omega}$ is the closure of Ω in V^n.

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THEOREM 1. *A homogeneous convex cone Ω not containing any full straight line is a self-dual cone if and only if R is parallel with respect to Γ.*(*)

THEOREM 2. *A homogeneous convex domain Ω not containing any full straight line is a self-dual cone if and only if Γ is parallel with respect to Γ.*

The necessary conditions of these theorems have been proved by O. S. Rothaus [2].

If Γ is parallel with respect to Γ, then by (4) R is parallel with respect to Γ. The converse is not true. For example, in the case of the interior of a paraboloid; $x^2+1-\frac{1}{2}(x^1)^2>0$, R is parallel but Γ is not parallel with respect to Γ.

We denote by ∇_X and L_X the covariant differentiation for Γ and the Lie differentiation in the direction of a vector field X respectively. We set

$$A_X = L_X - \nabla_X. \tag{5}$$

Then A_X is a derivation of the algebra of tensor fields and for a vector field Y we have

$$A_X Y = -\nabla_Y X. \tag{6}$$

If X and Y are Killing vector fields, then we know [1]

$$R(X, Y) = [A_X, A_Y] - A_{[X,Y]}. \tag{7}$$

For vector fields $X=\sum_i \xi_i \frac{\partial}{\partial x^i}$ and $Y=\sum_i \eta^i \frac{\partial}{\partial x^i}$ we define a vector field $X \square Y$ by

$$X \square Y = -\sum_{i,j,k} \Gamma^i{}_{jk} \xi^j \eta^k \frac{\partial}{\partial x^i}, \tag{8}$$

and we put

$$S_X Y = X \square Y. \tag{9}$$

The condition that Γ is parallel with respect to Γ is equivalent to

$$\nabla_Z (X \square Y) = (\nabla_Z X) \square Y + X \square (\nabla_Z Y). \tag{10}$$

We shall now recall the construction of clans from affine homogeneous convex domains [5]. It is known that a homogeneous convex domain Ω not containing any full straight line admits a simply transitive triangular affine Lie group T. Let $\mathfrak{t}$ denote the Lie algebra of T. We fix a point $e \in \Omega$ and choose an affine coordinate system $\{x^1, \cdots, x^n\}$ such that $x^1(e) = \cdots = x^n(e) = 0$. Identifying $X \in \mathfrak{t}$ with the vector field induced by the one parameter group of transformations $\exp(-tX)$, X

(*) T. Tsuji obtained the same result independently [7].

has an expression $X=\sum_i (\sum_j a^i{}_j x^j + a^i)\frac{\partial}{\partial x^i}$, where $a^i{}_j$ and a^i are constants. Let V denote the tangent space of Ω at e. Since T acts simply transitively on Ω, for each $a \in V$ there exists a unique element $X_a \in \mathfrak{t}$ such that the values of X_a at e is equal to a. For $a, b \in V$ we define a multiplication $a \vartriangle b$ in V by

$$a \vartriangle b = \sum_i (\sum_j a^i{}_j b^j)\left(\frac{\partial}{\partial x^i}\right)_e, \tag{11}$$

where $a^i{}_j$ and b^j are constants given by

$$X_a = \sum_i (\sum_j a^i{}_j x^j + a^i)\frac{\partial}{\partial x^i} \text{ and } X_b = \sum_i (\sum_j b^i{}_j x^j + b^i)\frac{\partial}{\partial x^i}.$$

Then we have

$$[X_a, X_b] = X_{b \vartriangle a - a \vartriangle b}. \tag{12}$$

Denoting by L_a the left multiplication by $a \in V$;

$$L_a b = a \vartriangle b, \tag{13}$$

we have

$$[L_a, L_b] = L_{a \vartriangle b - b \vartriangle a}. \tag{14}$$

Let $\langle\ ,\ \rangle$ denote the inner product on V induced by g and we put

$$s(a) = \operatorname{Tr} L_a. \tag{15}$$

Then we know

$$\langle a, b\rangle = s(a \vartriangle b).$$

The algebra V together with the linear form s is said to be the *clan* of Ω with respect to $e \in \Omega$ and the simply transitive triangular group T and is denoted by $V(\Omega)$.

PROPOSITION 1. *For $a, b \in V$ we denote by S_a, A_a, and $R(a, b)$ the values of S_{X_a}, A_{X_a}, and $R(X_a, X_b)$ at e respectively. Then we have*

(i) $S_a = \frac{1}{2}(L_a + {}^tL_a)$, $\quad S_a b = S_b a$,

(ii) $A_a = -\frac{1}{2}(L_a - {}^tL_a)$,

(iii) $R(a, b) = -[S_a, S_b]$,

where ${}^tL_a$ is the transpose of L_a with respect to $\langle\ ,\ \rangle$.

PROOF. We may assume $\phi(e)=1$. Since $v=\phi\, dx^1 \wedge \cdots \wedge dx^n$ is invariant under the one parameter group of transformations $\operatorname{Exp} tX_a$ generated by X_a, we have $\phi((\operatorname{Exp} tX_a)e)=\exp(-t \operatorname{Tr} L_a)$ and so

$$\log \phi((\operatorname{Exp} tX_a)e) = -ts(a). \tag{16}$$

Expanding the left side in a power series of t and evaluating the terms of the first, the second and the third orders, we have

$$\sum_i \frac{\partial \log \phi}{\partial x^i}(e)a^i = -s(a), \tag{17}$$

$$\sum_{i,j} \frac{\partial^2 \log \phi}{\partial x^i \partial x^j}(e)a^i a^j = \langle a, a\rangle = s(a \vartriangle a), \tag{18}$$

$$\sum_{i,j,k} \frac{\partial^3 \log \phi}{\partial x^i \partial x^j \partial x^k}(e)a^i a^j a^k = -2\langle a, a \vartriangle a\rangle = -2s(a \vartriangle (a \vartriangle a)), \tag{19}$$

where $a=\sum_i a^i\left(\frac{\partial}{\partial x^i}\right)_e$. Taking $a+b$ and $a+b+c$ instead of a in the formulae (18) and (19) respectively we obtain

$$\sum_{i,j} \frac{\partial^2 \log \phi}{\partial x^i \partial x^j}(e)a^i b^j = \langle a, b\rangle = s(a \vartriangle b), \tag{18'}$$

$$\begin{aligned} &3\sum_{i,j,k} \frac{\partial^3 \log \phi}{\partial x^i \partial x^j \partial x^k}(e)a^i b^j c^k \\ &= -(\langle a, b \vartriangle c\rangle + \langle a, c \vartriangle b\rangle + \langle b, c \vartriangle a\rangle + \langle b, a \vartriangle c\rangle + \langle c, a \vartriangle b\rangle + \langle c, b \vartriangle a\rangle). \end{aligned} \tag{19'}$$

By (14) and (18′) we have

$$\langle a \vartriangle b, c\rangle + \langle b, a \vartriangle c\rangle = \langle b \vartriangle a, c\rangle + \langle a, b \vartriangle c\rangle. \tag{20}$$

Using this we get

$$\begin{aligned} \sum_{i,j,k} \frac{\partial^3 \log \phi}{\partial x^i \partial x^j \partial x^k}(e)a^i b^j c^k &= -\langle a \vartriangle b, c\rangle - \langle b, a \vartriangle c\rangle \\ &= -\langle (L_a + {}^tL_a)b, c\rangle. \end{aligned} \tag{19''}$$

On the other hand it follows from (3) (8) (9) that

$$\begin{aligned} \sum_{i,j,k} \frac{\partial^3 \log \phi}{\partial x^i \partial x^j \partial x^k}(e)a^i b^j c^k &= 2\sum_{i,j,k,l} (g_{kl}\Gamma^l{}_{ji})(e)a^i b^j c^k \\ &= -2\langle S_a b, c\rangle. \end{aligned}$$

Thus we have

$$S_a b = \frac{1}{2}(L_a + {}^tL_a)b$$

and (i) is proved.

For $X_a=\sum_i(\sum_j a^i{}_jx^j+a^i)\frac{\partial}{\partial x^i}$ and $X_b=\sum_i(\sum_j b^i{}_jx^j+b^i)\frac{\partial}{\partial x^i}$ we have

$$\nabla_{X_a}X_b=\sum_i\{\sum_p b^i{}_p(\sum_q a^p{}_qx^p+a^p)$$

$$+\sum_{r,s}\Gamma^i{}_{rs}(\sum_p a^r{}_px^p+a^r)(\sum_q b^s{}_qx^q+b^s)\}\frac{\partial}{\partial x^i}.$$

Since $x^i(e)=0$, by (8), (11) and (i) the value $(\nabla_{X_a}X_b)_e$ of $\nabla_{X_a}X_b$ at e is reduced to

$$\begin{aligned}(\nabla_{X_a}X_b)_e&=\sum_i(\sum_p b^i{}_pa^p+\sum_{r,s}\Gamma^i{}_{rs}a^rb^s)\left(\frac{\partial}{\partial x^i}\right)_e\\&=b\vartriangle a-b\,\square\,a\\&=\frac{1}{2}(L_b-{}^tL_b)a.\end{aligned}$$

Therefore by (6) we get

$$A_ab=(A_{X_a}X_b)_e=-(\nabla_{X_b}X_a)_e=-\frac{1}{2}(L_a-{}^tL_a)b,$$

and (ii) is proved.

By (7) we have

$$R(X_a, X_b)=[A_{X_a}, A_{X_b}]-A_{[X_a,X_b]}.$$

Using (12), (14), (i) and (ii) we obtain

$$\begin{aligned}R(a,b)&=[A_a, A_b]-A_{b\triangle a-a\triangle b}\\&=\left[-\frac{1}{2}(L_a-{}^tL_a),\ -\frac{1}{2}(L_b-{}^tL_b)\right]+\frac{1}{2}(L_{b\triangle a-a\triangle b}-{}^tL_{b\triangle a-a\triangle b})\\&=\frac{1}{4}[L_a-{}^tL_a, L_b-{}^tL_b]-\frac{1}{2}([L_a, L_b]-{}^t[L_a, L_b])\\&=-[S_a, S_b],\end{aligned}$$

and so (iii) is proved. Q. E. D.

PROPOSITION 2. (i) *If Γ is parallel with respect to Γ, then we have*

$$[A_a, S_b]=S_{A_ab}.$$

(ii) *If R is parallel with respect to Γ, then we have*

$$[A_a, [S_b, S_c]]=[S_{A_ab}, S_c]+[S_b, S_{A_ac}].$$

PROOF. Since $X_a=\sum_i\xi^i\frac{\partial}{\partial x^i}$ is an infinitesimal affine transformation with respect to Γ, we have

$$0=\frac{\partial^2\xi^i}{\partial x^j\partial x^k}+\sum_p\left(\xi^p\frac{\partial\Gamma^i{}_{jk}}{\partial x^p}+\Gamma^i{}_{pk}\frac{\partial\xi^p}{\partial x^j}+\Gamma^i{}_{jp}\frac{\partial\xi^p}{\partial x^k}-\Gamma^p{}_{jk}\frac{\partial\xi^i}{\partial x^p}\right)$$

$$=\frac{\partial^2\xi^i}{\partial x^j\partial x^k}+L_{X_a}\Gamma^i{}_{jk},$$

where $L_{X_a}\Gamma^i{}_{jk}$ is the Lie derivative of the tensor field $\Gamma^i{}_{jk}$ by X_a. Since $\xi^i=\sum_j a^i{}_jx^j+a^i$, we get

(21) $$L_{X_a}\Gamma^i{}_{jk}=0.$$

From this we have

(22) $$L_{X_a}(X_b\,\Box\,X_c)=(L_{X_a}X_b)\,\Box\,X_c+X_b\,\Box\,(L_{X_a}X_c).$$

Therefore by (5), (10) and (22) the condition $\nabla_{X_a}\Gamma=0$ is equivalent to

$$A_{X_a}(X_b\,\Box\,X_c)=(A_{X_a}X_b)\,\Box\,X_c+X_b\,\Box\,(A_{X_a}X_c).$$

This implies (i). By (4) and (21) it follows

$$L_{X_a}R=0.$$

Thus by (5) the condition $\nabla_{X_a}R=0$ is equivalent to $A_{X_a}R=0$. Since A_{X_a} is a derivation of the algebra of tensor fields, we have

$$\begin{aligned}((A_{X_a}R)(X_b,X_c))X_d&=A_{X_a}(R(X_b,X_c)X_d)-R(A_{X_a}X_b,X_c)X_d\\&\quad-R(X_b,A_{X_a}X_c)X_d-R(X_b,X_c)A_{X_a}X_d\\&=([A_{X_a},R(X_b,X_c)]-R(A_{X_a}X_b,X_c)-R(X_b,A_{X_a}X_c))X_d\end{aligned}$$

and so by Proposition 1 (iii)

$$\begin{aligned}((A_aR)(b,c))d&=([A_a,R(b,c)]-R(A_ab,c)-R(b,A_ac))d\\&=([A_a,-[S_b,S_c]]+[S_{A_ab},S_c]+[S_b,S_{A_ac}])d\end{aligned}$$

This proves (ii). Q.E.D.

LEMMA 1. *If Γ is parallel with respect to Γ, then Ω is a cone.*

PROOF. It is known that if the clan $V(\Omega)$ of Ω has a unit element then Ω is a cone [5]. Therefore by Proposition 2 (i) it suffices to show that if $[A_a,S_b]=S_{A_ab}$ holds for all $a,b\in V(\Omega)$, then $V(\Omega)$ has a unit element. Let u be the principal idempotent of the clan $V(\Omega)$, i.e., u is an element in $V(\Omega)$ determined by $\langle u,a\rangle=s(a)$ for all $a\in V(\Omega)$. Then we get the principal decomposition

$$V(\Omega)=V_0+N,$$

where $V_0=\{a\in V(\Omega)\,;\,u\triangle a=a\}$ and $N=\{a\in V(\Omega)\,;\,u\triangle a=\frac{1}{2}a\}$. The principal decomposition $V(\Omega)=V_0+N$ is orthogonal with respect to the inner product $\langle\ ,\ \rangle$ and the following relations hold [5]

$$V_0 \triangle V_0 \subset V_0, \qquad V_0 \triangle N \subset N,$$
$$N \triangle V_0=\{0\}, \qquad N \triangle N \subset V_0. \tag{23}$$

Let p be an element in N. By our assumption and Proposition 1 we have

$$0=\langle([A_p, S_p]-S_{A_pp})p, p\rangle=\langle A_pS_pp-2S_pA_pp, p\rangle$$
$$=-3\langle S_pp, A_pp\rangle=\frac{3}{4}\langle(L_p+{}^tL_p)p,\ (L_p-{}^tL_p)p\rangle$$
$$=\frac{3}{4}(\langle L_pp, L_pp\rangle-\langle {}^tL_pp, {}^tL_pp\rangle).$$

Therefore by the orthogonality of the principal decomposition and by (23) we get

$$\langle p \triangle p, p \triangle p\rangle=\langle {}^tL_pp, {}^tL_pp\rangle=\langle p, L_p{}^tL_pp\rangle=0.$$

This means $p \triangle p=0$, $\langle p, p\rangle=s(p \triangle p)=0$ and so $p=0$. Thus we have $V(\Omega)=V_0$ and u is a unit element of the clan $V(\Omega)$. Q. E. D.

We shall now recall the notion of T-algebras [5] [6].

A *matrix algebra with involution* is an algebra over the real number field $\mathbf{R}$ which is bigraded by the subspaces $\mathfrak{A}_{ij}$ $(i, j=1, \cdots, m)$ and provided with an involutive anti-automorphism $*$ in such a way that

$$\mathfrak{A}_{ij}\mathfrak{A}_{jk} \subset \mathfrak{A}_{ik},$$
$$\mathfrak{A}_{ij}\mathfrak{A}_{lk}=\{0\} \quad \text{if} \quad j \neq l,$$
$$\mathfrak{A}_{ij}^*=\mathfrak{A}_{ji}.$$

The general element of $\mathfrak{A}_{ij}$ will be denoted by a_{ij}, b_{ij}, etc..

A matrix algebra with involution is said to be a *T-algebra* if the following axioms are satisfied:

(T. 1) For any i the algebra $\mathfrak{A}_{ii}$ is one-dimensional and admits an isomorphism $\rho: \mathfrak{A}_{ii} \to \mathbf{R}$ with the following properties.

(T. 2) $a_{ii}b_{ij}=\rho(a_{ii})b_{ij}$;

(T. 3) $n_i\rho(a_{ij}b_{ji})=n_j\rho(b_{ji}a_{ij})$, where $n_i=1+\frac{1}{2}\sum_{s \neq i} \dim \mathfrak{A}_{is}$;

(T. 4) $\rho(a_{ij}a_{ij}^*)>0$ if $a_{ij} \neq 0$;

(T. 5) $a_{ij}(b_{jk}c_{ki})=(a_{ij}b_{jk})c_{ki}$;

(T. 6) $a_{ij}(b_{jk}c_{kl})=(a_{ij}b_{jk})c_{kl}$ if $i<j<k$ and $j<l$;

(T. 7) $a_{ij}(b_{jk}b_{jk}^*)=(a_{ij}b_{jk})b_{jk}^*$ if $i<j<k$.

Let $\mathfrak{X}$ denote the space of hermitian matrices in the T-algebra $\mathfrak{A}$; $\mathfrak{X}=\{a \in \mathfrak{A};$

$a^*=a\}$. For each $a=\sum_{i,j} a_{ij} \in \mathfrak{X}$ we put

$$\hat{a}=\frac{1}{2}\sum_i a_{ii}+\sum_{i<j} a_{ij},$$

$$\underset{\vee}{a}=\frac{1}{2}\sum_i a_{ii}+\sum_{i>j} a_{ij},$$

We define a multiplication $L_a b=a \vartriangle b$ in $\mathfrak{X}$ by the formula

$$a \vartriangle b=\hat{a}b+b\underset{\vee}{a},$$

Then $\mathfrak{X}$ is a clan with unit element and we denote this clan by $\mathfrak{X}(\mathfrak{A})$. Let $\Omega(\mathfrak{A})$ be the set of matrices which are expressible in the form tt^*, where t is an upper triangular matrix with positive elements on the diagonal. Then $\Omega(\mathfrak{A})$ is a homogeneous convex cone in $\mathfrak{X}(\mathfrak{A})$. For every homogeneous convex cone Ω there exists a T-algebra $\mathfrak{A}$ such that Ω is isomorphic to $\Omega(\mathfrak{A})$ and a clan $V(\Omega)$ of Ω is isomorphic to $\mathfrak{X}(\mathfrak{A})$.

Now we return to the proof of our theorems. By the above fact we may assume $V(\Omega)=\mathfrak{X}(\mathfrak{A})$. Then it is known that for $a, b \in \mathfrak{X}(\mathfrak{A})$

$$\operatorname{Tr} L_a=\operatorname{Spur} a,$$

$$\langle a, b\rangle=\operatorname{Spur} ab,$$

where Spur $a=\sum_i n_i \rho(a_{ii})$. It is easy to see

$${}^t L_a b=\underset{\vee}{a}b+b\hat{a}.$$

Therefore we get

$$S_a b=\frac{1}{2}(ab+ba), \tag{24}$$

$$A_a b=-\frac{1}{2}\{(\hat{a}-\underset{\vee}{a})b-b(\hat{a}-\underset{\vee}{a})\}. \tag{25}$$

Let n_{ij} denote the dimension of $\mathfrak{A}_{ij}$. We define inductively an equivalence relation $\bar{R}$ in the set $\{1, \cdots, m\}$ of indices:

(1) $i \equiv i \pmod{\bar{R}}$ for all i,

(2) if we have already determined whether the i, j such that $|i-j|<r$ are comparable modulo $\bar{R}$ or not, then for $|i-j|=r$ we define $i \equiv j \pmod{\bar{R}}$ if and only if (i) $n_{ij} \neq 0$, (ii) $n_{ik}=n_{jk}$ for all $k \neq i, j$, and (iii) for all k lying between i and j (except i and j) either $n_{ik}=n_{kj}=0$ or $i \equiv k \pmod{\bar{R}}$ and $k \equiv j \pmod{\bar{R}}$.

We put

$$\mathfrak{A}_{ij}^c=\begin{cases}\mathfrak{A}_{ij} & \text{if } i \equiv j \pmod{\bar{R}} \\ \{0\} & \text{if } i \not\equiv j \pmod{\bar{R}}.\end{cases}$$

$$\mathfrak{A}^c = \sum_{i,j} \mathfrak{A}_{ij}^c .$$

Then $\mathfrak{A}^c$ is a T-algebra and the homogeneous convex cone $\Omega(\mathfrak{A}^c)$ corresponding to $\mathfrak{A}^c$ is self-dual.

LEMMA 2. *If the clan $\mathfrak{X}(\mathfrak{A})$ corresponding to a T-algebra $\mathfrak{A}$ satisfies the condition*

$$[A_a, [S_b, S_c]] = [S_{A_a b}, S_c] + [S_b, S_{A_a c}],$$

then we have $\mathfrak{A} = \mathfrak{A}^c$.

PROOF. By the condition we have

$$[[A_a, S_b] - S_{A_a b}, S_c] = [[A_a, S_c] - S_{A_a c}, S_b].$$

Let $a_{ij} \in \mathfrak{A}_{ij}$, $b_{jk} \in \mathfrak{A}_{jk}$ and $e_i \in \mathfrak{A}_{ii}$, where $i<j$, $k \neq i, j$ and $\rho(e_i)=1$. We put $a = a_{ij} + a_{ij}^*$, $b = b_{jk} + b_{jk}^*$ and $c = e_i$ and calculate the following formula

$$[[A_a, S_b] - S_{A_a b}, S_c] b = [[A_a, S_c] - S_{A_a c}, S_b] b$$

Using (24) and (25), the left side is equal to

$$\frac{1}{4} \{ a_{ij}(b_{jk} b_{jk}^*) - (a_{ij} b_{jk}) b_{jk}^* + (b_{jk} b_{jk}^*) a_{ij}^* - b_{jk}(b_{jk}^* a_{ij}^*) \}$$

and the right side is reduced to 0. Considering $\mathfrak{A}_{ij}$-component, by (T. 2) we get

$$(a_{ij} b_{jk}) b_{jk}^* = a_{ij}(b_{jk} b_{jk}^*) = \rho(b_{jk} b_{jk}^*) a_{ij} .$$

Multiplying both sides on the right by a_{ij}^* we obtain

$$((a_{ij} b_{jk}) b_{jk}^*) a_{ij}^* = \rho(b_{jk} b_{jk}^*) a_{ij} a_{ij}^* .$$

Therefore, by (T. 2) we have

$$\rho((a_{ij} b_{jk})(a_{ij} b_{jk})^*) = \rho(((a_{ij} b_{jk}) b_{jk}^*) a_{ij}^*) = \rho(b_{jk} b_{jk}^*) \rho(a_{ij} a_{ij}^*). \tag{26}$$

Assume $n_{ij} \neq 0$. For $a_{ij} \neq 0 \in \mathfrak{A}_{ij}$, by (26) the linear mapping given by $\mathfrak{A}_{jk} \ni b_{jk} \to a_{ij} b_{jk} \in \mathfrak{A}_{ik}$ is injective and so $n_{jk} \leqq n_{ik}$. In the same way we have $n_{ik} \leqq n_{jk}$. Therefore we have $n_{ik} = n_{jk}$ for all $k \neq i, j$. This implies that $i \equiv j \pmod{\bar{R}}$ if $n_{ij} \neq 0$. Thus we have $\mathfrak{A} = \mathfrak{A}^c$. Q. E. D.

Since the homogeneous convex cone $\Omega(\mathfrak{A}^c)$ determined by $\mathfrak{A}^c$ is self-dual, in view of Proposition 2, Lemma 1 and 2 the sufficient conditions of our theorems are proved.

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Department of Mathematics
Yamaguchi University
Yamaguchi, 753 Japan