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ON SYMMETRY OF KNOTS

By

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§1. Introduction

A knot K in a 3-sphere S^3 is said to have period n [9] [13] (or to be a *periodic* knot of order n) if there is a rotation of S^3 with period n and axis A, where $A\cap K=\phi$, which leaves K invariant. (This definition is now equivalent to the original definition due to the positive solution to Smith Conjecture.)

One of the main problems is to determine periods of a given knot and so far, several necessary conditions for K to have period n have been found. See [1], [5], [7], [9], [13].

In this paper, we prove a few additional conditions using the covering linkage invariants which will be explained below.

Let J_n be a set of n letters, 1, 2, $\cdots$, n, and $\mathcal{S}(J_n)$ the groups of all permutations on J_n. Thus $\mathcal{S}(J_n)$ is isomorphic to the symmetric group of order n!

Let Γ be a finite transitive permutation group, i.e., Γ is a transitive subgroup of $\mathcal{S}(J_n)$.

An epimorphism $\theta: G\rightarrow\Gamma\leqq\mathcal{S}(J_n)$ is called, in this paper, a representation of G of degree n.

Two representations $\theta_1, \theta_2: G\rightarrow\Gamma$ will be called *equivalent* [4], is symbols $\theta_1\equiv\theta_2$, if there is an inner automorphism $\rho: \mathcal{S}(J_n)\rightarrow\mathcal{S}(J_n)$ such that $\rho\theta_1=\theta_2$.

Let M be a 3-manifold and $G=\pi_1(M)$. To each representation of G of degress n, there is defined uniquely (up to homeomorphism) an n-sheeted covering space $\tilde{M}$ of M. Equivalent representations define homeomorphic covering spaces.

Let K be a knot in S^3 and let $M=S^3-K$. In this paper, $\pi_1(S^3-K)$ is denoted by $G(K)$. A representation $\theta: G(K)\rightarrow\Gamma\leqq\mathcal{S}(J_n)$ defines the covering space $\tilde{M}$ of M, called the *unbranched* covering space of K in S^3. It is known that $\tilde{M}$ is of the form $M^*-\tilde{K}$ for some orientable closed 3-manifold M^* and a knot (or link) $\tilde{K}$ in M^*. The "completion" M^* of $\tilde{M}$ is called the *branched covering space* of

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S^3 branched along K, and $\tilde{K}$ is called the knot (or link) in M^* that covers K. If $\tilde{K}$ has more than one components, $\tilde{K}_1, \tilde{K}_2, \cdots, \tilde{K}_r$, say, it may be possible to consider the linking number $\alpha_{ij}=lk(\tilde{K}_i, \tilde{K}_j)$ between $\tilde{K}_i$ and $\tilde{K}_j$ in M^*. The set $\{\alpha_{i,j}, 1\leqq i<j\leqq r\}$ is called the *convering linkage invariant* of K associated with θ [6]. If $lk(\tilde{K}_i, \tilde{K}_j)$ does not exist, we simply write $\alpha_{ij}=-$.

For representations over the dihedral groups D_n, n being odd, this invariant has been used in [7], [10].

We will use this invariant to find some conditions for K to have period n. As applications of our theorems, it will be proven that knots 8_{10}, 8_{20}, 9_{24}, 9_{35}, 9_{46} [12] cannot have certain periods, for which all previously known conditions fail to rule out these periods.

At the end of the paper, we list all prime periods of knots with less than 10 crossing points.

§ 2. Preliminaries

Let G be an arbitrary group. Let $\theta_i: G\to\Gamma\leqq\mathcal{S}(J_n)$ $(i=1, 2)$ be representations of degree n, and $\phi: G\to G$ an automorphism.

PROPOSITION 2.1. *If* $\theta_1\equiv\theta_2$, *then* $\theta_1\phi\equiv\theta_2\phi$.

PROOF. Since $\theta_1\equiv\theta_2$, there exists an inner automorphism ρ of $\mathcal{S}(J_n)$ such that $\rho\theta_1=\theta_2$, and hence $\rho\theta_1\phi=\theta_2\phi$. Thus $\theta_1\phi\equiv\theta_2\phi$.

PROPOSITION 2.2. *Let* $\phi: G\to G$ *be an automorphism of order* p, *a prime. Let* $\theta: G\to\Gamma\leqq\mathcal{S}(J_n)$ *be a representation. Then either*

(1) $\theta\phi\equiv\theta$, *or*

(2) *no two of* p *representations* $\theta, \theta\phi, \cdots, \theta\phi^{p-1}$ *are equivalent.*

PROOF. Assume $\theta\phi\not\equiv\theta$ and $\theta\phi^k\equiv\theta\phi^l$, $0\leqq k<l<p$. Then $\theta\equiv\theta\phi^{l-k}$ by Proposition 2.1. Since g.c.d. $(p, l-k)=1$, there exist integers α, β such that $\alpha p+\beta(l-k)=1$. Then $\phi=\phi^{\alpha p}\phi^{\beta(l-k)}=\phi^{\beta(l-k)}$ yields $\theta\phi=\theta\phi^{\beta(l-k)}\equiv\theta$, since $\theta\equiv\theta\phi^{l-k}$. This contradicts $\theta\phi\not\equiv\theta$.

COROLLARY 2.3. *Under the same assumption as in Proposition* 2.2, *if* $\Gamma\lhd\mathcal{S}(J_n)$ *and* Γ *has no automorphism of order* p, *then* $\theta\phi\equiv\theta$ *yields* $\theta\phi=\theta$.

PROOF. Since $\theta\phi\equiv\theta$, there is an inner automorphism of $\mathcal{S}(J_n)$ such that $\theta\phi=\rho\theta$. Then $\theta\phi^p=\rho^p\theta$ and hence $\theta=\rho^p\theta$, since $\phi^p=id$. Therefore $\rho^p|\Gamma=id$. Since Γ does not have an automorphism of order p, and $\Gamma\lhd\mathcal{S}(J_n)$, it follows that $\rho|\Gamma=id$, i.e., $\theta\phi=\theta$.

§ 3. Covering linkage invariants.

We consider S^3 as $R^3 \cup \{\infty\}$, and use cylindrical coordinates (r, θ, t) for R^3. Let τ_m be the rotation of S^3 about the t-axis T through $2\pi/m$, i.e.,

$$\begin{cases} \tau_m(r, \theta, t) = \left(r, \theta + \dfrac{2\pi}{m}, t\right) \\ \tau_m(\infty) = \infty . \end{cases}$$

Let K be a periodic knot of order m. Applying an isotopy deformation if necessary, we may assume without loss of generality that $\tau_m(K)=K$ and $K \cap T = \phi$. Such a rotation τ_m will be called the rotation associated with K.

In the following, a pair (K, τ_m) will also be called a periodic knot.

Now the identification space S^3/τ is again a 3-sphere Σ^3 and $K/\tau = \hat{K}$ becomes a knot in Σ^3. Let N_m be a 3-ball $\{(r, \theta, t) \mid 0 \leqq \theta \leqq 2\pi/m\} \cup \{\infty\}$. Then the presentations of $G(K)=\pi_1(S^3-K)$ and $G(\hat{K})=\pi_1(\Sigma^3-\hat{K})$ can be obtained from that of $\pi_1(N_m-K)$ as follows. (Also see Example 1.)

First, give an orientation to K. Let $A_1, A_2, \cdots, A_d$ be the points of intersection $K \cap \{(r, 0, t) \mid -\infty < r, t < \infty\} \subset \partial N_m$. Denote $B_i = \tau_m(A_i) \subset \partial N_m$, $i=1, 2, \cdots, d$.

Let $\mathcal{P}_0 = \langle x_1, x_2, \cdots, x_g \mid R_1, R_2, \cdots, R_{g-d} \rangle$ be a Writinger presentation of $\pi_1(N_m-K)$. Each generator x_i is represented by a small oriented loop once around an arc in a positive direction, and each relation R_i is of the form: $x_j^{\varepsilon_i} x_l x_j^{-\varepsilon_i} x_r^{-1} = 1$, $\varepsilon_i = +1$ or -1. For simplicity, we assume that the first d generators $x_1, \cdots, x_d$ and the last d generators $x_{q+1}, \cdots, x_{q+d}$ $(q+d=g)$ correspond to arcs intersecting ∂N_m at A_i and arcs intersecting ∂N_m at B_i, respectively. With these conventions, $G(\hat{K})$ has a presentation

$$\hat{\mathcal{P}} = \langle x_1, x_2, \cdots, x_g \mid R_1, R_2, \cdots, R_q, x_1 = x_{q+1}, \cdots, x_d = x_{q+d}, (q+d=g) \rangle .$$

Denote by $\mathcal{F}(u_1, u_2, \cdots, u_k)$ the free group generated freely by $u_1, u_2, \cdots, u_k$.

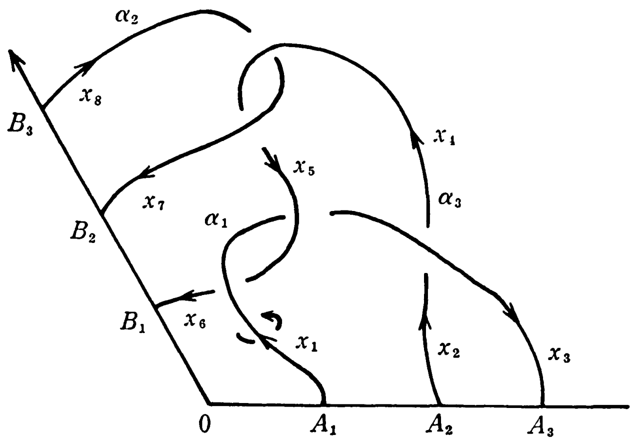
Now R_i $(i=1, 2, \cdots, q)$ is an element of $\mathcal{F}(x_1, x_2, \cdots, x_g)$. We define $R_{j,i}$ $(j=1, 2, \cdots, m)$ as an element of $\mathcal{F}(x_{j,1}, x_{j,2}, \cdots, x_{j,g})$ obtained from R_i by replacing every x_k^{ε} appearing in R_i by $x_{j,k}^{\varepsilon}$. Then $G(K)$ has a presentation:

$$\mathcal{P} = \langle x_{j,i} \mid R_{j,l}, x_{j,q+k} = x_{j+1,k} \rangle$$

where $1 \leqq j \leqq m$, $1 \leqq i \leqq g$, $1 \leqq l \leqq q$, $1 \leqq k \leqq d$ and j is taken modulo m.

The following example illustrates these presentations. For simplicity, we use x_i, y_i, z_i for $x_{1,i}, x_{2,i}, x_{3,i}$. Also we note that $\tau_{m*}(x_{j,i}) = x_{j+1,i}$.

EXAMPLE 1.



$N_m - K$

Fig. 1.

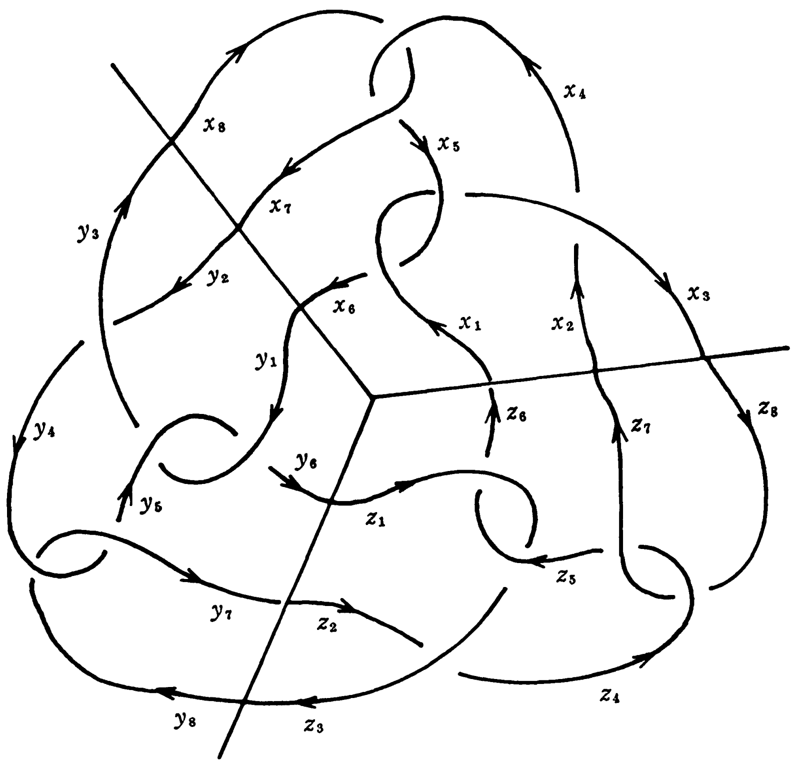


Fig. 2.

$$\mathcal{P}_0 = \langle x_1, x_2, x_3, \cdots, x_8 \mid R_1 = 1, \cdots, R_5 = 1 \rangle,$$

where

$$R_1 = x_5^{-1} x_1 x_5 x_3^{-1}, \qquad R_2 = x_3^{-1} x_2 x_3 x_4^{-1}, \qquad R_3 = x_4^{-1} x_8 x_4 x_7^{-1},$$

$$R_4 = x_7^{-1} x_4 x_7 x_5^{-1}, \qquad R_5 = x_1^{-1} x_5 x_1 x_6^{-1}.$$

$$\hat{\mathcal{P}}=\langle x_1, x_2, \cdots, x_8 | R_1=1, \cdots, R_5=1, x_1=x_6, x_2=x_7, x_3=x_8\rangle,$$

and

$$\mathcal{P}=\langle x_1, \cdots, x_8, y_1, \cdots, y_8, z_1, \cdots, z_8 | R_{1,1}=1, \cdots, R_{3,5}=1$$

$$x_{5+j}=y_j, y_{5+j}=z_j, z_{5+j}=x_j, j=1, 2, 3\rangle,$$

where $R_{1,i}=R_i$, $R_{2,1}=y_5^{-1}y_1y_5y_3^{-1}, \cdots, R_{3,5}=z_1^{-1}z_5z_1z_6^{-1}$.

PROPOSITION 3.1. *Let (K, τ) be a periodic knot of order m, and $\tau_*: G(K) \to G(K)$ an automorphism induced from τ.*

Let $\theta: G(K)\to\Gamma\leqq\mathcal{S}(J_n)$ be a representation of $G(K)$. Then

(1) *τ_* is not identity, and*

(2) *θ induces the representation of $G(\hat{K})=G(K/\tau)$ onto Γ if $\theta\tau_*=\theta$. Conoersely, if there is a representation $\hat{\theta}: G(\hat{K})\to\Gamma\leqq\mathcal{S}(J_n)$, then there is a representation $\theta: G(K)\to\Gamma\leqq\mathcal{S}(J_n)$ such that $\theta\tau_*=\theta$.*

PROOF. Since (1) is well-known [13], we only need to show (2).

Suppose $\theta\tau_*=\theta$. Using presentations $\mathcal{P}$, $\hat{\mathcal{P}}$, define $\hat{\theta}: G(\hat{K}) \to \Gamma\leqq\mathcal{S}(J_n)$ by $\hat{\theta}(x_i)=\theta(x_{1,i})$. $\hat{\theta}$ is well-defined. To see $\hat{\theta}$ is onto, it suffices to check that $\{\theta(x_{1,i}), 1\leqq i\leqq g\}$ generates Γ, since $\theta(x_{1,i})=\theta(x_{j,i})$ for any j. The converse is obvious.

To each knot K in S^3, we can assign a meridian-longitude pairs, (μ_K, l_K), μ_K, $l_K\in G(K)$. For a periodic knot (K, τ), we always choose, $x_{1,1}$ as μ_K.

THEOREM 3.2. *Let (K, τ) be a periodic knot of period m. Let $\theta: G(K)\to \Gamma\leqq\mathcal{S}(J_n)$ be a representation of $G(K)$. Suppose $\theta\tau_*=\theta$. Let $\hat{\theta}: G(K)\to\Gamma$ be the homomorphism induced by θ. Assume that $\theta(l_K)=id$ and $\hat{\theta}(l_{\hat{K}})=id$.*) Let M^* be the branched covering space of S^3 branched along K associated with θ.*

Suppose that K is covered by r knots $\tilde{K}_1, \tilde{K}_2, \cdots, \tilde{K}_r$, $r>1$, and let $\{\alpha_{i,j}\}$ be the covering linkage invariants of K. Let b be the order of the torsion group of $H_1(M^*; Z)$. Then if $\alpha_{ij}=lk(\tilde{K}_i, \tilde{K}_j)$ exists, α_{ij} is of the form am/b for some integer a. In particular, if g.c.d. $(b, m)=1$, then $\alpha_{ij}\equiv 0 \pmod m$.

PROOF. First we need precise expressions for longitudes l_K and $l_{\hat{K}}$ of K and $\hat{K}$.

Consider $(N_m, N_m\cap K)$. $N_m\cap K$ consists of d arcs $\alpha_1, \alpha_2, \cdots, \alpha_d$ and $\partial N_m\cap K$ consists of $2d$ points $\mathcal{A}=\{A_1, A_2, \cdots, A_d\}$ and $\mathcal{B}=\{B_1, B_2, \cdots, B_d\}$ (See Fig. 1). α_i connects two points in $\mathcal{A}\cup\mathcal{B}$. We may assume that $A_i=(r_i, 0, 0)$ and $B_i=(r_i, 2\pi/m, 0)$. Define an involution $\nu: \mathcal{A}\cup\mathcal{B}\to\mathcal{A}\cup\mathcal{B}$ by $\nu(A_i)=B_i$ and $\nu(B_i)=A_i$.

*) These conditions are always satisfied if Γ is metabelian, i.e. $\Gamma''=1$.

Since α_i is oriented, α_i has two end points, called the *initial* point X_i and the *terminal* point Y_i. If A_1 is the terminal point of some arc α_i, then reverse the orientation of K so that A_1 is the initial points of α_i. Now we rearrange $\alpha_1, \alpha_2, \cdots, \alpha_d$ as follows. α_1 is the arc whose initial point is A_1, and inductively, α_j is the arc whose initial point is $\nu(Y_{j-1})$, $j=2, 3, \cdots, d$, where Y_{j-1} is the terminal point of α_{j-1} (new).

To get $l_{\hat{K}}$, choose $2d$ line segments $\gamma(C)$, $C \in \mathcal{A} \cup \mathcal{B}$ on ∂N_m that connect $(0, 0, 1)$ to C.

Let W_j be the element in $\pi_1(N_m - K)$ that represents the loop $\gamma(X_j)\alpha_j\gamma(Y_j)^{-1}$. Then $l_{\hat{K}}$ is given by

$$l_{\hat{K}} = W_1 W_2 \cdots W_d x_1^{\sigma}, \tag{3.1}$$

where σ is an integer.
For example, for the knot $\hat{K}$ in Fig. 1,

$$l_{\hat{K}} = W_1 W_2 W_3 x_1^{-5} = (x_5)(x_4)(x_3 x_7 x_1) x_1^{-5}.$$

Let $\phi : J_m \times J_d \to J_m$ be a function defined by

$$\begin{cases} \phi(j, i) = j-1 \pmod m & \text{if } \ Y_i \in \mathcal{A} \\ \phi(j, i) = j+1 \pmod m & \text{if } \ Y_i \in \mathcal{B}, \end{cases} \tag{3.2}$$

where Y_i is the terminal point of α_i.

EXAMPLE 1 (Continued).

$$\begin{array}{lll} \phi(1, 1)=3, & \phi(1, 2)=2, & \phi(1, 3)=2 \\ \phi(2, 1)=1, & \phi(2, 2)=3, & \phi(2, 3)=3 \\ \phi(3, 1)=2, & \phi(3, 2)=1, & \phi(3, 3)=1. \end{array}$$

Let $\mathcal{X}_i$ be the set $\{x_{i,1}, x_{i,2}, \cdots, x_{i,g}\}$ and $\mathcal{F}(\mathcal{X}_i)$ the free group freely generated by $\mathcal{X}_i$.

Using $l_{\hat{K}}$, we can show that a longitude l_K of K is given by

$$\begin{aligned} l_K = & W_1(\mathcal{X}_{\nu(1,1)}) W_2(\mathcal{X}_{\nu(1,2)}) \cdots W_d(\mathcal{X}_{\nu(1,d)}) \\ & W_1(\mathcal{X}_{\nu(2,1)}) \cdots W_d(\mathcal{X}_{\nu(2,d)}) \\ & \cdots\cdots\cdots\cdots \\ & W_1(\mathcal{X}_{\nu(m,1)}) \cdots W_d(\mathcal{X}_{\nu(m,d)}) \\ = & \prod_{j=1}^{m} [W_1(\mathcal{X}_{\nu(j,1)}) W_2(\mathcal{X}_{\nu(j,2)}) \cdots W_d(\mathcal{X}_{\nu(j,d)})] x_{1,1}^{m\sigma} \end{aligned} \tag{3.3}$$

where indices $\nu(j, i)$ is defined, inductively, by

$$\begin{cases} \nu(1,1)=1 \\ \nu(j,1)=\nu(j-1,d)+1 & \text{for } j>1 \\ \nu(j,i)=\phi(\nu(j,i-1),i-1) & \text{for } i>1, \end{cases}$$

and $W_i(\mathcal{X}_k)$ is the element of $\mathcal{F}(\mathcal{X}_k)$ that is obtained from W_i by replacing every x_j^ε appearing in W_i by $x_{k,j}^\varepsilon$.

EXAMPLE 1 (Continued). $l_K=W_1(\mathcal{X}_1)W_2(\mathcal{X}_3)W_3(\mathcal{X}_1)W_1(\mathcal{X}_2)W_2(\mathcal{X}_1)W_3(\mathcal{X}_2)W_1(\mathcal{X}_3)W_2(\mathcal{X}_2)W_3(\mathcal{X}_3)x_1^{-15}=(x_5z_4x_3x_7x_1)\cdot(y_5x_4y_3y_7y_1)\cdot(z_5y_4z_3z_7z_1)x_1^{-15}$.

Now, $\theta\tau_*=\theta$, by assumption, and since $\tau_*(x_{1,i})=x_{2,i}$, it follows that $\theta(x_{j,i})=\theta\tau_*^{j-1}(x_{1,i})=\theta(x_{1,i})$. Therefore, $\theta W_k(\mathcal{X}_{\nu(r,s)})=\theta W_k(\mathcal{X}_1)$ and hence

$$\begin{aligned}\theta(l_K)&=\prod_{j=1}^{m}\theta[W_1(\mathcal{X}_{\nu(j,1)})W_2(\mathcal{X}_{\nu(j,2)})\cdots W_d(\mathcal{X}_{\nu(j,d)})]\theta(x_{1,1})^{m\sigma}\\ &=\prod_{j=1}^{m}\theta(W_1(\mathcal{X}_1)W_2(\mathcal{X}_1)\cdots W_d(\mathcal{X}_1))\cdot\theta(x_{1,1})^{m\sigma}\\ &=[\theta(W_1(\mathcal{X}_1)W_2(\mathcal{X}_1)\cdots W_d(\mathcal{X}_1)x_{1,1}^{\sigma})]^m\\ &[\hat{\theta}(W_1W_2\cdots W_dx_1^{\sigma})]^m\\ &=\hat{\theta}(l_{\hat{K}})^m.\end{aligned}$$

Thus, we obtain

$$\theta(l_K)=\hat{\theta}(l_{\hat{K}})^m. \tag{3.4}$$

Let $\{O_1, O_2, \cdots, O_r\}$ be the set of orbits of J_n under the action of $\theta(x_{1,1})$ (and $\theta(l_K)=id$). Assume that O_i corresponds to a covering knot $\tilde{K}_i$.

Since $lk(\tilde{K}_i, \tilde{K}_j)$ exists, there is a linking homomorphism

$$\xi:\mathcal{F}(\{x_{j,i,k}\,;\,1\leqq j\leqq m,\ 1\leqq i\leqq g=q+d,\ 1\leqq k\leqq n\})\longrightarrow Q.$$

[6, Proposition 7.1].

More precisely, let b be the order of the torsion group of $H_1(M^*;Z)$ and Q_P the additive group of rationals of the form a/b with an integer a. Then ξ is, in fact, a homomorphism

$$\xi:\mathcal{F}(\{x_{j,i,k}\})\longrightarrow Q_P$$

satisfying

$$\sum_{k\in O_i}\xi(x_{1,1,k})=1 \quad\text{and}\quad \sum_{\substack{l\in O_t\\ t\neq i}}(x_{1,1,l})=0. \tag{3.5}$$

Since $x_{j,i}$ and $x_{1,1}$ are conjugate, we note that

$$\sum_{k\in O_i}\xi(x_{j,s,k})=1 \quad\text{and}\quad \sum_{\substack{l\in O_t\\ t\neq i}}\xi(x_{j,s,l})=0. \tag{3.6}$$

We will show that there is a linking homomorphism

$$\eta : \mathcal{F}(\{x_{j,i,k}\}) \longrightarrow Q_P \qquad \text{such that}$$

(3.7) $$\text{(1)} \quad \sum_{k \in O_i} \eta(x_{1,1,k}) = 1 \quad \text{and} \quad \sum_{\substack{l \in O_t \\ t \neq i}} \eta(x_{1,1,l}) = 0$$

and

$$\text{(2)} \quad \eta(x_{1,i,k}) = \eta(x_{2,i,k}) = \cdots = \eta(x_{m,i,k})$$

for all i and k.

To obtain η, we define inductively

$$\begin{cases} \xi_1 = \xi \\ \xi_l(x_{j,i,k}) = \xi_{l-1}(x_{j-1,i,k}), & 2 \leqq l \leqq m. \end{cases}$$

Then each ξ_l is also a linking homomorphism. This follows from the specific presentation $\mathcal{P}$ of $G(K)$. Therefore, it follows from (3.6) by induction that

(3.8) $$\sum_{k \in O_i} \xi_l(x_{i,s,k}) = 1 \quad \text{and} \quad \sum_{\substack{k \in O_t \\ t \neq i}} \xi_l(x_{j,s,k}) = 0$$

for all l.

Define

$$\eta(x_{j,s,k}) = \frac{1}{m} \sum_{l=1}^{m} \xi_l(x_{j,s,k}).$$

Then

$$\sum_{k \in O_i} \eta(x_{j,s,k}) = 1 \quad \text{and} \quad \sum_{\substack{k \in O_t \\ t \neq i}} \eta(x_{j,s,k}) = 0$$

Further, we have

$$\begin{aligned} \eta(x_{j,s,k}) &= \frac{1}{m} \{\xi_1(x_{j,s,k}) + \xi_2(x_{j,s,k}) + \cdots + \xi_m(x_{j,s,k})\} \\ &= \frac{1}{m} \{\xi_1(x_{j,s,k}) + \xi_1(x_{j-1,s,k}) + \xi_1(x_{j-2,s,k}) + \cdots + \xi_1(x_{j+1,s,k})\} \\ &= \frac{1}{m} \sum_{q=1}^{m} \xi_1(x_{q,s,k}), \end{aligned}$$

and hence, $\eta(x_{j,s,k}) = \eta(x_{l,s,k})$ for all l, j, which proves (3.7).

Now, since η is a linking homomorphism of $\mathcal{F}(\{x_{j,i,k}\})$, $\eta(x_{j,i,k})$ are obtained as solutions of a certain system of linear equations [6, p. 1328] and the coefficient matrix of the system is exactly the relation matrix of $H_1(M^*; Z)$, and hence, $\eta(x_{i,j,k})$ belongs to Q_P.

Further, since $\eta(x_{j,s,k}) = \eta(x_{l,s,k})$ for all j, l, η also defines a linking homomorphism

$$\hat{\eta} : \mathcal{F}(\{x_{i,k} | 1 \leqq i \leqq g,\ 1 \leqq k \leqq n\}) \longrightarrow Q_P$$

by putting $\hat{\eta}(x_{i,k}) = \eta(x_{1,i,k})$.

Now Corollary 7.3 in [6] shows that

$$\eta(\mathcal{D}_k l_K) = lk(\tilde{K}_i, \tilde{K}_j) \qquad \text{for } k \in O_j.$$

(For definition of $\mathcal{D}_k$, see [6, p. 1318].)

Since $\theta(l_K) = \hat{\theta}(l_{\hat{K}})^m$, it follows that $\eta(\mathcal{D}_k l_K) = m\hat{\eta}(\mathcal{D}_k l_{\hat{K}})$.

Since $\hat{\eta}\mathcal{D}_k(l_{\hat{K}})$ is of the form a/b, $\eta(\mathcal{D}_k l_K)$ is of the fhrm ma/b, and hence $lk(\tilde{K}_i, \tilde{K}_j)$ is of the form ma/b. This proves Theorem 3.2.

REMARK. An analogous theorem for a dihedral representation $\theta : G(K) \rightarrow D_n \leqq \mathcal{S}(J_n)$ has been proven in [7] under complicated conditions, which confirm the existence of the dihedral covering linkage invariant. R. I. Hartley also obtains a similar result.

COROLLARY 3.3. *Under the same assumption as in Theorem* 3.2,

$$lk(\tilde{K}_i, \tilde{K}_j) = m\, lk_{\hat{M}}(\hat{K}_i, \hat{K}_j),$$

where $\hat{M}$ is the branched covering space of S^3 branched along $\hat{K}$ associated with $\hat{\theta}$, and $\hat{K}_1, \cdots, \hat{K}_r$ are knots that cover $\hat{K}$ in $\hat{M}$, and $lk_{\hat{M}}(\hat{K}_i, \hat{K}_j)$ denotes the linking number between $\hat{K}_i$ and $\hat{K}_j$ in $\hat{M}$, and assume that $\tilde{K}_i$ and $\hat{K}_i$ correspond to the same orbit in J_n.

PROOF. This is essentially what we have shown in the proof of Theorem 3.2.

§ 4. Equivalent representations.

Let (K, τ) be a periodic knot of order m.

Let $\theta : G(K) \rightarrow \Gamma \leqq \mathcal{S}(J_n)$ be a representation. Then, $\theta, \theta\tau_*, \cdots, \cdots, \theta\tau_*^{m-1}$ are also representations of $G(K)$. They may be equivalent to each other. Even, if they are not equivalent, the corresponding spaces are homeomorphic. In particular, we have

PROPOSITION 4.1. *Covering linkage invariants obtained from these representations are identical as sets.*

EXAMPLE 2. Let K_0 be a trefoil knot and let $K = K_0 \# K_0$. Obviously, K has period 2. Let $\theta : G(K) \rightarrow D_3 \leqq \mathcal{S}(J_3)$ be a representation given in Fig. 3. Then the second representation $\theta\tau_* : G(K) \rightarrow D_3 \leqq \mathcal{S}(J_3)$ is given in Fig. 4. Since there is no inner automorphism ρ of $\mathcal{S}(J_3)$ with $\theta\tau_* = \rho\theta$, θ is not equivalent to $\theta\tau_*$. However, the covering linkage invariants are {2} for both θ and $\theta\tau_*$.

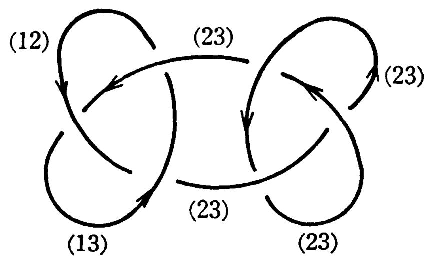


Fig. 3.

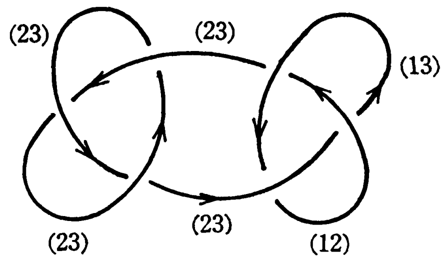


Fig. 4.

In this section, we prove that under certain conditions, $\theta\tau_*$ and θ cannot be equivalent for many θ.

THEOREM 4.2. *Let (K, τ) be a periodic knot of order p, an odd prime. Suppose that $G(K)$ has a representation $\tilde{\theta}$ on D_p of degree p such that $\tilde{\theta}(x_{1,1})=(1\ p-1)(2\ p-2)\cdots((p-1)/2\ (p+1)/2)$. If $G(\hat{K})=G(K/\tau)$ does not have a representation on D_p, then there is one and only one representation θ such that $\theta\tau_*\equiv\theta$.*

PROOF. We use the same notation and symbols as those used in § 3. We study a representation θ such that $\theta\tau_*\equiv\theta$. Since $G(\hat{K})\not\to D_p$, it follows from Proposition 3.1 (2) that $\theta\tau_*\neq\theta$, and hence there is an inner automorphism ρ of $\mathcal{S}(J_p)$ such that $\theta\tau_*=\rho\theta$. Since τ_* has order p, ρ must have order p and $\rho\neq id$. Therefore, ρ is a conjugation by a cycle $(12\cdots p)^\lambda$ for some $\lambda\neq 0$.

Since $\tau_*(x_{j,i})=x_{j+1,i}$, we have $\theta(x_{j+1,i})=\theta\tau_*(x_{j,i})=\theta\tau_*^j(x_{1,i})=\rho^j\theta(x_{1,i})$ and hence, θ is completely determined if $\theta_0:\pi_1(N_p-K)\to D_p\leqq\mathcal{S}(J_p)$ is given subject to $\theta_0(x_{1,q+i})(=\theta\tau_*(x_{1,i}))=\rho\theta_0(x_{1,i})$, since $x_{2,i}=x_{1,q+i}$ in $G(K)$. Therefore, we study θ_0. Now, a slightly modified argument used in [3, p. 160-162] shows that each representation $\theta_0:\pi_1(N_p-K)\to D_p\leqq\mathcal{S}(J_p)$ corresponds to a solution of the system of linear equations (4.1), (4.2) over the field $Z/(p)$:

$$(4.1)\quad \begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1,g}x_g\equiv 0 \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2,g}x_g\equiv 0 \\ \quad\cdots\cdots \\ a_{q,1}x_1+a_{q,2}x_2+\cdots+a_{q,g}x_g\equiv 0 \end{cases} \pmod{p}$$

$$(4.2)\quad \begin{cases} x_1+\lambda\equiv x_{q+1}, \\ x_2+\lambda\equiv x_{q+2} \\ \vdots \\ x_d+\lambda\equiv x_{q+d}, \qquad (q+d=g) \end{cases} \pmod{p}$$

where the coefficient matrix $A=\|a_{ij}\|_{1\leq i\leq q, 1\leq j\leq g}$, of (4.1), is the Jacobian matrix

$$\left\|\frac{\partial R_i}{\partial x_j}\right\|$$

evaluated at $x_1=x_2=\cdots=x_g=-1$. The correspondence between a solution and a representation will be given as follows.

Let $(x_1, x_2, \cdots, x_g)=(c_1, c_2, \cdots, c_g)$ be a non-trivial solution of (4.1), (4.2). Let $\langle a, b \mid a^2=b^p=(ab)^2=1\rangle$ be a presentation of D_p. Then $\theta_0: \pi_1(N_p-K)\to D_p$ is given by

$$\text{(4.3)} \qquad \begin{cases} \theta_0(x_i)=b^{-c_i}ab^{c_i}=(c_i)(c_i-1,\ c_i+1)(c_i-2,\ c_i+2) \\ \qquad\qquad \cdots\left(c_i-\dfrac{p-1}{2},\ c_i+\dfrac{p-1}{2}\right) \\ \theta_0(x_{q+i})=\rho\theta_0(x_i). \end{cases}$$

Now eliminate unknowns $x_{q+1}, \cdots, x_{q+d}$ using (4.2) to obtain

$$\text{(4.4)} \qquad \begin{cases} (a_{11}+a_{1,q+1})x_1+\cdots+(a_{1,d}+a_{1,g})x_d+a_{1,d+1}x_{d+1} \\ \qquad +\cdots+a_{1,q}x_q=-\lambda(a_{1,q+1}+\cdots+a_{1,g}) \\ \cdots\cdots \\ (a_{q,1}+a_{q,q+1})x_1+\cdots+(a_{q,d}+a_{q,g})x_d+a_{q,d+1}x_{d+1} \\ \qquad +\cdots+a_{q,q}x_q=-\lambda(a_{q,q+1}+\cdots+a_{q,g}). \end{cases}$$

The coefficient matrix B of (4.4) is exactly the Alexander matrix of $G(K)$ evaluated at $x_1=x_2=\cdots=x_q=-1$.

Let

$$C=\begin{bmatrix} -\lambda(a_{1,q+1}+\cdots a_{1,g}) \\ \vdots \\ -\lambda(a_{q,q+1}+\cdots a_{q,g}) \end{bmatrix}.$$

Since $\theta\tau_*\equiv\theta$, there is at least one representation $\theta_0: \pi_1(N_p-K)\to D_p$. Therefore, there is at least one solution for (4.4), and hence, rank $B=$rank (BC) over $Z/(p)$.

By a property of the Alexander matrix, det $B=0$ (see, for example, [3, p. 162]) and thus rank $B\leqq q-1$. However since $G(\hat{K})\not\to D_p$, it follows from [3] that rank $B\geqq q-1$ over $Z/(p)$, and hence rank $B=$rank $(BC)=q-1$. Therefore, there are exactly p distinct solutions. We claim that representations corresponding to these solutions are equivalent.

Let $v=(\alpha_1, \alpha_2, \cdots, \alpha_q)$ and $w=(\beta_1, \beta_2, \cdots, \beta_q)$ be two solutions of (4.4).

Then $v-w$ is a solution of the system of homogeneous linear equations

$$\text{(4.5)} \qquad B\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{pmatrix}\equiv 0 \qquad (\text{mod } p)$$

Since $v \neq w$, $\alpha_i - \beta_i \neq 0$ for some i.

Consider the system of equations with $q-1$ unknowns

$$(4.6) \qquad B \begin{pmatrix} x_1 \\ \vdots \\ x_{i-1} \\ \alpha_i - \beta_i \\ x_{i+1} \\ \vdots \\ x_q \end{pmatrix} \equiv 0 \qquad (\text{mod } p)$$

Since rank $B = q-1$, (4.6) has a unique solution, if it exists. Now, since $\sum_{i=1}^{q} a_{j,i} = 0$ for $j = 1, 2, \cdots, q$, (this is a property of the Alexander matrix of a knot), one (obvious) solution of (4.6) is $(\alpha_i - \beta_i, \cdots, \alpha_i - \beta_i)$. Therefore, $\alpha_t - \beta_t = \alpha_i - \beta_i = l$, say, for all t, and $w = (\alpha_1 - l, \alpha_2 - l, \cdots, \alpha_q - l)$. Then the representations $\hat{\theta}_1$, $\hat{\theta}_2$ corresponding to v, w are:

$$\hat{\theta}_1(x_k) = b^{-\alpha_k} a b^{\alpha_k}$$

$$\hat{\theta}_2(x_k) = b^{-(\alpha_k - l)} a b^{\alpha_k - l} = b^{-l} \hat{\theta}_1(x_k) b^l$$

and hence $\hat{\theta}_1 \equiv \hat{\theta}_2$.
Therefore, if $\theta_1 \tau_* \equiv \theta_1$ and $\theta_2 \tau_* \equiv \theta_2$, then $\theta_1 \equiv \theta_2$. To show that there exists θ such that $\tau\theta_* \equiv \theta$, it only suffices to note [4] that $G(K)$ has $p^s - 1/p - 1$ ($\equiv 1 \not\equiv 0$ (mod p)) representations on D_p for some integer s.

This proves Theorem 4.2.

COROLLARY 4.3. *Under the same assumption as in Theorem 4.2, if $G(K)$ has more than one representations on D_p, then there is a representation θ such that $\theta, \theta\tau_*, \theta\tau_*^2, \cdots, \theta\tau_*^{p-1}$ are all inequivalent.*

EXAMPLE 3. A knot $K = 9_{47}$ (Fig. 5 below) has a period 3 and $\hat{K}$ is a trivial knot. $G(K)$ has 4 representations on $D_3 \leqq \mathcal{S}(J_3)$. The covering linkage invariants of K are {2/3}, {2/3}, {2/3}, {−2/3} [4, p. 200]. The last covering linkage invariant corresponds to a representation θ such that $\theta\tau_* \equiv \theta$.

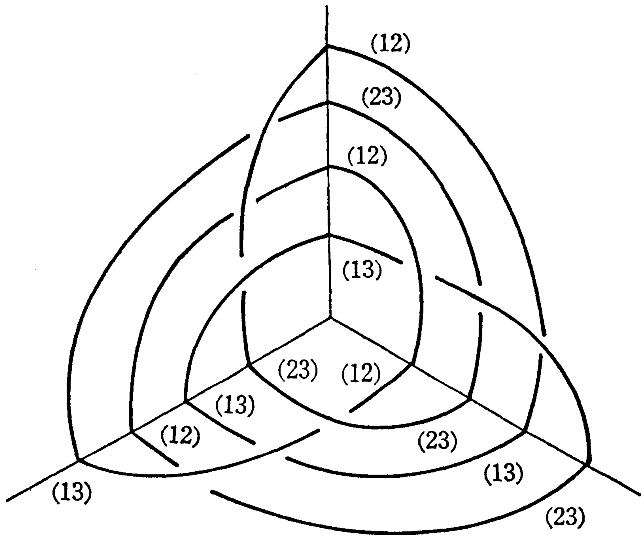


Fig. 5.

§ 5. Applications.

In order to show that a knot K does not have period p, first try to find a representation of $G(K)$ onto $\Gamma \leqq \mathcal{S}(J_n)$ which has no automorphisms of order p. If there is only one representation (up to equivalent) and $\Gamma \lhd \mathcal{S}(J_n)$, then $\theta\tau_* \equiv \theta$ yields $\theta\tau_* = \theta$. Therefore θ induces a representation $\hat{\theta}: G(\hat{K}) \to \Gamma \leqq \mathcal{S}(J_n)$. Further, if the covering linkage invariant is defined, we can apply Theorem 3.2 or Theorem 4.2.

Besides these theorems, the following proposition will be used frequently.

PROPOSITION 5.1 [9, Theorem 1, p. 169].
Let (K, τ) be a periodic knot of order p, a prime.
Let $\Delta(t)$ and $\hat{\Delta}(t)$ be the Alexander polynomials of K and $\hat{K}=K/\tau$. Then
(1) *$\hat{\Delta}(t)$ divides $\Delta(t)$ and,*
(2) $\Delta(t) \equiv \hat{\Delta}(t)^p(1+t+\cdots+t^{\lambda-1})^{p-1}$ (mod p),
where λ is a positive integer such that g.c.d. $(\lambda, p)=1$.

PROPOSITION 5.2. *A knot 10_{137} (Fig. 6 below) cannot have period* 5.

PROOF. Since the Alexander polynomial of K is $\Delta(t)=1-6t+11t^2-6t^3+t^4$ $=(1-3t+t^2)^2$, there is a representation $\theta: G(K) \to A_4 \leqq \mathcal{S}(J_4)$, where A_4 is the alternating group on 4 letters. In fact, $\theta(a)=(123)$, $\theta(b)=(134)$, $\theta(c)=(243)$, is one of such representations, and $G(K)$ does not have other representations (up to

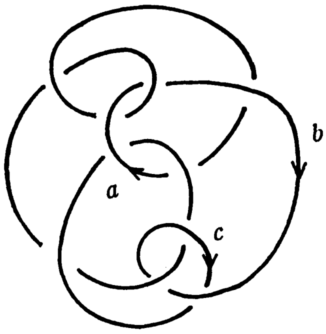


Fig. 6.

equivalent). Suppose that (K, τ) is a periodic knot of order 5. Then $\theta\tau_*=\theta$ by Corollary 2.3. Therefore, by Proposition 3.1, there exists $\hat{\theta}: G(\hat{K})\to A_4 \leqq \mathcal{S}(J_4)$. Now it follows from Proposition 5.1 that $\hat{\Delta}(t)=1$. But, then $G(\hat{K})$ cannot have a representation onto A_4 [11, p. 609]. Therefore, K cannot have period 5.

PROPOSITION 5.3. *A knot $K=9_{35}$ cannot have period* 7.

PROOF. Suppose that (K, τ) is a periodic knot of order 7. There is only one representation $\theta: G(K)\to A_4 \leqq \mathcal{S}(J_n)$ [11] and hence $\theta\tau_*\equiv\theta$. Since A_4 has no automorphism of order 7, $\theta\tau_*\equiv\theta$ yields $\theta\tau_*=\theta$. Now, the covering linkage invariant of K associated with θ is defined and it is $\{3/4\}$. Since $H_1(M^*; Z)=Z_4$, it follows from Theorem 3.2, that $3/4\equiv 0 \pmod 7$ which obviously fails. Therefore, 9_{35} cannot have period 7.

PROPOSITION 5.4. *A knot $K=9_{46}$ connot have period* 3.

PROOF. Suppose that (K, τ) is a periodic knot of order 3. Since $\Delta(t)=2-5t+2t^2$, $\hat{\Delta}(t)$ of $\hat{K}=K/\tau$ must be 1 by Proposition 5.1. Therefore, $G(\hat{K})\not\to D_3$ [3]. However, $G(K)$ has 4 representations onto D_3 [4, p. 200]. Thus, Theorem 4.2 implies that there are two representations θ_1 and θ_2 such that no two of $\theta_1(\equiv\theta_1\tau_*)$, θ_2, $\theta_2\tau_*$, $\theta_2\tau_*^2$ are equivalent. But the covering linkage invariants corresponding to θ_2, $\theta_2\tau_*$, $\theta_2\tau_*^2$ must coincide by Proposition 4.1. This is not the case, because they are $\{-2/3\}$, $\{-2/3\}$, $\{2/3\}$, $\{-\}$ [4, p. 200]. Therefore, 9_{46} cannot have period 3.

Finally, to prove that knots 8_{10}, 8_{20}, 9_{24} cannot have certain periods, we study their possible orbit knots $\hat{K}$.

PROPOSITION 5.5. *Let (K, τ) be a periodic knot. If K is a fibre knot, then $\hat{K}=K/\tau$ is either a fibre knot or a trivial knot.*

PROOF. Let $G=G(K)=\pi_1(S^3-K)$ and $H=\pi_1(S^3-\hat{K})$. Then $H=G/N$ for some normal subgroup N in G. Since K is a fibre knot, G', the commutator subgroup of G, is finitely generated, and hence, $H'=(G/N)'=G'N/N\cong G'/N\cap G'$ is also finitely generated. Therefore, if $H'\neq 1$, then $\hat{K}$ is a fibre knot. If $H'=1$, then $\hat{K}$ is unknotted, since H is abelian.

PROPOSITION 5.6. *Knots* 8_{10}, 8_{20}, 9_{24} *are fibre knots.*

PROOF. 8_{10} and 9_{24} are alternating knots and their Alexander polynomials are monic. Therefore, by Theorem 1.1 in [8], they are fibre knots. The fact that 8_{20} is also a fibre knot is known, but it is also easy to show that the commutator subgroup of the group of 8_{20} is free of rank 4.

PROPOSITION 5.7. *Knots* 8_{10}, 8_{20} *cannot have any period.*

PROOF. The Alexander polynomials of 8_{10} and 8_{20} are, respectively,

$$\Delta(t)=1-3t+6t^2-7t^3+6t^4-3t^5+t^6=(1-t+t^2)^3$$

and

$$\Delta(t)=1-2t+3t^2-2t^3+t^4=(1-t+t^2)^2\,.$$

Therefore, it follows from Proposition 5.1 that 8_{10} can have only prime periods 2 or 3 with $\Delta(\hat{K})=1-t+t^2$ for both cases, and 8_{20} can have only prime period 2 with $\Delta(\hat{K})=1-t+t^2$. Since 8_{10} and 8_{20} are fibre knots, $\hat{K}$ must be a fibre knot with $\Delta(\hat{K})=1-t+t^2$. Such a knot $\hat{K}$ must be the trefoil knot [2, p. 245].

Now for $K=8_{10}$ or 8_{20}, $G(K)$ has a (unique) representation onto D_3 and $G(\hat{K})$ also has a (unique) representation. Then, by Corollary 3.3, we have $lk_{\tilde{M}}(\tilde{K}_1, \tilde{K}_2)=plk_{\hat{M}}(\hat{K}_1, \hat{K}_2)$, for $p=2$, or 3. But it is known [4, p. 200] that $lk_{\hat{M}}(\hat{K}_1, \hat{K}_2)=\pm 2$ and $lk_{\tilde{M}}(\tilde{K}_1, \tilde{K}_2)=0$. This proves Proposition 5.7,

PROPOSITON 5.8. *A knot* $K=9_{24}$ *cannot have any period.*

PROOF. Since $\Delta(t)=(1-t+t^2)^2(1-3t+t^2)$, the possible prime period is 2 and $\Delta(\hat{K})=1-t+t^2$ or $1-3t+t^2$ by Proposition 5.1. Since K is a fibre knot, so is $\hat{K}$ and then, $\hat{K}$ is either a trefoil knot or the figure eight knot [2, p. 245], noting that the latter has property (P). Now, suppose that $\hat{K}$ is a trefoil knot. Then each of $G(K)$ and $G(\hat{K})$ has a (unique) representation onto D_3, and Corollary 3.3 implies that

$$lk_{\tilde{M}}(\tilde{K}_1, \tilde{K}_2)=2lk_{\hat{M}}(\hat{K}_1, \hat{K}_2)\,.$$

But,

$$lk_{\tilde{M}}(\tilde{K}_1, \tilde{K}_2)=0\,, \qquad \text{while } lk_{\hat{M}}(\hat{K}_1, \hat{K}_2)=\pm 2\,,$$

[4, p. 200], a contradiction.

Suppose that $\hat{K}$ is the figure eight knot. Then each of $G(K)$ and $G(\hat{K})$ has a (unique) representation onto D_5. Then $lk_{\tilde{M}}(\tilde{K}_1, \tilde{K}_2)=2lk_{\hat{M}}(\hat{K}_1, \hat{K}_2)$, where $\tilde{K}_1$, $\hat{K}_1$ are knots with covering index 1. A simple computation shows, however, that $lk_{\tilde{M}}(\tilde{K}_1, \tilde{K}_2)=0$ and $lk_{\hat{M}}(\hat{K}_1, \hat{K}_2)=\pm 2$. This contradiction proves Proposition 5.8.

The following table lists all prime periods of knots with less than 10 crossing points. The number in a circle indicates the *possible* period whose existence is not confirmed.

KNOT	PERIODS	KNOT	PERIODS	KNOT	PERIODS	KNOT	PERIODS
3_1	2, 3	8_8	2	9_8	2	9_{29}	—
4_1	2	8_9	2	9_9	2	9_{30}	—
5_1	2, 5	8_{10}	—	9_{10}	2	9_{31}	2
5_2	2	8_{11}	2	9_{11}	2	9_{32}	—
6_1	2	8_{12}	2	9_{12}	2	9_{33}	—
6_2	2	8_{13}	2	9_{13}	2	9_{34}	—
6_3	2	8_{14}	2	9_{14}	2	9_{35}	②, 3
7_1	2, 7	8_{15}	2	9_{15}	2	9_{36}	—
7_2	2	8_{16}	—	9_{16}	2	9_{37}	②
7_3	2	8_{17}	—	9_{17}	2	9_{38}	—
7_4	2	8_{18}	2	9_{18}	2	9_{39}	—
7_5	2	8_{19}	2, 3	9_{19}	2	9_{40}	2, 3
7_6	2	8_{20}	—	9_{20}	2	9_{41}	3
7_7	2	8_{21}	2	9_{21}	2	9_{42}	—
8_1	2	9_1	2, 3	9_{22}	—	9_{43}	—
8_2	2	9_2	2	9_{23}	2	9_{44}	—
8_3	2	9_3	2	9_{24}	—	9_{45}	—
8_4	2	9_4	2	9_{25}	③	9_{46}	2
8_5	2	9_5	2	9_{26}	2	9_{47}	3
8_6	2	9_6	2	9_{27}	2	9_{48}	②
8_7	2	9_7	2	9_{28}	2	9_{49}	3

References

[1] Burde, G., Über periodische Knoten, Archiv der Math **30** (1978), 487-492.

[2] Burde, G. and Zieschang, H., Neuwirthsche Knoten und Flächenabbildungen, Abh. Math. Sem. Hamburg, **31** (1967) 239-246.

[3] Fox, R. H., A quick trip through knot theory, Topology of Three Manifolds and Related Topics (Prentice Hall, 1962) 120-167.

[4] Fox, R. H., Metacyclic invariants of knots and links, Can. J. Math. 22 (1970) 193-201.

[5] Gordon, C. McA., Litherlands, R. A. and Murasugi, K., Signatures of covering links. (to appear)

[6] Hartley, R. I., and Murasugi, K., Covering linkage invariants, Can. J. Math. **29** (1977) 1312-1339.
[7] Lüdicke, U., Darstellungen der Verkettungsgruppe und zyklische Knoten, Dissert. Univ. of Frankfurt, (1978).
[8] Murasugi, K., On a certain subgroup of the groups of an alternating link, Amer. J. Math. **85** (1963) 544-550.
[9] Murasugi, K., On periodic Knots, Comm. Math. Helv. **4** (1971) 162-174.
[10] Perko, K., On dihedral covering spaces of knots, Inventiones Math. **34** (1976) 77-82.
[11] Riley, R., Homomorphisms of knot groups on finite groups, Math. of Computation **25** (1971) 603-619.
]12] Rolfsen, D., Knots and links, Publish or Perish, Inc. Math. Lecture Series, **7** (1976).
[13] Trotter, H. F., Periodic Automorphisms of groups and knots, Duke Math. J. **28** (1961) 553-557.

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