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HOMEOMORPHISMS OF INFINITE-DIMENSIONAL FIBRE BUNDLES

By

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0. Introduction

Throughout in this paper, *all spaces are metrizable* and E denotes a locally convex linear metric space homeomorphic ($\cong$) to its own countably infinite product E^{ω} or the subspace E_f^{ω} of E^{ω} consisting of all elements whose almost all coordinates are zero.

A *manifold modeled on* E, briefly E-*manifold*, is a metric space M admitting an open cover by sets homeomorphic to open subsets of E. We assume the following:

Each E-*manifold has the same weight as* E.

Hence if E is separable, then E-manifolds means separable E-manifolds. It is well-known that *each connected* E-*manifold has the same weight as* E.

An E-*manifold bundle* is a locally trivial fibre bundle with fibre an E-manifold. An E-manifold bundle with fibre M is briefly called an M-*bundle*. Then an E-bundle is a locally trivial fibre bundle with fibre E. It is proved by T. A. Chapman [Ch_3] that *each* E-*bundle is trivial*, that is, bundle isomorphic a product bundle. In this paper (Section 3 and 4), we show that each E-manifold bundle over B can be embedded in the product bundle $B\times E$ as a closed or/and an open sub-bundle.

Let $p: X\to B$ be an E-manifold bundle. By Bundle Stability Theorem [Sa_2], $p: X\to B$ is bundle isomorphic to $p\circ\mathrm{proj}: X\times E\to B$. A subset K of X is said to be B-*preservingly* E-*deficient* if there exists a bundle homeomorphism $h: X\to X\times E$ such that $h(K)\subset X\times\{0\}$. From 5-2 in [Sa_2], we can require h to satisfy that $h(x)=(x, 0)$ for each $x\in K$. In this paper, we research several properties of B-preservingly E-deficient sets.

In Section 2, we show that a B-preservingly E-deficient locally closed set K is negligible in X, that is, $p|X\setminus K: X\setminus K\to B$ is also an E-manifold bundle which is bundle isomorphic to $p: X\to B$. And in Section 4, we show that if Y is a B-preservingly E-deficient closed set in X and $p|Y: Y\to B$ is also an E-manifold bundle, then Y is B-preservingly collared in X, that is, there is an open embedding

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$g: Y\times[0,1)\to X$ such that $pg=p\circ\mathrm{proj}$ and $g(y,0)=y$ for each $y\in Y$.

Moreover we establish two Approximation Theorems and Hemeomorphism Extension Theorem. In Section 5, we show that any bundle map of an E-manifold bundle to another over same base can be approximated by both closed and open bundle embeddings. In Section 6, we prove that if $f: K\to X$ is a B-preserving embedding of a B-preserving E-deficient closed set K in X which is homotopic to the inclusion by a B-preserving homotopy, then f can extend to a bundle homeomorphism $h: X\to X$ ambiently invertibly bundle isotopic to the identity.

Thus we extend results of T. A. Chapman and R. Y. T. Wong [C-W] to the case of arbitraly metric base spaces. It was raised as an open question in their paper [C-W]. Our approach is different from theirs.

Finally, we prove that each bundle homotopy equivalence between E-manifold bundles over same base is bundle homotopic to a bundle homeomorphism.

1. Preliminary

Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of X. We say that $\mathcal{U}$ *refines* $\mathcal{V}$, denote $\mathcal{U}<\mathcal{V}$, provided each $U\in\mathcal{U}$ is contained in some $V\in\mathcal{V}$. For $A\subset X$, define $\mathrm{st}(A;\mathcal{U})=\cup\{U\in\mathcal{U}|A\cap U\neq\emptyset\}$. And for a collection $\mathcal{W}$ of subsets of X, $\mathrm{st}(\mathcal{W};\mathcal{U})=\{\mathrm{st}(W;\mathcal{U})|W\in\mathcal{W}\}$ and denote $\mathrm{st}(\mathcal{U})=\mathrm{st}(\mathcal{U};\mathcal{U})$. If $\mathcal{U}$ and $\mathcal{V}$ are covers of X and $\mathcal{U}$ *refines* $\mathcal{V}$, then we call $\mathcal{U}$ a *refinement* of $\mathcal{V}$, and if $\mathrm{st}(\mathcal{U})$ *refines* $\mathcal{V}$, then we call $\mathcal{U}$ a *star-refinement* of $\mathcal{V}$. We say that a map $f: Y\to X$ is *$\mathcal{U}$-near* to a map $g: Y\to X$ or f and g are *$\mathcal{U}$-near* if $\{\{f(y),g(y)\}|y\in Y\}$ refines $\mathcal{U}\cup\{\{x\}|x\in X\}$, and a homotopy $h: Y\times\boldsymbol{I}\to X$ is *$\mathcal{U}$-limited* if $\{h(\{y\}\times\boldsymbol{I})|y\in Y\}$ refines $\mathcal{U}\cup\{\{x\}|x\in X\}$. A $\mathcal{U}$-limited homotopy (isotopy) is called a *$\mathcal{U}$-homotopy* (*$\mathcal{U}$-isotopy*).

A map $f: B\times X\to B\times Y$ (or $f: X\times B\to Y\times B$) is said to be *B-preserving* if $\pi_B f=\pi_B$, where π_B is the *projection* onto B. Then, for each $b\in B$, define $f_b: X\to Y$ by $f_b(x)=f(b,x)$ (or $=f(x,b)$). Let $p: X\to B$ and $q: Y\to B$ be maps. A map $f: X\to Y$ is *B-preserving* if $qf=p$. A map $g: X\times Z\to Y$ (or $g: X\to Y\times Z$) is *B-preserving* if $qg=p\pi_X$ (or $q\pi_Y g=p$). And a homotopy $h: X\times\boldsymbol{I}\to Y$ is *B-preserving* if $qh_t=p$ for $t\in\boldsymbol{I}$. If $p: X\to B$ and $q: Y\to B$ are bundles, then a B-preserving continuous map (embedding, homeomorphism, etc.) $f: X\to Y$ is called a *bundle map* (a *bundle embedding*, a *bundle homeomorphism*, etc.) and a B-preserving homotopy (isotopy) $h: X\times\boldsymbol{I}\to Y$ is a called a *bundle homotopy* (a *bundle isotopy*.)

For topological properties of the linear metric space E, we refer readers to see the Bessaga and Pełczyński's Book [B-P].

In [Mi], E. Micheal established two useful criterion for a property $\mathcal{P}$ in order that a topological space X has $\mathcal{P}$ if each point of X has a neighbourhood which has $\mathcal{P}$. The first is one for a property of open sets and the second is one for a property of closed sets.

Let $\mathcal{P}$ be a property of open sets in a topological space X. Then $\mathcal{P}$ is *G-hereditary* if $\mathcal{P}$ satisfies the following conditions:

a) If U is an open set in X which has property $\mathcal{P}$, then every open subset of U has also property $\mathcal{P}$.

b) A union of two open sets both of which have property $\mathcal{P}$ has also property $\mathcal{P}$.

c) A discrete union of open sets all of which have property $\mathcal{P}$ has also property $\mathcal{P}$.

1-1 THEOREM (Micheal [Mi]): *Let $\mathcal{P}$ be a G-hereditary property of open sets in a paracompact (Hausdorff) space X. If each point of X has an open neighbourhood which has property $\mathcal{P}$, then X has property $\mathcal{P}$.*

Let $\mathcal{P}$ be a property of closed sets in a topological space X. Then $\mathcal{P}$ is *F-hereditary* if $\mathcal{P}$ satisfies the following conditions:

a) If A is a closed set in X which has property $\mathcal{P}$, then every closed subset of A has also property $\mathcal{P}$.

b) If A' and A'' have property $\mathcal{P}$ and $A=A'\cup A''=\mathrm{int}_A A'\cup \mathrm{int}_A A''$, then A has also property $\mathcal{P}$.

c) A discrete union of closed sets all of which have property $\mathcal{P}$ has also property $\mathcal{P}$.

1-2 THEOREM (Micheal [Mi]): *Let $\mathcal{P}$ be an F-hereditary property of closed sets in a paracompact (Hausdorff) space X. If each point of X has a closed neighbourhood which has property $\mathcal{P}$, then X has property $\mathcal{P}$.*

2. Negligibility of $\boldsymbol{E}$-deficient sets in Bundles

Let $p: X\to B$ be a map. A subset K of X is said to be *B-preservingly negligible in X* (with respect to p) if there exists a B-preserving homeomorphism $f: X\to X\setminus K$. If for each open cover $\mathcal{U}$ of X, there is a B-preserving homeomorphism $f: X\to X\setminus K$ $\mathcal{U}$-near to id, then we say that K is *B-preservingly strongly negligible in X* (with respect to p). A *B-preserving extractor pushing K off X* is an invertible B-preserving isotopy $h: X\times \boldsymbol{I}\to X$ such that $h_0=\mathrm{id}$, $h_1(X)=X\setminus K$ and h_t is onto for each $t\in[0,1)$. If h is $\mathcal{U}$-limited for an open cover $\mathcal{U}$ of K in X, h is called a *B-preserving $\mathcal{U}$-extractor pushing K off X*. A subset K of X is said to be *B-preservingly extractible from X* (with respect to p) if for each open cover $\mathcal{U}$ of K in X, there is a B-preserving $\mathcal{U}$-extractor pushing K off X.

In this section, we will show that if $p: X \to B$ is an E-manifold bundle, then each B-preservingly E-deficient locally closed set in X is B-preservingly extractible from X, hence B-preservingly strongly negligible in X. First, we will see the equivalence of E-deficiency and $\boldsymbol{R}^\omega$- (or $\boldsymbol{R}_f^\omega$-) deficiency.

2-1 THEOREM: *Let $p: X \to B$ be an E-manifold bundle and $K \subset X$. Then the following are equivalent:*

i) *K is B-preservingly E-deficient in X.*

ii) *K is B-preservingly $\boldsymbol{R}^\omega$-deficient or $\boldsymbol{R}_f^\omega$-deficient in X, according as $E \cong E^\omega$ or $\cong E_f^\omega$.*

iii) *There is a bundle homeomorphism $h: X \to X \times [0, 1)$ (or $h: X \to X \times [0, 1]$) such that $h(K) \subset X \times \{0\}$ (more strongly, $h(x) = (x, 0)$ for each $x \in K$).*

PROOF: The equivalence of i) and ii) is proved as same as Theorem 3.1 in [Ch₁] and Theorem 2-6 in [Sa₁]. Since

$$(\boldsymbol{R}^\omega \times \boldsymbol{R}^\omega, \boldsymbol{R}^\omega \times \{0\}) \cong (\boldsymbol{R}^\omega \times [0, 1), \boldsymbol{R}^\omega \times \{0\}) \cong (\boldsymbol{R}^\omega \times [0, 1], \boldsymbol{R}^\omega \times \{0\})$$

and

$$(\boldsymbol{R}_f^\omega \times \boldsymbol{R}_f^\omega, \boldsymbol{R}_f^\omega \times \{0\}) \cong (\boldsymbol{R}_f^\omega \times [0, 1), \boldsymbol{R}_f^\omega \times \{0\}) \cong (\boldsymbol{R}_f^\omega \times [0, 1], \boldsymbol{R}_f^\omega \times \{0\}),$$

the equivalence of ii) and iii) is clear. (cf. 5-2 in [Sa₂]) □

Using above theorem, we can prove the following theorem similarly as Lemma 3 in [Cu] (cf. Corollary 2-7 in [Sa₁]).

2-2 THEOREM: *Let $p: X \to B$ be an E-manifold bundle. Then any B-preservingly E-deficient locally closed set in X is B-preservingly extractible from X.*

Particulary, $p|X \setminus K: X \setminus K \to B$ is also an E-manifold bundle and it is bundle isomorphic to $p: X \to B$.

2-3 THEOREM: *Assume that E is completely metrizable. Let $p: X \to B$ be an E-manifold bundle. Then a countable union of B-preservingly E-deficient locally closed sets in X is B-preservingly extractible from X.*

PROOF: Locally closed sets are F_σ and subsets of B-preservingly E-deficient sets are B-preservingly E-deficient, so we may show that a countable union of B-preservingly E-deficient closed sets is B-preservingly extractible. By complete metrizability of the fibre and locally triviality of p, there exists a metric d on X such that each fibre $p^{-1}(b)$ is d-complete. From the same proof as Lemma 4 in [Cu], it follows that if $\{K_i\}_{i \in \boldsymbol{N}}$ is a sequence of closed subsets of X such that

$K_n \cap (X \setminus \bigcup_{i=1}^{n-1} K_i)$ is B-preservingly extractible from $X \setminus \bigcup_{i=1}^{n-1} K_i$ for each $n \in \boldsymbol{N}$, then $\bigcup_{i=1}^{\infty} K_i$ is also B-preservingly extractible from X. Thus the result follows from 5-2 in [Sa_2] and above 2-2. □

3. Bundle Embedding Theorem

In this section, we will prove that each E-manifold bundle over B can be embedded in the product bundle $B \times E$ as a closed sub-bundle. To prove it, we need the parametric version of the technique of Klee [Kl].

3-1 THEOREM: *Let K_1 and K_2 be B-preservingly E-deficient closed sets in $B \times E$. If $f: K_1 \to K_2$ is a B-preserving homeomorphism, then there is a B-preserving homeomorphism $\tilde{f}: B \times E \to B \times E$ such that $\tilde{f}|K_1 = f$.*

Using Micheal's Theorem (Theorem 1.2) and above Klee's Theorem, we will prove the following Embedding Theorem:

3-2 BUNDLE EMBEDDING THEOREM: *Let $p: X \to B$ be an E-manifold bundle. Then there exists a bundle embedding $f: X \to B \times E$ such that $f(X)$ is B-preservingly E-deficient closed in $B \times E$.*

PROOF: Define the Property $\mathcal{E}$ for closed subsets of B as follows:

($\mathcal{E}$) A closed subset A of B has Property $\mathcal{E}$ if there exists an B-preserving embedding $f: p^{-1}(A) \to B \times E$ such that $f(p^{-1}(A))$ is B-preservingly E-deficient closed in $B \times E$.

Since the fibre of $p: X \to B$ can be embedded in E as an E-deficient closed set, it follows from local triviality of $p: X \to B$ that each point of B has a closed neighbourhood which has Property $\mathcal{E}$.

Now we will see that Property $\mathcal{E}$ is F-hereditary. Then the result follows from Theorem 1-2. We may see the condition b) of F-hereditary property. Let $A = A' \cup A'' = \mathrm{int}_A A' \cup \mathrm{int}_A A''$ where A' and A'' have Property $\mathcal{E}$. There are B-preserving embeddings $f': p^{-1}(A') \to B \times E$ and $f'': p^{-1}(A'') \to B \times E$ such that $f'(p^{-1}(A'))$ and $f''(p^{-1}(A''))$ are B-preservingly E-deficient closed in $B \times E$. Since $f'(p^{-1}(A' \cap A''))$ and $f''(p^{-1}(A' \cap A''))$ are B-preservingly E-deficient closed in $B \times E$, using Theorem 3-1, we have a B-preserving homeomorphism $h: B \times E \to B \times E$ such that $h|f'p^{-1}(A' \cap A'') = f''f'^{-1}$. Define an embedding $f: p^{-1}(A) \to B \times E$ by $f|p^{-1}(A') = hf'$ and $f|p^{-1}(A'') = f''$. By 5-6 in [Sa_2], $f(p^{-1}(A)) = hf'(p^{-1}(A')) \cup f''(p^{-1}(A''))$ is a B-preservingly E-deficient closed set in $B \times E$. □

4. Collaring Theorem for Bundles

Using Micheal's Theorem (Theorem 1.1), M. Brown [Br] proved that a locally collared subset of metric space is collared. (R. Connelly [Co] gave a new proof of this Brown's Collaring Theorem.) A bundle version can be proved in a same way. In case of product bundles, it is mentioned in [C–F] and [Fe].

Let $p: X \to B$ be a map. A subset K of X said to be *B-preservingly collared* in X (with respect to p) if there exists a B-preserving open embedding $g: K \times [0, 1) \to X$ such that $g(x, 0) = x$ for each $x \in K$. If each $x \in K$ has an open neighbourhood in K which is B-preservingly collared in X, then we say that K is *locally B-preservingly collared in X*.

4–1 THEOREM: *Let $p: X \to B$ be a map. Each locally B-preservingly collared subset of X is B-preservingly collared.*

From this theorem, we have the following:

4–2 COLLARING THEOREM: *Let $p: X \to B$ be an E-manifold bundle. If Y is a B-preservingly E-deficient closed set in X and $p|Y: Y \to B$ is an E-manifold bundle, then Y is B-preservingly collared in X.*

PROOF: By the above theorem, we may show that Y is locally B-preservingly collared in X.

Using locally triviality of $p: X \to B$ and $p|Y: Y \to B$, for each $y \in Y$ there are an open neighbourhood U of $p(y)$ in B, U-preserving open embeddings $g: U \times E \times [0, 1) \to X$ and $f: U \times E \to Y \cap g(U \times E \times [0, 1))$ such that $y \in f(U \times E)$. Since $Y \setminus f(U \times E)$ is a B-preservingly E-deficient closed set in X, we can assume that $f(U \times E) = Y \cap g(U \times E \times [0, 1))$ from 2–2. By 5–2 in [Sa$_2$], $g^{-1}(f \times \mathrm{id}): U \times E \times \{0\} \to U \times E \times [0, 1)$ is a U-preservingly E-deficient closed embedding. Using 3–1, construct a U-preserving homeomorphism $h: U \times E \times [0, 1) \to U \times E \times [0, 1)$ such that $h|U \times E \times \{0\} = g^{-1}(f \times \mathrm{id})$. Then $gh(f^{-1} \times \mathrm{id}): f(U \times E) \times [0, 1) \to X$ is a B-preserving open embedding such that

$$gh(f^{-1} \times \mathrm{id})(x, 0) = x \qquad \text{for each } x \in f(U \times E).$$

Hence Y is locally B-preservingly collared in X. □

From Bundle Embedding Theorem 3–3 and Collaring Theorem 4–2, it follows

4–3 COROLLARY: *Let $p: X \to B$ be an E-manifold bundle. Then there exists an open embedding $g: X \times [0, 1) \to B \times E$ such that $\pi_B g = p\pi_X$ and $g(X \times \{0\})$ is B-preservingly E-deficient closed in $B \times E$.*

Note $E\times[0,1)\cong E$, then each E-manifold bundle $p:X\to B$ is bundle isomorphic to $p\pi_X:X\times[0,1)\to B$ by Bundle Stability Theorem 4-2 in [Sa_2]. Hence we have a bundle version of Henderson Open Embedding Theorem [H_1].

4-4 Bundle Open Embedding Theorem: *Let $p:X\to B$ be an E-manifold bundle and K a B-preservingly E-deficient closed set in X. Then there exists a bundle embedding $g:X\to B\times E$ such that $g(X)$ is open in $B\times E$ and $g(K)$ is B-preservingly E-deficient closed in $B\times E$.*

Proof: By 2-1, there is a bundle homeomorphism $h:X\to X\times[0,1)$ such that $h(K)\subset X\times\{0\}$. The result follows from above 4-3. □

5. Approximation Theorems

In this section, we will show that any bundle map of an E-manifold bundle to another is approximated by bundle embeddings. We prove two Approximation Theorems. The first (5-1) is an approximation by closed embeddings and the second (5-2) is one by open embeddings.

5-1 First Approximation Theorem: *Let $p:X\to B$ and $q:Y\to B$ be E-manifold bundles. If $f:Y\to X$ is a bundle map such that $f|Z$ is a closed embedding and $f(Z)$ is B-preservingly E-deficient in X for a closed set Z in Y, then for each open cover $\mathcal{U}$ of X, there exists a bundle $\mathcal{U}$-homotopy $f^*:Y\times \boldsymbol{I}\to X$ such that*

i) $f_0^*=f$,

ii) $f_t^*|Z=f|Z$ *for each* $t\in\boldsymbol{I}$,

iii) $f_1^*:Y\to X$ *is a bundle embedding, and*

iv) $f_1^*(Y)$ *is B-preservingly E-deficient closed in X.*

Proof: By Bundle Embedding Theorem 3-3, we can assume that $Y\supset Z$ are closed subsets of $B\times E$ and $q=\pi_B|Y$. Then the result follows from Mapping Replacement Theorem 6-2 in [Sa_2]. □

5-2 Second Approximation Theorem: *Let $p:X\to B$ and $q:Y\to B$ be E-manifold bundles and Z a B-preservingly E-deficient closed set in Y. If $f:Y\to X$ is a bundle map such that $f|Z$ is a closed embedding and $f(Z)$ is B-preservingly E-deficient in X, then for each open cover $\mathcal{U}$ of X, there exists a bundle $\mathcal{U}$-homotopy $f^{**}:Y\times\boldsymbol{I}\to X$ such that*

i) $f_0^{**}=f$,

ii) $f_t^{**}|Z=f|Z$ *for each* $t\in\boldsymbol{I}$,

iii) $f_1^{**}:Y\to X$ *is a bundle embedding and*

iv) $f_1^{**}(Y)$ *is open in* X.

PROOF: Let $\mathcal{U}$ be an open cover of X such that $\mathrm{st}(\mathcal{V})<\mathcal{U}$ and let $f^*: Y\times \boldsymbol{I}\to X$ be the bundle $\mathcal{V}$-homotopy obtained by the above first theorem. Using Collaring Theorem 4-2, there is an open embedding $g: Y\times[0,1)\to X$ such that $pg=q\pi_Y$ and $gi=f_1^*$, where $i: Y\to Y\times[0,1)$ is the injection defined by $i(y)=(y,0)$. We can assume that each $g(\{y\}\times[0,1))$ is contained in some member of $\mathcal{V}$.

Recall $E\times[0,1)\cong E$. Using Bundle Stability Theorem 4-2 (and 5-1) in [Sa₂], we have a $(gi)^{-1}(\mathcal{U})$-homotopy $h: Y\times[0,1)\times I\to Y$ such that $h_0=\pi_Y, h_1$ is a homeomorphism, $qh_t=q\pi_Y$ and $h_ti|Z=\mathrm{id}$ for each $t\in \boldsymbol{I}$. Then $h_1^{-1}|Z=i$ and $gh_1^{-1}: Y\to X$ is an open embedding. Define a homotopy $k: Y\times[0,1)\times \boldsymbol{I}\to Y\times[0,1)$ by

$$k(y,s,t)=(y,st).$$

Then $k_0=i\pi_Y$, $k_1=\mathrm{id}$ and $k_ti=i$ for each $t\in \boldsymbol{I}$. It is easy to see that a homotopy $f^{**}: Y\times \boldsymbol{I}\to X$ defined by

$$f_t^{**}(y)=\begin{cases} f_{3t}^*(y) & \text{if } 0\leqq t\leqq 1/3 \\ gih_{2-3t}h_1^{-1}(y) & \text{if } 1/3\leqq t\leqq 2/3 \\ gk_{3t-2}h_1^{-1}(y) & \text{if } 2/3\leqq t\leqq 1 \end{cases}$$

fulfills our requirements. □

6. Bundle Homeomorphism Extension

In this section, we establish a bundle version of Homeomorphism Extension Theorem due to R. D. Anderson and J. D. McCharen [A-M]. In the case of polyhedral base spaces, it has been proved by T. A. Chapman and R. Y. T. Wong [C-W].

6-1 BUNDLE HOMEOMORPHISM EXTENSION THEOREM: *Let $p: X\to B$ be an E-manifold bundle, K a B-preservingly E-deficient closed set in X and $\mathcal{U}$ an open cover of X. If $h: K\times \boldsymbol{I}\to X$ is a $\mathcal{U}$-homotopy such that $ph=p\pi_K$, $h_0=\mathrm{id}$, h_1 is an embedding and $h_1(K)$ is B-preservingly E-deficient closed in X, then for each open cover $\mathcal{V}$ of X, there exists an ambient invertible bundle $\mathrm{st}(\mathcal{U};\mathcal{V})$-isotopy $h^*: X\times \boldsymbol{I}\to X$ such that $h_0^*=\mathrm{id}$ and $h_1^*|K=h_1$.*

PROOF: Let $\mathcal{W}$ be a star-refinement of $\mathcal{V}$. Note that $K\cup h_1(K)$ is B-preservingly E-deficient in X by Theorem 5-6 in [Sa₂]. By the same arguments in the proof of HET in [Sa₁], there is an ambient invertible bundle $\mathcal{W}$-isotopy $f: X\times \boldsymbol{I}\to X$ such that $f_0=\mathrm{id}$ and

$$f_1(K\cup h_1(K))\cap(K\cup h_1(K))=\emptyset.$$

Next, using Anderson-McCharen's trick, we will construct an ambient invertible bundle $\mathrm{st}(\mathrm{st}(\mathcal{U};\mathcal{W});\mathcal{W})$-isotopy $g: X\times \boldsymbol{I}\to X$ such that $g_0=\mathrm{id}$ and $g_1|K=f_1h_1$.

Then a bundle isotopy $h^*: X\times \boldsymbol{I}\to X$ defined by $h_t^*=f_t^{-1}g_t$ $(t\in \boldsymbol{I})$ is a desired isotopy.

Construction of g: Using Mapping Replacement Theorem 6–2 in [Sa$_2$], we have an embedding $k: K\times \boldsymbol{I}\to X$ such that $pk=p\pi_K$, $k_i=f_ih_i$ $(i=0, 1)$, (i.e. $k_0=\mathrm{id}$ and $k_1=f_1h_1$), $k(K\times \boldsymbol{I})$ is B-preservingly E-deficient closed in X and k_t is $\mathcal{W}$-near to f_th_t (hence k is a st(st($\mathcal{U}$; $\mathcal{W}$) ; $\mathcal{W}$)-isotopy). From Bundle Open Embedding Theorem 4–4, we can assume that X is an open subset of $B\times E$, $p=\pi_B|X$ and $k(K\times \boldsymbol{I})$ is B-preservingly E-deficient closed in $B\times E$ (hence so is K). Note that $E\times \boldsymbol{R}\cong E$. Then using Theorem 3–1, construct a bundle embedding $j: X\to B\times E\times \boldsymbol{R}$ such that $j(X)$ is open in $B\times E\times \boldsymbol{R}$ and $jk=\mathrm{id}$.

Let $\mathcal{U}^*=\mathrm{st}(\mathrm{st}(\mathcal{U}\,;\mathcal{W})\,;\mathcal{W})$. For each $x\in K$, $\{x\}\times \boldsymbol{I}=jk(\{x\}\times \boldsymbol{I})$ is contained in some member of $j(\mathcal{U}^*)$. Then there is a closed neighbourhood N of K in $B\times E$ such that for each $x\in N$, $\{x\}\times \boldsymbol{I}$ is contained in some member of $j(\mathcal{U}^*)$. From Dowker's Theorem ([Du] p. 171), there is a continuous map $a: N\to(0, 1)$ such that

$$a(x)<\sup\{s\in(0, 1]\,|\,\{x\}\times[-s, 1+s]\subset j(U) \text{ for some } U\in\mathcal{U}^*\}.$$

Take a continuous map $b: N\times \boldsymbol{R}\to \boldsymbol{I}$ such that $b(\mathrm{bd}\,N\times \boldsymbol{R})=0$ and $b(K)=1$.

Now define an ambient invertible N-preserving $j(\mathcal{U}^*)$-isotopy $g': N\times \boldsymbol{R}\times \boldsymbol{I}\to N\times \boldsymbol{R}$ by

$$g'(x, s, t)=\begin{cases} \left(x, \dfrac{a(x)+tb(x)}{a(x)}s+tb(x)\right) & \text{if} \quad -a(x)\leqq s\leqq 0 \\ \left(x, \dfrac{1+a(x)-tb(x)}{1+a(x)}s+tb(x)\right) & \text{if} \quad 0\leqq s\leqq 1+a(x) \\ (x, s) & \text{otherwise.} \end{cases}$$

Then $g_0'=\mathrm{id}$ and $g_1'(x, 0)=(x, 1)$ for each $x\in K$. Since $g_t'|\mathrm{bd}\,N\times \boldsymbol{R}=\mathrm{id}$ for each $t\in \boldsymbol{I}$, g' has the extension $g'': B\times E\times \boldsymbol{R}\times \boldsymbol{I}\to B\times E\times \boldsymbol{R}$ such that $g_t''|B\times E\setminus N=\mathrm{id}$ for each $t\in \boldsymbol{I}$. Observe that $g_t''(j(X))=j(X)$ for each $t\in \boldsymbol{I}$. Finally, define an ambient invertible bundle $\mathcal{U}^*$-isotopy $g: X\times \boldsymbol{I}\to X$ by $g_t=j^{-1}g_t''j$ for each $t\in \boldsymbol{I}$. Then $g_0=\mathrm{id}$ and each $x\in K$,

$$\begin{aligned} g_1(x)&=j^{-1}g_1''j(k(x, 0))=j^{-1}g_1''(x, 0)\\ &=j^{-1}(x, 1)=k(x, 1)=f_1h_1(x). \quad \square \end{aligned}$$

7. Classification of Bundles

D. W. Henderson [H$_8$] (Henderson and Schori [H-S]) shows that each homotopy equivalence between E-manifolds is homotopic to a homeomorphism. Now we can prove its bundle version.

7–1 Bundle Classification Theorem: *Let $p: X\to B$ and $q: Y\to B$ be E-*

manifold bundles. If $f: X \to Y$ is a bundle homotopy equivalence, then there exists a bundle homeomorphism $h: X \to Y$ such that h is bundle homotopic to f.

Using the previous results and the following bundle version of Lemma 5.1 in [H_2], we can prove the theorem as same as Theorem 4 in [H_2] and Theorem C in [H-S].

7-2 LEMMA: *Let $p: X \to B$ be an E-manifold bundle and for each $n \in \boldsymbol{N}$, let $g_n: X \times [0, \infty) \to X \times [0, \infty)$ is a bundle embedding such that $g_n(X \times [0, \infty)) = X \times [0, 1)$, $g_n | X \times \{0\} = \mathrm{id}$ and for each $t > 0$, $g_n(X \times [0, t))$ is open and $g_n(X \times [0, t])$ is closed in $X \times [0, \infty)$. Then there exists a bundle homeomorphism h of $X \times [0, \infty)$ onto the direct limit of*

$$X \times [0, \infty) \xrightarrow{g_1} X \times [0, \infty) \xrightarrow{g_2} X \times [0, \infty) \xrightarrow{g_3} \cdots\cdots$$

such that h is bundle isotopic to the inclusion of the first $X \times [0, \infty)$ into the direct limit.

To make sure, we write proofs of 7-1 and 7-2 after Henderson.

PROOF of THEOREM 7-1: Let $g: Y \to X$ be a bundle homotopy inverse of f. From Approximation Theorem 5-1, f is bundle homotopic to a bundle closed embedding $f': X \to Y$. By Collaring Theorem 4-2, there is an open embedding $f^*: X \times [0, \infty) \to Y \times [0, 1)$ such that $f^*(x, 0) = (f'(x), 0)$ for each $x \in X$. Observe for each $t \in [0, \infty)$, $f^*(X \times [0, t])$ is closed in $Y \times [0, \infty)$ and f^* is bundle homotopic to $f \times \mathrm{id}$.

Similarly, there exists an open embedding $g^*: Y \times [0, \infty) \to X \times [0, \infty)$ such that $g^*(Y \times \{0\}) \subset X \times \{0\}$, $g^*(Y \times [0, \infty)) \subset X \times [0, 1)$, $g^*(Y \times [0, t])$ is closed in $X \times [0, \infty)$ for each $t \in [0, \infty)$ and g^* is bundle homotopic to $g \times \mathrm{id}$.

Note that $g^* f^*$ is bundle homotopic to id and $g^* f^*(X \times \{0\})$ is a closed subset of $X \times \{0\}$ hence B-preservingly E-deficient closed in $X \times [0, \infty)$. Using Bundle Homeomorphism Extension Theorem 6-1, for each $n \in \boldsymbol{N}$, construct a bundle homeomorphism $k_n: X \times [0, \infty) \to X \times [0, \infty)$ such that k_n is bundle isotopic to id, $k_n(g^* f^*)^n | X \times \{0\} = \mathrm{id}$ and $k_n | X \times [1, \infty) = \mathrm{id}$. Then note that $g^* f^* k_n^{-1} | X \times \{0\} = (g^* f^*)^{n+1} | X \times \{0\}$. Lemma 7-2 applies to the direct system

$$X \times [0, \infty) \xrightarrow{k_1 g^* f^*} X \times [0, \infty) \xrightarrow{k_2 g^* f^* k_1^{-1}} X \times [0, \infty) \xrightarrow{k_3 g^* f^* k_2^{-1}} \cdots\cdots$$

which is isomorphic by $\{\mathrm{id}, k_1^{-1}, k_2^{-1}, \cdots\}$ to the direct system

$$X \times [0, \infty) \xrightarrow{g^* f^*} X \times [0, \infty) \xrightarrow{g^* f^*} X \times [0, \infty) \xrightarrow{g^* f^*} \cdots\cdots.$$

Thus there exists a bundle homeomorphism h_1 of $X \times [0, \infty)$ onto the direct limit of

$$(*) \qquad X\times[0,\infty)\xrightarrow{f^*}Y\times[0,\infty)\xrightarrow{g^*}X\times[0,\infty)\xrightarrow{f^*}Y\times[0,\infty)\xrightarrow{g^*}\cdots$$

such that h_1 is bundle homotopic to the inclusion i_1 of the first $X\times[0,\infty)$ into the limit.

By similar reasoning, there exists a bundle homeomorphism h_2 of $Y\times[0,\infty)$ onto the direct limit of (*) such that h_2 is bundle homotopic to the inclusion i_2 of the second $Y\times[0,\infty)$ into the limit.

Then $h'=h_2^{-1}h_1:X\times[0,\infty)\to Y\times[0,\infty)$ is a bundle homeomorphism which is bundle homotopic to $h_2^{-1}i_1=h_2^{-1}i_2f^*$, hence bundle homotopic to f^*, so to $f\times\mathrm{id}$. By Strong Bundle Stability Theorem, the projections $\pi_X:X\times[0,\infty)\to X$ and $\pi_Y:Y\times[0,\infty)\to Y$ are bundle homotopic to bundle homeomorphisms. Then the result follows. □

Proof of Lemma 7-2: For each continuous map $r:X\to[0,\infty)$, briefly denote by X_r the variable product of $[0,\infty)$ by X, that is,

$$X_r=X\times_r[0,\infty)=\{(x,t)\in X\times[0,\infty)\,|\,t\leqq r(x)\}.$$

For $t\in[0,\infty)$, $X_t=X\times[0,t]$. And $X\times[0,\infty)$ is denoted by X_∞.

If $a,b,c,d:X\to(0,\infty)$ are continuous maps such that $a<b<d$ and $a<c<d$, then define the X-preserving homeomorphism $s(a,b,c,d):X_\infty\to X_\infty$ by

$$s(a,b,c,d)(x,t)=\begin{cases}\left(x,\dfrac{c(x)-a(x)}{b(x)-a(x)}(t-a(x))+a(x)\right) & \text{if}\quad a(x)\leqq t\leqq b(x)\\[2ex] \left(x,\dfrac{d(x)-c(x)}{d(x)-b(x)}(t-d(x))+d(x)\right) & \text{if}\quad b(x)\leqq t\leqq d(x)\\[2ex] (x,t) & \text{otherwise.}\end{cases}$$

Observe that $s(a,b,c,d)^{-1}=s(a,c,b,d)$, $s(a,b,c,d)(X_b)=X_c$ and $s(a,b,c,d)|X_a\cup(X_\infty\setminus X_d)=\mathrm{id}$.

Using Dowker's Theorem ([Du] p. 171), construct continuous maps $a,b:X\to(0,\infty)$ such that

$$X_0\subset g_1(X_b)\subset X_{a/2}\subset X_a\subset g_1(X_1)\subset X_1.$$

Define an open bundle embedding $h_1:X_\infty\to X_\infty$ by

$$h_1|s\left(\frac{b}{2},b,1,2\right)\circ g_1^{-1}\circ s\left(\frac{a}{2},1,a,2\right)\circ g_1(X_2)$$

$$=g_1\circ s\left(\frac{b}{2},b,1,2\right)\circ g_1^{-1}\circ s\left(\frac{a}{2},a,1,2\right)\circ g_1\circ s\left(\frac{b}{2},1,b,2\right)$$

and

$$h_1|s\left(\frac{b}{2}, b, 1, 2\right)\circ g_1^{-1}\circ s\left(\frac{a}{2}, 1, a, 2\right)\circ g_1(X_\infty\setminus X_2)$$

$$=s\left(\frac{a}{2}, a, 1, 2\right)\circ g_1\circ s\left(\frac{b}{2}, 1, b, 2\right).$$

It is easy to check that $h_1|X_1=g_1|X_1$ and $g_1(X_\infty)\subset X_1\subset h_1(X_2)$. Similarly define an open embedding $h_2: X_\infty\to X_\infty$ such that $h_2|X_2=g_2h_1$ and $g_2(X_\infty)\subset h_2(X_3)$; and continue likewise constructing at the n-th stage an open embedding $h_n: X_\infty\to X_\infty$ such that $h_n|X_n=g_nh_{n-1}$ and $g_n(X_\infty)\subset h_n(X_{n+1})$.

Let j_n be the inclusion of the n-th stage into the direct limit dir $\lim(X_\infty, g_n)$. Then $h_\infty: X_\infty\to$ dir $\lim(X_\infty, g_n)$ defined by $h_\infty|X_{n+1}=j_{n+1}h_n|X_{n+1}$ is a bundle homeomorphism whose inverse is the limit of $h_n^{-1}g_n$. Since X deforms into X_1 by a bundle isotopy, h_∞ is bundle isotopic to j_1.

$$\begin{array}{ccccccccccc}
X_1 & \hookrightarrow & X_2 & \hookrightarrow & X_3 & \hookrightarrow & X_4 & \hookrightarrow & \cdots & & X_\infty \\
\text{id}\downarrow & & \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & & & \downarrow h_\infty \\
X_\infty & \xrightarrow[g_1]{} & X_\infty & \xrightarrow[g_2]{} & X_\infty & \xrightarrow[g_3]{} & X_\infty & \xrightarrow[g_4]{} & \cdots & & \text{dir}\lim(X_\infty, g_n)
\end{array}\ \square.$$

Finally we remark that a bundle version of Theorem 1 in [Ch₂] is valid.

7-3 THEOREM: *Let $p: X\to B$ and $q: Y\to B$ be E-manifold bundles. If $f, g: X\to Y$ are bundle homotopic bundle homeomorphism, then they are ambiently invertibly bundle isotopic.*

PROOF (after Chapman): It is sufficient to prove that a bundle homeomorphism $f: X\times(0, 1]\to X\times(0, 1]$ bundle homotopic to id is ambiently invertibly bundle isotopic to id. (By Bundle Stability Theorem in [Sa₂], $p: X\to B$ is bnudle isomorphic to $p\pi_X: X\times(0, 1]\to B$.)

Note that $X\times\{1\}$ is a B-preservingly E-deficient closed set in $X\times(0, 1]$ and $f|X\times\{1\}$ is a B-preserving homeomorphism of $X\times\{1\}$ onto a B-preservingly E-deficient closed set in $X\times(0, 1]$ which is B-preservingly homotopic to id. Using Bundle Homeomorphism Extension Theorem 6-1, f is ambiently invertibly bundle isotopic to a bundle homeomorphism $f': X\times(0, 1]\to X\times(0, 1]$ such that $f'|X\times\{1\}=\text{id}$.

Using the Alexander trick, let $f^*: X\times(0, 1]\times I\to X\times(0, 1]$ be defined by

$$f_t^*(x, s)=\begin{cases}(x, s) & \text{if } t\leqq s\\ t\cdot f'(x, s/t) & \text{if } s<t\end{cases}$$

where $t\cdot(x, u)=(x, tu)$. Then f^* is an ambient invertible bundle isotopy satisfying

$f_0^*=\mathrm{id}$ and $f_1^*=f'$. □

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