TSUKUBA J. MATH.
Vol. 1 (1977), 195—211

# A THEOREM ON THE FORMALIZED ARITHMETIC WITH FUNCTION SYMBOLS ′ AND +.

By

Tsuyoshi YUKAMI

## Introduction.

Let $L_0$ be a first order language with the following primitive symbols: 1. Function symbol: ′, +. 2. Individual constant: 0. 3. Predicate symbol: =. Let $L$ be the first order language obtained from $L_0$ by adding predicate symbols $P, Q, \cdots$.

By $N$ we denote the theory in $L$ whose axioms and axiom schemata are:

$(N_1)$ $$\forall x(0\neq x').$$

$(N_2)$ $$\forall x\forall y(x'=y'\supset x=y),$$

$(N_3)$ $$\forall x(x+0=x),$$

$(N_4)$ $$\forall x\forall y(x+y'=(x+y)'),$$

$(N_5)$ $$\forall xP(x,0,0),$$

$(N_6)$ $$\forall x\forall y\forall z\{P(x,y,z)\supset P(x,y',z+x)\},$$

$(N_7)$ $$\forall x\forall y\forall z\forall w\{(P(x,y,z)\wedge P(x,y,w))\supset z=w\},$$

$(N_8)$ $$\forall x(x=x),$$

$(N_9)$ $$\forall x\forall y\{x=y\supset(\mathfrak{A}(x)\supset\mathfrak{A}(y))\},$$

$(N_{10})$ $$\{\mathfrak{A}(0)\wedge\forall x(\mathfrak{A}(x)\supset\mathfrak{A}(x'))\}\supset\forall x\mathfrak{A}(x),$$

$(N_{11})$ $$s=t, \text{ where } s=t \text{ is valid.}$$

We define $\mathrm{b}(t)$, $\mathrm{b}(\mathfrak{A})$ inductively as follows: 1. If $t=0$ or $t$ is a free variable, then $\mathrm{b}(t)=0$. 2. If $t$ is a bound variable, then $\mathrm{b}(t)=1$. 3. $\mathrm{b}(t')=\mathrm{b}(t)$. 4. $\mathrm{b}(s+t)=\mathrm{b}(s)+\mathrm{b}(t)$. 5. $\mathrm{b}(Q(t_1,\cdots,t_\nu))=\max(\mathrm{b}(t_1),\cdots,\mathrm{b}(t_\nu))$. 6. $\mathrm{b}(\neg\mathfrak{A})=\mathrm{b}(\mathfrak{A})$. 7. $\mathrm{b}(\mathfrak{A}\wedge\mathfrak{B})=\max(\mathrm{b}(\mathfrak{A}),\mathrm{b}(\mathfrak{B}))$. 8. $\mathrm{b}(\forall x\mathfrak{A}(x))=\mathrm{b}(\mathfrak{A}(x))$.

The purpose of this paper is to prove the following theorem.

*For any formula* $\mathfrak{A}(a)$ *of* $L$; *if there is a number* $m$ *such that, for any natural number* $n$, *there exists a proof* $\mathfrak{P}$ *of* $\mathfrak{A}(\bar{n})$ *in* $N$ *with the following properties* (1) *and*

Received June 25, 1977. Revised November 22, 1977.

(2), *then* $\forall x\mathfrak{A}(x)$ *is provable in N.*

(1) *The length of* $\mathfrak{P}$ *is less than m.*

(2) *For any induction schema* $\mathfrak{B}$ *in* $\mathfrak{P}$ *which is not a formula of* $L_0$, $\mathrm{b}(\mathfrak{B})<m$.

We consider the system N* obtained from N by omitting the axioms $(N_1)$-$(N_8)$, and by replacing the axiom schemata $(N_9)$-$(N_{11})$ by the following inference rules:

(*)
$$\frac{\mathfrak{A}(s), \Gamma\to\Delta}{\mathfrak{A}(t), \Gamma\to\Delta} \qquad \frac{\Gamma\to\Delta, \mathfrak{A}(s)}{\Gamma\to\Delta, \mathfrak{A}(t)}$$

where $s=t$ is valid.

(**) (induction inference)
$$\frac{\{\mathfrak{A}(0)\wedge\forall x(\mathfrak{A}(x)\supset\mathfrak{A}(x'))\}\supset\forall x\mathfrak{A}(x), \Gamma\to\Delta}{\Gamma\to\Delta}$$

where the formula $\{\mathfrak{A}(0)\wedge\forall x(\mathfrak{A}(x)\supset\mathfrak{A}(x'))\}\supset\forall x\mathfrak{A}(x)$ is called induction formula of the inference.

(***)
$$\frac{\Gamma\to\Delta, s=t \quad \mathfrak{A}(s), \Pi\to\Sigma}{\Gamma, \mathfrak{A}(t), \Pi\to\Delta, \Sigma} \qquad \frac{\Gamma\to\Delta, s=t \quad \mathfrak{A}(t), \Pi\to\Sigma}{\Gamma, \mathfrak{A}(s), \Pi\to\Delta, \Sigma}$$

$$\frac{\Gamma\to\Delta, s=t \quad \Pi\to\Sigma, \mathfrak{A}(s)}{\Gamma, \Pi\to\Delta, \Sigma, \mathfrak{A}(t)} \qquad \frac{\Gamma\to\Delta, s=t \quad \Pi\to\Sigma, \mathfrak{A}(t)}{\Gamma, \Pi\to\Delta, \Sigma, \mathfrak{A}(s)}$$

Because we can consider $(N_1)\wedge\cdots\wedge(N_8)\supset\mathfrak{A}(a)$ in place of $\mathfrak{A}(a)$, it is sufficient for our purpose to prove:

THEOREM. *For any formula* $\mathfrak{A}(a)$ *of L; if there is a number m such that, for each natural number n, there exists a proof* $\mathfrak{P}$ *of* $\mathfrak{A}(\bar{n})$ *in N* with the following properties* (1) *and* (2), *then* $\forall x\mathfrak{A}(x)$ *is provable in N.*

(1) *The length of* $\mathfrak{P}$ *is less than m.*

(2) *For any induction formula* $\mathfrak{B}$ *of induction inference in* $\mathfrak{P}$, *if* $\mathfrak{B}$ *is not a formula of* $L_0$, *then* $\mathrm{b}(\mathfrak{B})<m$.

The proof of the theorem consists of three parts §1, §2 and §3. Section 2 is the main part of it.

## §1.

To prove the theorem, we assume that, for each natural number $n$, there exists a proof $\mathfrak{P}_n$ of $\mathfrak{A}(\bar{n})$ with the properties in the theorem. In this section, we divide the set $\{\mathfrak{P}_n|n\in\omega\}$ into finite groups.

An equivalence relation $\mathfrak{A}\sim\mathfrak{B}$ between formulas is defined, by induction, as follows:

1. If $\mathfrak{A}$ and $\mathfrak{B}$ are atomic formulas with the same predicate symbol, then $\mathfrak{A}$

$\sim\mathfrak{B}$.

2-3. If $\mathfrak{A}\sim\mathfrak{B}$ and $\mathfrak{C}\sim\mathfrak{D}$, then $\neg\mathfrak{A}\sim\neg\mathfrak{B}$ and $\mathfrak{A}\wedge\mathfrak{C}\sim\mathfrak{B}\wedge\mathfrak{D}$.

4. If $\mathfrak{A}\sim\mathfrak{B}$, then $\forall x\mathfrak{A}\sim\forall y\mathfrak{B}$.

The above relation $\sim$ is extended for proofs in N* as follows:

1. If $\mathfrak{P}=\mathfrak{A}\rightarrow\mathfrak{A}$ and $\mathfrak{Q}=\mathfrak{B}\rightarrow\mathfrak{B}$, then $\mathfrak{P}\sim\mathfrak{Q}$.

2. If $\mathfrak{P}=\mathfrak{P}_0/\Gamma\rightarrow\Delta$, $\mathfrak{Q}=\mathfrak{Q}_0/\Pi\rightarrow\Sigma$, $\mathfrak{P}_0\sim\mathfrak{Q}_0$, and the last inference rules of $\mathfrak{P},\mathfrak{Q}$ are the same type, then $\mathfrak{P}\sim\mathfrak{Q}$.

3. If $\mathfrak{P}=\mathfrak{P}_1\mathfrak{P}_2/\Gamma\rightarrow\Delta$, $\mathfrak{Q}=\mathfrak{Q}_1\mathfrak{Q}_2/\Pi\rightarrow\Sigma$, $\mathfrak{P}_1\sim\mathfrak{Q}_1$, $\mathfrak{P}_2\sim\mathfrak{Q}_2$, and the last inference rules of $\mathfrak{P},\mathfrak{Q}$ are the same type, then $\mathfrak{P}\sim\mathfrak{Q}$.

The proofs of the following two lemmas are easy and so we omit them.

LEMMA 1.1. *If* $\mathfrak{P}\sim\mathfrak{Q}$, *and* $\mathfrak{P}$ *and* $\mathfrak{Q}$ *are proofs of* $\mathfrak{A}_1,\cdots,\mathfrak{A}_\mu\rightarrow\mathfrak{B}_1,\cdots,\mathfrak{B}_\nu$ *and of* $\mathfrak{C}_1,\cdots,\mathfrak{C}_\kappa\rightarrow\mathfrak{D}_1,\cdots,\mathfrak{D}_\lambda$, *respectively, then* $\mu=\kappa$ *and* $\nu=\lambda$.

LEMMA 1.2. *The number of equivalence classes by* $\sim$ *of proofs with length* $\leq m$ *are finite.*

*The first division*: *We divide the set* $\{\mathfrak{P}_n|n\in\omega\}$ *by the equivalence relation* $\sim$. Note that, by Lemma 1.2, these classes are finite.

A function $\mathfrak{F}(\mathfrak{A},\mathfrak{T})$ ($\mathfrak{A}$ is a formula and $\mathfrak{T}$ is a set of formulas) is defined inductively as follows:

1. If $\mathfrak{A}$ and elements of $\mathfrak{T}$ are all atomic formulas with the same predicate symbol, then $\mathfrak{F}(\mathfrak{A},\mathfrak{T})=\mathfrak{A}$.

2. If $\mathfrak{A}=\mathfrak{A}_1\wedge\mathfrak{A}_2$ and $\mathfrak{T}=\{\mathfrak{B}_1{}^\lambda\wedge\mathfrak{B}_2{}^\lambda|\lambda\in\Lambda\}$, then

$$\mathfrak{F}(\mathfrak{A},\mathfrak{T})=\mathfrak{F}(\mathfrak{A}_1,\{\mathfrak{B}_1{}^\lambda|\lambda\in\Lambda\})\wedge\mathfrak{F}(\mathfrak{A}_2,\{\mathfrak{B}_2{}^\lambda|\lambda\in\Lambda\})\,.$$

3. If $\mathfrak{A}=\neg\mathfrak{A}_0$ and $\mathfrak{T}=\{\neg\mathfrak{B}_\lambda|\lambda\in\Lambda\}$, then $\mathfrak{F}(\mathfrak{A},\mathfrak{T})=\neg\mathfrak{F}(\mathfrak{A}_0,\{\mathfrak{B}_\lambda|\lambda\in\Lambda\})$.

4. If $\mathfrak{A}=\forall x\mathfrak{A}_0$ and $\mathfrak{T}=\{\forall x_\lambda\mathfrak{B}_\lambda|\lambda\in\Lambda\}$, then

$$\mathfrak{F}(\mathfrak{A},\mathfrak{T})=\forall x\mathfrak{F}(\mathfrak{A}_0,\{\mathfrak{B}_\lambda|\lambda\in\Lambda\})\,.$$

5. Otherwise $\mathfrak{F}(\mathfrak{A},\mathfrak{T})=0=0$.

We define:

$$\mathfrak{F}(\mathfrak{A}_1,\cdots,\mathfrak{A}_\mu\rightarrow\mathfrak{B}_1,\cdots,\mathfrak{B}_\nu,\{\mathfrak{C}_1{}^\lambda,\cdots,\mathfrak{C}_\mu{}^\lambda\rightarrow\mathfrak{D}_1{}^\lambda,\cdots,\mathfrak{D}_\nu{}^\lambda|\lambda\in\Lambda\})$$
$$=\mathfrak{F}(\mathfrak{A}_1,\{\mathfrak{C}_1{}^\lambda|\lambda\in\Lambda\}),\cdots\rightarrow\mathfrak{F}(\mathfrak{B}_1,\{\mathfrak{D}_1{}^\lambda|\lambda\in\Lambda\}),\cdots\,.$$

The above function $\mathfrak{F}$ is extended for proofs as follows:

1. If $\mathfrak{P}=\mathfrak{A}\rightarrow\mathfrak{A}$ and $\mathfrak{Q}_\lambda=\mathfrak{B}_\lambda\rightarrow\mathfrak{B}_\lambda(\lambda\in\Lambda)$, then

$$\mathfrak{F}(\mathfrak{A}\rightarrow\mathfrak{A},\{\mathfrak{Q}_\lambda|\lambda\in\Lambda\})=\mathfrak{F}(\mathfrak{A},\{\mathfrak{B}_\lambda|\lambda\in\Lambda\})\rightarrow\mathfrak{F}(\mathfrak{A},\{\mathfrak{B}_\lambda|\lambda\in\Lambda\})\,.$$

2. If $\mathfrak{P}=\mathfrak{P}_0/\Gamma\rightarrow\Delta$, $\mathfrak{Q}_\lambda=\mathfrak{Q}_0{}^\lambda/\Pi_\lambda\rightarrow\Sigma_\lambda(\lambda\in\Lambda)$, and the last inference rules of $\mathfrak{P}$ and

$\mathfrak{Q}_\lambda$ are all the same type, then

$$\mathfrak{F}(\mathfrak{P}, \{\mathfrak{Q}_\lambda|\lambda\in\Lambda\})=\frac{\mathfrak{F}(\mathfrak{P}_0, \{\mathfrak{Q}_0{}^\lambda|\lambda\in\Lambda\})}{\mathfrak{F}(\Gamma\to\Delta, \{\Pi_\lambda\to\Sigma_\lambda|\lambda\in\Lambda\})}$$

3. If $\mathfrak{P}=\mathfrak{P}_1\mathfrak{P}_2/\Gamma\to\Delta$, $\mathfrak{Q}_\lambda=\mathfrak{Q}_1{}^\lambda\mathfrak{Q}_2{}^\lambda/\Pi_\lambda\to\Sigma_\lambda$, and the last inference rules of $\mathfrak{P}$ and $\mathfrak{Q}_\lambda$ are all the same type, then

$$\mathfrak{F}(\mathfrak{P}, \{\mathfrak{Q}_\lambda|\lambda\in\Lambda\})=\frac{\mathfrak{F}(\mathfrak{P}_1, \{\mathfrak{Q}_1{}^\lambda|\lambda\in\Lambda\})\quad\mathfrak{F}(\mathfrak{P}_2, \{\mathfrak{Q}_2{}^\lambda|\lambda\in\Lambda\})}{\mathfrak{F}(\Gamma\to\Delta, \{\Pi_\lambda\to\Sigma_\lambda|\lambda\in\Lambda\})}.$$

4. Otherwise $\mathfrak{F}(\mathfrak{P}, \{\mathfrak{Q}_\lambda|\lambda\in\Lambda\})=0=0\to 0=0$.

LEMMA 1.3.

(1) $\deg(\mathfrak{F}(\mathfrak{A}, \{\mathfrak{B}_\lambda|\lambda\in\Lambda\}))\leq\min(\deg(\mathfrak{A}), \{\deg(\mathfrak{B}_\lambda)|\lambda\in\Lambda\})$, *where* $\deg(\mathfrak{A})$ *denotes the number of occurrences of logical symbols in* $\mathfrak{A}$.

(2) $\mathfrak{F}(\mathfrak{A}(\bar{n}), \{\mathfrak{A}(\bar{m}_\lambda)|\lambda\in\Lambda\})=\mathfrak{A}(\bar{n})$.

(3) $\mathfrak{F}(\mathfrak{A}, \{\mathfrak{B}_\lambda(x_\lambda)|\lambda\in\Lambda\})=\mathfrak{F}(\mathfrak{A}, \{\mathfrak{B}_\lambda(t_\lambda)|\lambda\in\Lambda\})$, *where* $t_\lambda(\lambda\in\Lambda)$ *are terms.*

(4) $\{\mathfrak{F}(\mathfrak{A}(x), \{\mathfrak{B}_\lambda|\lambda\in\Lambda\})\}\begin{pmatrix}x\\t\end{pmatrix}=\mathfrak{F}(\mathfrak{A}(t), \{\mathfrak{B}_\lambda|\lambda\in\Lambda\})$, *where* $t$ *is a term.*

LEMMA 1.4. *If* $\mathfrak{A}=\{\mathfrak{B}(0)\wedge\forall x(\mathfrak{B}(x)\supset\mathfrak{B}(x'))\}\supset\forall x\mathfrak{B}(x)$,

$$\mathfrak{C}_\lambda=\{\mathfrak{D}_\lambda(0)\wedge\forall x_\lambda(\mathfrak{D}_\lambda(x_\lambda)\supset\mathfrak{D}_\lambda(x_\lambda'))\}\supset\forall x_\lambda\mathfrak{D}_\lambda(x_\lambda),$$

*and* $\mathfrak{F}(\mathfrak{A}, \{\mathfrak{C}_\lambda|\lambda\in\Lambda\})=\mathfrak{E}$, *then*

$$\mathfrak{E}=\{\mathfrak{F}(0)\wedge\forall x(\mathfrak{F}(x)\supset\mathfrak{F}(x'))\}\supset\forall x\mathfrak{F}(x)$$

*for some* $\mathfrak{F}(x)$.

LEMMA 1.5. *If* $\mathfrak{P}\sim\mathfrak{Q}_\lambda(\lambda\in\Lambda)$, *and* $\Gamma\to\Delta$ *and* $\Pi_\lambda\to\Sigma_\lambda$ *are end-sequents of* $\mathfrak{P}$ *and* $\mathfrak{Q}_\lambda$, *respectively, then* $\mathfrak{F}(\mathfrak{P}, \{\mathfrak{Q}_\lambda|\lambda\in\Lambda\})\sim\mathfrak{P}$ *and the end-sequent of* $\mathfrak{F}(\mathfrak{P}, \mathfrak{Q}_\lambda|\lambda\in\Lambda\})$ *is* $\mathfrak{F}(\Gamma\to\Delta, \{\Pi_\lambda\to\Sigma_\lambda|\lambda\in\Lambda\})$.

By replacing $\mathfrak{P}_n$ by $\mathfrak{F}(\mathfrak{P}_n, \{\mathfrak{P}_\lambda|\lambda\in\Lambda\})$, where $\{\mathfrak{P}_\lambda|\lambda\in\Lambda\}$ is the equivalence class which contains $\mathfrak{P}_n$, we can assume that given proofs $\mathfrak{P}_n(n\in\omega)$ have, in addition to the properties in the theorem, the following properties: (1) There exists a number $m_1$ such that, for any natural number $n$, $\deg(\mathfrak{B})<m_1$ for any formula $\mathfrak{B}$ which occurs in $\mathfrak{P}_n$. (2) There exists a finite set of predicate symbols which, for each natural number $n$, contains all predicate symbols occurring in $\mathfrak{P}_n$.

Hence we can assume, without loss of generality, that formulas in beginning sequents in $\mathfrak{P}_n$ are all atomic formulas.

Further we can also assume, without loss of generality, that $\mathfrak{P}_n$ is cut-free, by the following lemma. Its proof is routine, and so we omit it.

LEMMA 1.6. *There exists a function* $\mathfrak{H}(m, n)$ *such that: if we have a proof* $\mathfrak{P}$ *of* $\Gamma \to \Delta$ *in* $N^*$ *with length* $\leq m$ *and* $\deg(\mathfrak{A}) \leq n$ *for any formula* $\mathfrak{A}$ *in* $\mathfrak{P}$, *then we can find a cut-free proof of* $\Gamma \to \Delta$ *in* $N^*$ *with length* $\leq \mathfrak{H}(m, n)$.

## § 2.

In this section, for each $n \in \omega$, we define a formula $\phi_{\mathfrak{S}}$ of the language $L_0$ for each sequent $\mathfrak{S}$ in the proof $\mathfrak{P}_n$. For the definition we must make some preparations.

Functions $\mathrm{a}(t)$, $\mathrm{d}(t)$, $\mathrm{e}(\mathfrak{A}, x)$, $\mathrm{f}(\mathfrak{A}, x)$ and $\mathrm{g}(\mathfrak{A})$ are defined inductively as follows:

1.1 If $t=0$, or $t$ is a free variable, then $\mathrm{a}(t)=t$.

1.2 If $t$ is a bound variable, then $\mathrm{a}(t)=0$.

1.3 $\mathrm{a}(t')=(\mathrm{a}(t))'$.

1.4 $\mathrm{a}(s+t)=\mathrm{a}(s)+\mathrm{a}(t)$.

2.1 $\mathrm{d}(P(t_1, \cdots, t_\nu))=\nu$.

2.2 $\mathrm{d}(\neg\mathfrak{A})=\mathrm{d}(\mathfrak{A})$.

2.3 $\mathrm{d}(\mathfrak{A} \wedge \mathfrak{B})=\mathrm{d}(\mathfrak{A})+\mathrm{d}(\mathfrak{B})$.

2.4 $\mathrm{d}(\forall x\mathfrak{A})=\mathrm{d}(\mathfrak{A})$.

3.1 $\mathrm{e}(P(t_1, \cdots, t_\nu), x)=\langle 1, \cdots, \nu\rangle$.

3.2 $\mathrm{e}(\mathfrak{A} \wedge \mathfrak{B}, x)=\mathrm{e}(\mathfrak{A}, x)*\langle \nu+m_1, \cdots, \nu+m_\mu\rangle$,

where $\mathrm{e}(\mathfrak{B}, x)=\langle m_1, \cdots, m_\mu\rangle$ and $\nu=\mathrm{d}(\mathfrak{A})$.

3.3 $\mathrm{e}(\neg\mathfrak{A}, x)=\mathrm{e}(\mathfrak{A}, x)$.

3.4 $\mathrm{e}(\forall y\mathfrak{A}, x)=\mathrm{e}(\mathfrak{A}, x)$.

3.5 $\mathrm{e}(\forall x\mathfrak{A}, x)=\langle\ \rangle$.

4.1 $\mathrm{f}(P(t_1, \cdots, t_\nu), x)=\langle\ \rangle$.

4.2 $\mathrm{f}(\mathfrak{A} \wedge \mathfrak{B}, x)=\mathrm{f}(\mathfrak{A}, x)*\langle \nu+m_1, \cdots, \nu+m_\mu\rangle$,

where $\mathrm{f}(\mathfrak{B}, x)=\langle m_1, \cdots, m_\mu\rangle$ and $\nu=\mathrm{d}(\mathfrak{A})$.

4.3 $\mathrm{f}(\neg\mathfrak{A}, x)=\mathrm{f}(\mathfrak{A}, x)$.

4.4 $\mathrm{f}(\forall y\mathfrak{A}, x)=\mathrm{f}(\mathfrak{A}, x)$.

4.5 $\mathrm{f}(\forall x\mathfrak{A}, x)=\langle 1, \cdots, \nu\rangle$, where $\nu=\mathrm{d}(\mathfrak{A})$.

5.1 $\mathrm{g}(\mathfrak{A})$ is a finite sequence of terms with length $\mathrm{d}(\mathfrak{A})$.

5.2 $\mathrm{g}(P(t_1, \cdots, t_\nu))(i) = t_i$.

5.3 $\mathrm{g}(\neg \mathfrak{A}) = \mathrm{g}(\mathfrak{A})$.

5.4 $\mathrm{g}(\mathfrak{A} \wedge \mathfrak{B})(i) = \begin{cases} \mathrm{g}(\mathfrak{A})(i) & \text{if } i \leq \mathrm{d}(\mathfrak{A}), \\ \mathrm{g}(\mathfrak{B})(j) & \text{if } i = j + \mathrm{d}(\mathfrak{A}). \end{cases}$

5.5 $\mathrm{g}(\forall x \mathfrak{A}) = \mathrm{g}(\mathfrak{A})$.

If any bound variable which occurs in a term $t$ occurs in $x_1, \cdots, x_\nu$, $\mathrm{c}(t, a, \langle x_1, \cdots, x_\nu \rangle)$ denotes

$$a + (\overbrace{x_1 + \cdots + x_1}^{m_1}) + \cdots + (\overbrace{x_\nu + \cdots + x_\nu}^{m_\nu}),$$

where $m_i$ is the number of occurrences of $x_i$ in $t$.

For a formula $\mathfrak{B}$, a finite sequence $\alpha$ of free variables with length $\mathrm{d}(\mathfrak{B})$, and a finite sequence $\xi$ of bound variables; if any bound variable which occurs free in $\mathfrak{B}$ occurs in $\xi$, then we define $\mathrm{c}(\mathfrak{B}, \alpha, \xi)$, by induction on $\mathfrak{B}$, in the following manner:

1. $\mathrm{c}(P(t_1, \cdots, t_\nu), \langle a_1, \cdots, a_\nu \rangle, \xi) = P(\mathrm{c}(t_1, a_1, \xi), \cdots, \mathrm{c}(t_\nu, a_\nu, \xi))$.
2. $\mathrm{c}(\neg \mathfrak{B}, \alpha, \xi) = \neg \mathrm{c}(\mathfrak{B}, \alpha, \xi)$.
3. $\mathrm{c}(\mathfrak{B} \wedge \mathfrak{C}, \alpha * \beta, \xi) = \mathrm{c}(\mathfrak{B}, \alpha, \xi) \wedge \mathrm{c}(\mathfrak{C}, \beta, \xi)$, where the length of $\alpha$ is $\mathrm{d}(\mathfrak{B})$.
4. If $x$ does not occur in $\xi$, then

$$\mathrm{c}(\forall x \mathfrak{B}(x), \alpha, \xi) = \forall x (\mathrm{c}(\mathfrak{B}(x), \alpha, \langle x \rangle * \xi)).$$

5. If $x$ occurs in $\xi$, then

$$\mathrm{c}(\forall x \mathfrak{B}(x), \alpha, \xi) = \forall x (\mathrm{c}(\mathfrak{B}(x), \alpha, \xi)).$$

We define a relation $\simeq$ between formulas as follows:

1. If $\mathfrak{A}$ and $\mathfrak{B}$ are atomic formulas with the same predicate symbol, then $\mathfrak{A} \simeq \mathfrak{B}$.

2–3. If $\mathfrak{A} \simeq \mathfrak{B}$ and $\mathfrak{C} \simeq \mathfrak{D}$, then $\neg \mathfrak{A} \simeq \neg \mathfrak{B}$ and $\mathfrak{A} \wedge \mathfrak{C} \simeq \mathfrak{B} \wedge \mathfrak{D}$.

4. If $\mathfrak{A} \simeq \mathfrak{B}$, then $\forall x \mathfrak{A} \simeq \forall x \mathfrak{B}$.

LEMMA 2.1

(I) *For any term $t$,*

$$\rightarrow \forall x_1 \cdots \forall x_\nu \left\{ t = \mathrm{c}(t, a, \langle x_1, \cdots, x_\nu \rangle) \begin{pmatrix} a \\ \mathrm{a}(t) \end{pmatrix} \right\}$$

*is provable in $N$, where $x_1, \cdots, x_\nu$ are all the bound variables which occur in $t$.*

(II) *For any term $t(x)$,* $\mathrm{a}(t(x)) = \mathrm{a}(t(0))$.

(III) *For any term $t(x)$,*

$$\rightarrow \mathrm{a}(t(x')) = (\mathrm{a}(t(x)) + \bar{m})$$

*is provable in $N$, where $m$ is the number of occurrences of $x$ in $t(x)$.*

(IV) *For any formula* $\mathfrak{B}(x)$, *if* $s=\mathrm{g}(\mathfrak{B}(x))(i)$ *and* $t=\mathrm{g}(\mathfrak{B}(0))(i)$, *then* $\mathrm{a}(s)=\mathrm{a}(t)$.

(V) *For any formula* $\mathfrak{B}(x)$, if $s=\mathrm{g}(\mathfrak{B}(x))(i)$ *and* $t=\mathrm{g}(\mathfrak{B}(x'))(i)$, *and* $i$ *occurs in* $\mathrm{e}(\mathfrak{B}(x), x)$, *then*

$$\rightarrow \mathrm{a}(t)=\mathrm{a}(s)+\bar{m}$$

*is provable in* $N$, *where* $m$ *is the number of occurrences of* $x$ *in* $s$.

(VI) *For any formula* $\mathfrak{B}(x)$, *if* $s=\mathrm{g}(\mathfrak{B}(x))(i)$ *and* $t=\mathrm{g}(\mathfrak{B}(x'))(i)$, *and* $i$ *occurs in* $\mathrm{f}(\mathfrak{B}(x), x)$, *then* $\mathrm{a}(t)=\mathrm{a}(s)$.

LEMMA 2.2

(I) $$a=b\rightarrow \forall y_1\cdots\{(\mathrm{c}(t(x), a, \xi)\begin{pmatrix} x \\ 0 \end{pmatrix}=\mathrm{c}(t(0), b, \xi)\}$$

*is provable in* $N$, *where* $y_1, \cdots$ *are all the bound variables except* $x$ *which occurs in* $t(x)$.

(II) $$a=b+\bar{m}\rightarrow \forall x \forall y_1\cdots\left\{(\mathrm{c}(t(x), b, \xi))\begin{pmatrix} x \\ x' \end{pmatrix}=\mathrm{c}(t(x'), a, \xi)\right\}$$

*is provable in* $N$, *where* $m$ *is the number of occurrences of* $x$ *in* $t(x)$.

(III) $$a_1=b_1\wedge\cdots\wedge a_\nu=b_\nu\rightarrow \mathrm{c}(\mathfrak{B}(0), \alpha, \xi)\equiv(\mathrm{c}(\mathfrak{B}(x), \beta, \xi))\begin{pmatrix} x \\ 0 \end{pmatrix}$$

*is provable in* $N$, *where*

$$\alpha=\langle a_1, \cdots, a_\nu\rangle \text{ and } \beta=\langle b_1, \cdots, b_\nu\rangle$$

*with* $\nu=\mathrm{d}(\mathfrak{B}(x))$.

(IV) If $\mathrm{e}(\mathfrak{B}(x), x)=\langle i_1, \cdots, i_\kappa\rangle$ *and* $\mathrm{f}(\mathfrak{B}(x), x)=\langle j_1, \cdots, j_\lambda\rangle$,

*then*

$$b_{i_1}=a_{i_1}+\bar{m}_1\wedge\cdots\wedge b_{i_\kappa}=a_{i_\kappa}+\bar{m}_\kappa\wedge b_{j_1}=a_{j_1}\wedge\cdots\wedge b_{j_\lambda}=a_{j_\lambda}$$
$$\rightarrow \forall x\left\{\mathrm{c}(\mathfrak{B}(x'), \beta, \xi)\equiv(\mathrm{c}(\mathfrak{B}(x), \alpha, \xi))\begin{pmatrix} x \\ x' \end{pmatrix}\right\}$$

*is provable in* $N$, *where* $\alpha=\langle a_1, \cdots, a_\nu\rangle$ *and* $\beta=\langle b_1, \cdots, b_\nu\rangle$ *with* $\nu=\mathrm{d}(\mathfrak{B}(x))$, *and* $m_k$ *is the number of occurrences of* $x$ *in* $\mathrm{g}(\mathfrak{B}(x))(i_k)(k=1, \cdots, \kappa)$.

For each sequent $\Gamma\rightarrow\Delta$ in the proof $\mathfrak{P}_n$, we define the following elements with the properties mentioned:

(a) Divisions $\Gamma_1, \Gamma_2$ and $\Delta_1, \Delta_2$ of $\Gamma$ and $\Delta$, respectively: Formulas in $\Gamma_2, \Delta_2$ are formulas of $L_0$.

(b) Free variables $a_1, \cdots, a_\nu$ and corresponding terms $t_1, \cdots, t_\nu$: Free variables which occur in $t_1, \cdots, t_\nu$ do not occur as eigen-variable in $\mathfrak{P}_n$ above the sequent $\Gamma\rightarrow\Delta$.

(c) Finite sequences of formulas $\Pi(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu)$ corresponding to $\Gamma_1$ and $\Delta_1$, respectively: For each formula $\mathfrak{B}(a_1, \cdots, a_\nu)$ in $\Pi(a_1, \cdots, a_\nu)$, $\Sigma(a_1, \cdots, a_\nu)$ corresponding to a formula $\mathfrak{C}$ in $\Gamma_1, \Delta_1$, $\mathfrak{B}(a_1, \cdots, a_\nu) \simeq \mathfrak{C}$.

(d) A set $\theta$ of equalities $r_1 = s_1, \cdots$: For each formula $\mathfrak{B}(a_1, \cdots, a_\nu)$ in $\Pi(a_1, \cdots, a_\nu)$, $\Sigma(a_1, \cdots, a_\nu)$ corresponding to a formula $\mathfrak{C}$ in $\Gamma_1, \Delta_1$, if $u(a_1, \cdots, a_\nu) = \mathrm{g}(\mathfrak{B}(a_1, \cdots, a_\nu))(i)$ and $v = \mathrm{g}(\mathfrak{C})(i)$, then

$$r_1 = s_1, \cdots \rightarrow \forall x_1 \cdots (u(t_1, \cdots, t_\nu) = v)$$

is provable in $N$, where $x_1, \cdots$ are all the bound variables which occur in $u(a_1, \cdots, a_\nu)$, $v$. Free variables which occur in equalities in $\theta$ do not occur as eigen-variable in $\mathfrak{P}_n$ above $\Gamma \rightarrow \Delta$.

The above elements are defined by induction, from end-sequent up to beginning sequents, in the following manner:

1. For the end-sequent $\rightarrow \mathfrak{A}(\bar{n})$;

(a) $\Delta_1 = \mathfrak{A}(\bar{n})$, $\Gamma_1 = \Gamma_2 = \Delta_2 = \phi$.

(b) $a$ and $\bar{n}$.

(c) $\Pi(a) = \phi$, $\Sigma(a) = \mathfrak{A}(a)$.

(d) $\theta = \phi$.

2. $$\frac{\Gamma \rightarrow \Delta, \mathfrak{B}}{\neg \mathfrak{B}, \Gamma \rightarrow \Delta}$$

2.1 If $\Gamma_1$ defined for the lower sequent is of the form $\neg \mathfrak{B}, \Gamma_3$, then we define for the upper one:

(a) $\Gamma_1 = \Gamma_3$, $\Delta_1 = \Delta_3, \mathfrak{B}$; $\Gamma_2 = \Gamma_4$ and $\Delta_2 = \Delta_4$, where $\Gamma_1 = \neg \mathfrak{B}, \Gamma_3$, $\Gamma_2 = \Gamma_4$, $\Delta_1 = \Delta_3$ and $\Delta_2 = \Delta_4$ are elements in (a) defined for the lower sequent.

(b) Elements in (b) are the same as for the lower sequent.

(c) $\Pi(a_1, \cdots, a_\nu) = \Pi^*(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu) = \Sigma^*(a_1, \cdots, a_\nu), \mathfrak{C}(a_1, \cdots, a_\nu)$, where $\Pi(a_1, \cdots, a_\nu) = \neg \mathfrak{C}(a_1, \cdots, a_\nu), \Pi^*(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu) = \Sigma^*(a_1, \cdots, a_\nu)$ for the lower sequent.

(d) Elements in (d) are the same as for the lower sequent.

2.2 If $\Gamma_2$ defined for the lower sequent is of the form $\neg \mathfrak{B}, \Gamma_4$, then we define for the upper one:

(a) $\Gamma_1 = \Gamma_3$, $\Gamma_2 = \Gamma_4$, $\Delta_1 = \Delta_3$ and $\Delta_2 = \Delta_4, \mathfrak{B}$, where $\Gamma_1 = \Gamma_3$, $\Gamma_2 = \neg \mathfrak{B}, \Gamma_4$; $\Delta_1 = \Delta_3$ and $\Delta_2 = \Delta_4$ are elements in (a) defined for the lower sequent.

(b)-(d) Elements in (b)-(d) are the same as for the lower sequent.

3. $$\frac{\mathfrak{B}(t), \Gamma \rightarrow \Delta}{\forall x \mathfrak{B}(x), \Gamma \rightarrow \Delta}$$

3.1 If $\Gamma_1$ defined for the lower sequent is of the form $\forall x \mathfrak{B}(x), \Gamma_3$, then we define for the upper one:

(a) $\Gamma_1=\mathfrak{B}(t), \Gamma_3$; $\Gamma_2=\Gamma_4$, $\Delta_1=\Delta_3$ and $\Delta_2=\Delta_4$, where $\Gamma_1=\forall x\mathfrak{B}(x), \Gamma_3$; $\Gamma_2=\Gamma_4$, $\Delta_1=\Delta_3$ and $\Delta_2=\Delta_4$ are the elements in (a) defined for the lower sequent.

(b) $a, a_1, \cdots, a_\nu$ and $t, t_1, \cdots, t_\nu$, where $a_1, \cdots, a_\nu$ and $t_1, \cdots, t_\nu$ are the elements in (b) defined for the lower sequent ($a$ is a new free variable).

(c) $\Pi(a, a_1, \cdots, a_\nu)=\mathfrak{C}(a, a_1, \cdots, a_\nu), \Pi^*(a_1, \cdots, a_\nu)$ and $\Sigma(a, a_1, \cdots, a_\nu)=\Sigma^*(a_1, \cdots, a_\nu)$, where $\Pi(a_1, \cdots, a_\nu)=\forall x\mathfrak{C}(x, a_1, \cdots, a_\nu), \Pi^*(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu)=\Sigma^*(a_1, \cdots, a_\nu)$ are the elements in (c) defined for the lower sequent.

(d) Elements in (d) are the same as for the lower sequent.

3.2 If $\Gamma_2$ defined for the lower sequent is of the form $\forall x\mathfrak{B}(x), \Gamma_4$, then we define for the upper one:

(a) $\Gamma_1=\Gamma_3$, $\Gamma_2=\mathfrak{B}(t), \Gamma_4$; $\Delta_1=\Delta_3$ and $\Delta_2=\Delta_4$.

(b)-(d) Elements in (b)-(d) are the same as for the lower sequent.

4. $$\frac{\Gamma\rightarrow\Delta, \mathfrak{B}(a)}{\Gamma\rightarrow\Delta, \forall x\mathfrak{B}(x)}$$

4.1 If $\Delta_1$ defined for the lower sequent is of the form $\Delta_3, \forall x\mathfrak{B}(x)$, then we define for the upper one:

(a) $\Gamma_1=\Gamma_3$, $\Gamma_2=\Gamma_4$, $\Delta_1=\Delta_3, \mathfrak{B}(a)$ and $\Delta_2=\Delta_4$, where $\Gamma_1=\Gamma_3$, $\Gamma_2=\Gamma_4$, $\Delta_1=\Delta_3, \forall x\mathfrak{B}(x)$ and $\Delta_2=\Delta_4$ are the elements in (a) defined for the lower sequent.

(b) $a, a_1, \cdots, a_\nu$ and $a, t_1, \cdots, t_\nu$, where $a_1, \cdots, a_\nu$ and $t_1, \cdots, t_\nu$ are the elements in (b) defined for the lower sequent ($a$ is the eigen-variable of the inference).

(c) $\Pi(a, a_1, \cdots, a_\nu)=\Pi^*(a_1, \cdots, a_\nu)$ and $\Sigma(a, a_1, \cdots, a_\nu)=\Sigma^*(a_1, \cdots, a_\nu), \mathfrak{C}(a, a_1, \cdots, a_\nu)$, where $\Pi(a_1, \cdots, a_\nu)=\Pi^*(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu)=\Sigma^*(a_1, \cdots, a_\nu), \forall x\mathfrak{C}(x, a_1, \cdots, a_\nu)$ are the elements in (c) defined for the lower sequent.

(d) Elements in (d) are the same as for the lower sequent.

4.2 If $\Delta_2$ defined for the lower sequent is of the form $\Delta_4, \forall x\mathfrak{B}(x)$, then we define for the upper one:

(a) $\Gamma_1=\Gamma_3$, $\Gamma_2=\Gamma_4$, $\Delta_1=\Delta_3$ and $\Delta_2=\Delta_4, \mathfrak{B}(a)$.

(b)-(d) Elements in (b)-(d) are the same as for the lower sequent.

5. $$\frac{\Gamma\rightarrow\Delta, r=s \quad \Xi\rightarrow\Omega, \mathfrak{B}(r)}{\Gamma, \Xi\rightarrow\Omega, \mathfrak{B}(s)}$$

5.1 Assume that $\Gamma_1=\Gamma_3, \Gamma_4$; $\Gamma_2=\Gamma_5, \Gamma_6$; $\Delta_1=\Delta_3, \Delta_4, \mathfrak{B}(s)$ and $\Delta_2=\Delta_5, \Delta_6$ are the elements in (a) defined for the lower sequent, where $\Gamma_3, \Gamma_5$; $\Gamma_4, \Gamma_6$; $\Delta_3, \Delta_5$ and $\Delta_4$, $\Delta_6$ are divisions of $\Gamma, \Xi, \Delta$ and $\Omega$, respectively.

5.1.1 We define for the left upper sequent:

(a) $\Gamma_1=\Gamma_3$, $\Gamma_2=\Gamma_5$, $\Delta_1=\Delta_3$ and $\Delta_2=\Delta_5, r=s$.

(b) Elements in (b) are the same as for the lower sequent.

(c) $\Pi(a_1, \cdots, a_\nu)=\Pi_1(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu)=\Sigma_1(a_1, \cdots, a_\nu)$, where $\Pi(a_1, \cdots, a_\nu)=\Pi_1(a_1, \cdots, a_\nu), \Pi_2(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu)=\Sigma_1(a_1, \cdots, a_\nu), \Sigma_2(a_1, \cdots, a_\nu), \mathfrak{C}(a_1, \cdots, a_\nu)$ are the elements in (c) defined for the lower sequent, and $\Pi_1(a_1, \cdots, a_\nu)$ and $\Sigma_1(a_1, \cdots, a_\nu)$ correspond to $\Gamma_3$ and $\Delta_3$, respectively.

(d) Elements in (d) are the same as for the lower sequent.

5.1.2 We define for the right upper sequent:

(a) $\Gamma_1=\Gamma_4$, $\Gamma_2=\Gamma_6$, $\Delta_1=\Delta_4, \mathfrak{B}(r)$ and $\Delta_2=\Delta_6$.

(b) Elements in (b) are the same as for the lower sequent.

(c) $\Pi(a_1, \cdots, a_\nu)=\Pi_2(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu)=\Sigma_2(a_1, \cdots, a_\nu), \mathfrak{C}(a_1, \cdots, a_\nu)$, where $\Pi(a_1, \cdots, a_\nu)=\Pi_1(a_1, \cdots, a_\nu), \Pi_2(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu)=\Sigma_1(a_1, \cdots, a_\nu), \Sigma_2(a_1, \cdots, a_\nu)$, $\mathfrak{C}(a_1, \cdots, a_\nu)$ are the elements in (c) defined for the lower sequent, and $\Pi_2(a_1, \cdots, a_\nu)$ and $\Sigma_2(a_1, \cdots, a_\nu), \mathfrak{C}(a_1, \cdots, a_\nu)$ correspond to $\Gamma_4$ and $\Delta_4, \mathfrak{B}(s)$, respectively.

(d) $\theta=r=s, \theta_0$, where $\theta=\theta_0$ is the element in (d) defined for the lower sequent.

5.2 Assume that $\Gamma_1=\Gamma_3, \Gamma_4$; $\Gamma_2=\Gamma_5, \Gamma_6$; $\Delta_1=\Delta_3, \Delta_4$ and $\Delta_2=\Delta_5, \Delta_6, \mathfrak{B}(s)$ are the elements in (a) defined for the lower sequent, and that $\Gamma_3, \Gamma_5$; $\Gamma_4, \Gamma_6$; $\Delta_3, \Delta_5$ and $\Delta_4, \Delta_6$ are divisions of $\Gamma, \Xi, \Delta$ and $\Omega$, respectively.

5.2.1 We define for the left upper sequent:

(a) $\Gamma_1=\Gamma_3$, $\Gamma_2=\Gamma_5$, $\Delta_1=\Delta_3$ and $\Delta_2=\Delta_5, r=s$.

(b) Elements in (b) are the same as for the lower sequent.

(c) $\Pi(a_1, \cdots, a_\nu)=\Pi_1(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu)=\Sigma_1(a_1, \cdots, a_\nu)$, where $\Pi(a_1, \cdots, a_\nu)=\Pi_1(a_1, \cdots, a_\nu), \Pi_2(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu)=\Sigma_1(a_1, \cdots, a_\nu), \Sigma_2(a_1, \cdots, a_\nu)$ are the elements in (c) defined for the lower sequent, and $\Pi_1(a_1, \cdots, a_\nu)$ and $\Sigma_1(a_1, \cdots, a_\nu)$ correspond to $\Gamma_3$ and $\Delta_3$, respectively.

(d) Elements in (d) are the same as for the lower sequent.

5.2.2 We define for the right upper sequent:

(a) $\Gamma_1=\Gamma_4$, $\Gamma_2=\Gamma_6$, $\Delta_1=\Delta_4$ and $\Delta_2=\Delta_6, \mathfrak{B}(r)$.

(b) Elements in (b) are same as for the lower sequent.

(c) $\Pi(a_1, \cdots, a_\nu)=\Pi_2(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu)=\Sigma_2(a_1, \cdots, a_\nu)$, where $\Pi(a_1, \cdots, a_\nu)=\Pi_1(a_1, \cdots, a_\nu), \Pi_2(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu)=\Sigma_1(a_1, \cdots, a_\nu), \Sigma_2(a_1, \cdots, a_\nu)$ are the elements in (c) defined for the lower sequent, and $\Pi_2(a_1, \cdots, a_\nu)$ and $\Sigma_2(a_1, \cdots, a_\nu)$ correspond to $\Gamma_4$ and $\Delta_4$, respectively.

(d) $\theta=r=s, \theta_0$, where $\theta=\theta_0$ is the element in (d) defined for the lower sequent.

6. $$\frac{\{\mathfrak{B}(0)\wedge\forall x(\mathfrak{B}(x)\supset\mathfrak{B}(x'))\}\supset\forall x\mathfrak{B}(x), \Gamma\rightarrow\Delta}{\Gamma\rightarrow\Delta}$$

6.1 If the induction formula of the inference is a formula of $L_0$, then we define for the upper sequent:

(a) $\Gamma_1=\Gamma_3$, $\Gamma_2=\{\mathfrak{B}(0)\wedge\forall x(\mathfrak{B}(x)\supset\mathfrak{B}(x'))\}\supset\forall x\mathfrak{B}(x), \Gamma_4$, $\Delta_1=\Delta_3$ and $\Delta_2=\Delta_4$, where $\Gamma_1=\Gamma_3$, $\Gamma_2=\Gamma_4$, $\Delta_1=\Delta_3$ and $\Delta_2=\Delta_4$ are the elements in (a) defined for the lower sequent.

(b)-(d) Elements in (b)-(d) are the same as for the lower sequent.

6.2 If the induction formula is not a formula of $L_0$, then we define for the upper sequent:

(a) $\Gamma_1=\{\mathfrak{B}(0)\wedge\forall x(\mathfrak{B}(x)\supset\mathfrak{B}(x'))\}\supset\forall x\mathfrak{B}(x), \Gamma_3$, $\Gamma_2=\Gamma_4$, $\Delta_1=\Delta_3$ and $\Delta_2=\Delta_4$, where $\Gamma_1=\Gamma_3$, $\Gamma_2=\Gamma_4$, $\Delta_1=\Delta_3$ and $\Delta_2=\Delta_4$ are the elements in (a) defined for the lower sequent.

(b) $b_1, \cdots, b_\mu, c_1, \cdots, c_\mu, d_1, \cdots, d_\mu, a_1, \cdots, a_\nu$ and $\mathrm{a}(s_1), \cdots, \mathrm{a}(s_\mu), \mathrm{a}(u_1), \cdots, \mathrm{a}(u_\mu), \mathrm{a}(v_1), \cdots, \mathrm{a}(v_\mu), t_1, \cdots, t_\nu$, where $a_1, \cdots, a_\nu$ and $t_1, \cdots, t_\nu$ are elements in (b) defined for the lower sequent, $b_1, \cdots, b_\mu, c_1, \cdots, c_\mu, d_1, \cdots, d_\mu$ are new distinct free variables with $\mu=\mathrm{d}(\forall x\mathfrak{B}(x))$, and $s_i=\mathrm{g}(\mathfrak{B}(0))(i)$, $u_i=\mathrm{g}(\mathfrak{B}(x))(i)$ and $v_i=\mathrm{g}(\mathfrak{B}(x'))(i)$.

(c) $\Pi(b_1, \cdots, b_\mu, c_1, \cdots, c_\mu, d_1, \cdots, d_\mu, a_1, \cdots, a_\nu)=\mathfrak{C}(b_1, \cdots, b_\mu, c_1, \cdots, c_\mu, d_1, \cdots, d_\mu)$, $\Pi^*(a_1, \cdots, a_\nu)$ and $\Sigma(b_1, \cdots, b_\mu, c_1, \cdots, c_\mu, d_1, \cdots, d_\mu, a_1, \cdots, a_\nu)=\Sigma^*(a_1, \cdots, a_\nu)$, where $\Pi(a_1, \cdots, a_\nu)=\Pi^*(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu)=\Sigma^*(a_1, \cdots, a_\nu)$ are the elements in (c) defined for the lower sequent, and $\mathfrak{C}(b_1, \cdots, b_\mu, c_1, \cdots, c_\mu, d_1, \cdots, d_\mu)=\mathrm{c}(\{\mathfrak{B}(0)\wedge\forall x(\mathfrak{B}(x)\supset\mathfrak{B}(x'))\}\supset\forall x\mathfrak{B}(x), \alpha, \langle\ \rangle)$ with $\alpha=\langle b_1, \cdots, b_\mu, c_1, \cdots, c_u, d_1, \cdots, d_\mu, c_1, \cdots, c_\mu\rangle$.

(d) Elements in (d) are the same as for the lower sequent.

7. We omit the definitions for the remaining cases. It can easily be understood how to treat them.

For each sequent $\mathfrak{S}$ in $\mathfrak{P}_n$, we define a formula $\phi_{\mathfrak{S}}$ of $L_0$, by induction from beginning sequents down to the end-sequent, as follows:

1. For a beginning sequent $P(t_1, \cdots, t_\mu)\rightarrow P(t_1, \cdots, t_\mu)$:

1.2 If $\Gamma_1=P(t_1, \cdots, t_\mu)$ and $\Delta_1=P(t_1, \cdots, t_\mu)$ for the sequent, then

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=u_1(a_1, \cdots, a_\nu)=v_1(a_1, \cdots, a_\nu)\wedge\cdots\wedge u_\mu(a_1, \cdots, a_\nu)=v_\mu(a_1, \cdots, a_\nu),$$

where $\Pi(a_1, \cdots, a_\nu)=P(u_1(a_1, \cdots, a_\nu), \cdots, u_\mu(a_1, \cdots, a_\nu))$ and $\Sigma(a_1, \cdots, a_\nu)=P(v_1(a_1, \cdots, a_\nu), \cdots, v_\mu(a_1, \cdots, a_\nu))$ are the elements in (c) defined for the sequent.

1.2 If $\Gamma_1=P(t_1, \cdots, t_\mu)$ and $\Delta_1=\phi$ for the sequent (in this case, from the fact that formulas in $\Delta_2$ are those of $L_0$, we can conclude that $\mu=2$ and $P$ is the equality symbol), then

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=\neg u(a_1, \cdots, a_\nu)=v(a_1, \cdots, a_\nu),$$

where $\Pi(a_1, \cdots, a_\nu)=u(a_1, \cdots, a_\nu)=v(a_1, \cdots, a_\nu)$ and $\Sigma(a_1, \cdots, a_\nu)=\phi$ are the elements in (c) defined for the sequent.

1.3 If $\Gamma_1=\phi$ and $\Delta_1=P(t_1, \cdots, t_\mu)$ for the sequent, then

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=u(a_1, \cdots, a_\nu)=v(a_1, \cdots, a_\nu),$$

where $\Pi(a_1, \cdots, a_\nu)=\phi$ and $\Sigma(a_1, \cdots, a_\nu)=u(a_1, \cdots, a_\nu)=v(a_1, \cdots, a_\nu)$ are the elements in (c) defined for the sequent.

1.4 If $\Gamma_1=\Delta_1=\phi$, then $\phi_{\mathfrak{S}}$ is not defined.

2. $$\frac{\Gamma\to\Delta,\mathfrak{B}}{\neg\mathfrak{B},\Gamma\to\Delta}$$

2.1 If $\phi_{\mathfrak{S}'}$ has been defined for the upper sequent, then $\phi_{\mathfrak{S}}=\phi_{\mathfrak{S}'}$ for the lower one.

2.2 If $\phi_{\mathfrak{S}'}$ is not defined for the upper sequent, then $\phi_{\mathfrak{S}}$ is not defined also for the lower one.

3. $$\frac{\mathfrak{B}(t),\Gamma\to\Delta}{\forall x\mathfrak{B}(x),\Gamma\to\Delta}$$

3.1 If the chief formula $\forall x\mathfrak{B}(x)$ belongs to $\Gamma_1$ defined for the lower sequent, and $\phi_{\mathfrak{S}'}(a, a_1, \cdots, a_\nu)$ has been defined for the upper one, then

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=\exists x\phi_{\mathfrak{S}'}(x, a_1, \cdots, a_\nu)$$

for the lower one.

3.2 If the chief formula $\forall x\mathfrak{B}(x)$ belongs to $\Gamma_2$ defined for the lower sequent, and $\phi_{\mathfrak{S}'}(a_1, \cdots, a_\nu)$ has been defined for the upper one, then

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=\phi_{\mathfrak{S}'}(a_1, \cdots, a_\nu)$$

for the lower one.

3.3 If $\phi_{\mathfrak{S}'}$ is not defined for the upper sequent, then $\phi_{\mathfrak{S}}$ is not defined also for the lower one.

4. $$\frac{\Gamma\to\Delta,\mathfrak{B}(a)}{\Gamma\to\Delta,\forall x\mathfrak{B}(x)}$$

4.1 If the chief formula $\forall x\mathfrak{B}(x)$ belongs to $\Delta_1$ defined for the lower sequent, and $\phi_{\mathfrak{S}'}(a, a_1, \cdots, a_\nu)$ has been defined for the upper one, then

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=\forall x\phi_{\mathfrak{S}'}(x, a_1, \cdots, a_\nu)$$

for the lower one.

4.2 If the chief formula $\forall x\mathfrak{B}(x)$ belongs to $\Delta_2$ defined for the lower sequent, and $\phi_{\mathfrak{S}'}(a_1, \cdots, a_\nu)$ has been defined for the upper one, then

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=\phi_{\mathfrak{S}'}(a_1, \cdots, a_\nu)$$

for the lower one.

4.3 If $\phi_{\mathfrak{S}'}$ is not defined for the upper sequent, then $\phi_{\mathfrak{S}}$ is not defined also for the lower one.

5. $$\frac{\Gamma\to\Delta, r=s \quad \Xi\to\Omega,\mathfrak{A}(r)}{\Gamma,\Xi\to\Delta,\Omega,\mathfrak{A}(s)}$$

5.1 If $\phi_{\mathfrak{S}_1}(a_1, \cdots, a_\nu)$ and $\phi_{\mathfrak{S}_2}(a_1, \cdots, a_\nu)$ have been defined for the left and right upper sequents, respectively, then

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=\phi_{\mathfrak{S}_1}(a_1, \cdots, a_\nu)\vee\phi_{\mathfrak{S}_2}(a_1, \cdots, a_\nu)$$

for the lower one.

5.2 If $\phi_{\mathfrak{S}_1}(a_1, \cdots, a_\nu)$ has been defined for the left upper sequent, but $\phi_{\mathfrak{S}_2}$ is not defined for the right upper one, then

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=\phi_{\mathfrak{S}_1}(a_1, \cdots, a_\nu)$$

for the lower one.

5.3 If $\phi_{\mathfrak{S}_2}(a_1, \cdots, a_\nu)$ has been defined for the right upper sequent, but $\phi_{\mathfrak{S}_1}$ is not defined for the left upper one, then

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=\phi_{\mathfrak{S}_2}(a_1, \cdots, a_\nu)$$

for the lower one.

5.4 If $\phi_{\mathfrak{S}_1}$ and $\phi_{\mathfrak{S}_2}$ are not defined for the left and right upper sequents, respectively, then $\phi_{\mathfrak{S}}$ is not defined also for the lower one.

6. $$\frac{\{\mathfrak{B}(0)\wedge\forall x(\mathfrak{B}(x)\supset\mathfrak{B}(x'))\}\supset\forall x\mathfrak{B}(x),\ \Gamma\rightarrow\Delta}{\Gamma\rightarrow\Delta}$$

6.1 If the induction formula of the inference is a formula of $L_0$, and $\phi_{\mathfrak{S}'}(a_1, \cdots, a_\nu)$ has been defined for the upper sequent, then

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=\phi_{\mathfrak{S}'}(a_1, \cdots, a_\nu)$$

for the lower one.

6.2 If $\phi_{\mathfrak{S}'}$ is not defined for the upper sequent, then $\phi_{\mathfrak{S}}$ is not defined also for the lower one.

6.3 If the induction formula is not a formula of $L_0$, and $\phi_{\mathfrak{S}'}(b_1, \cdots, b_\mu, c_1, \cdots, c_\mu, d_1, \cdots, d_\mu, a_1, \cdots, a_\nu)$ has been defined for the upper sequent, then

$$\begin{aligned}\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=\exists x_1\cdots\exists x_\mu\exists y_1\cdots\exists y_\mu\exists z_1\cdots\exists z_\mu\\ \{\phi_{\mathfrak{S}'}(x_1, \cdots, x_\mu, y_1, \cdots, y_\mu, z_1, \cdots, z_\mu, a_1, \cdots, a_\nu)\\ \wedge x_{j_1}=y_{j_1}=z_{j_1}\wedge\cdots\wedge x_{j_\lambda}=y_{j_\lambda}=z_{j_\lambda}\\ \wedge x_{i_1}=y_{i_1}\wedge\cdots\wedge x_{i_\kappa}=y_{i_\kappa}\wedge z_{i_1}=y_{i_1}+\bar{m}_1\wedge\cdots\wedge z_{i_\kappa}=y_{i_\kappa}+\bar{m}_\kappa\}\end{aligned}$$

for the lower one, where $\mathrm{e}(\mathfrak{B}(x), x)=\langle i_1, \cdots, i_\kappa\rangle$, $\mathrm{f}(\mathfrak{B}(x), x)=\langle j_1, \cdots, j_\lambda\rangle$, and $m_k$ is the number of occurrences of $x$ in $\mathrm{g}(\mathfrak{B}(x))(i_k)(k=1, \cdots, \kappa)$.

7. $$\frac{\Gamma\rightarrow\Delta, \mathfrak{B}\quad \Gamma\rightarrow\Delta, \mathfrak{C}}{\Gamma\rightarrow\Delta, \mathfrak{B}\wedge\mathfrak{C}}$$

7.1 If $\mathfrak{B}\wedge\mathfrak{C}$ belongs to $\Delta_1$ defined for the lower sequent, and $\phi_{\mathfrak{S}_1}(a_1, \cdots, a_\nu)$ and $\phi_{\mathfrak{S}_2}(a_1, \cdots, a_\nu)$ have been defined for the left and right upper ones, respectively, then

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=\phi_{\mathfrak{S}_1}(a_1, \cdots, a_\nu)\wedge\phi_{\mathfrak{S}_2}(a_1, \cdots, a_\nu)$$

for the lower one.

7.2 If $\mathfrak{B}\wedge\mathfrak{C}$ belongs to $\Delta_2$, and $\phi_{\mathfrak{S}_1}(a_1, \cdots, a_\nu)$ and $\phi_{\mathfrak{S}_2}(a_1, \cdots, a_\nu)$ have been defined for the left and right upper sequents, respectively, then

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=\phi_{\mathfrak{S}_1}(a_1, \cdots, a_\nu)\vee\phi_{\mathfrak{S}_2}(a_1, \cdots, a_\nu)$$

for the lower one.

7.3 If $\mathfrak{B}\wedge\mathfrak{C}$ belongs to $\varDelta_2$, and $\phi_{\mathfrak{S}'}$ has been defined for one of the upper sequents, but $\phi_{\mathfrak{S}}$ is not defined for another one, then

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)=\phi_{\mathfrak{S}'}(a_1, \cdots, a_\nu)$$

for the lower one.

7.4 If $\mathfrak{B}\wedge\mathfrak{C}$ belongs to $\varDelta_1$, and $\phi_{\mathfrak{S}'}$ is not defined for one of the upper sequents, then $\phi_{\mathfrak{S}}$ is not defined also for the lower one.

7.5 If $\phi_{\mathfrak{S}'}$ is not defined for both of the upper sequents, then $\phi_{\mathfrak{S}}$ is not defined also for the lower one.

8. We omit definitions for the remainig cases. It can easily be understood how to treat them.

We can easily prove the following lemma, by induction from beginning sequents down to the end-sequent. In the induction step of the proof, we use Lemma 2.2 in the case where we consider induction inferences.

LEMMA 2.3 *Let $\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)$ be defined for a sequent $\mathfrak{S}$ in $\mathfrak{P}_n$, and let $\Pi(a_1, \cdots, a_\nu), \Sigma(a_1, \cdots, a_\nu)$ be elements in (c) defined for $\mathfrak{S}$. Then*

$$\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu), \Pi(a_1, \cdots, a_\nu)\rightarrow\Sigma(a_1, \cdots, a_\nu)$$

*is provable in N.*

The following lemma also is easy to prove by induction from beginning sequents down to the end-sequent. We use the fact that the elements in (b), (d) have the properties mentioned in (b), (d) in the definition of elements (a), (b), (c), (d). Further in the induction step of the proof, we use Lemma 2.1 in the case where we consider induction inferences.

LEMMA 2.4 (I) *Let $\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)$ be defined for a sequent $\mathfrak{S}$ in $\mathfrak{P}_n$, and let $\Gamma_2, \varDelta_2$; $t_1, \cdots, t_\nu$; $\theta$ be elements in (a), (b), (d), respectively, defined for $\mathfrak{S}$. Then*

$$\theta, \Gamma_2\rightarrow\phi_{\mathfrak{S}}(t_1, \cdots, t_\nu), \varDelta_2$$

*is provable in N.*

(II) *If $\phi_{\mathfrak{S}}$ is not defined for a sequent $\mathfrak{S}$ in $\mathfrak{P}_n$, and $\Gamma_2, \varDelta_2$ are elements in (a) defined for $\mathfrak{S}$, then $\Gamma_2\rightarrow\varDelta_2$ is provable in N.*

From Lemma 2.4 (II) and the consistency of $N$, we can conclude that $\phi_{\mathfrak{S}}(a)$ is defined for the end-sequent of $\mathfrak{P}_n$. Hence, from Lemma 2.3, we can conclude that $\phi_{\mathfrak{S}}(a)\rightarrow\mathfrak{A}(a)$ is provable in $N$. And, from Lemma 2.4 (I), we can conclude that $\rightarrow\phi_{\mathfrak{S}}(\bar{n})$ is provable in $N$.

§ 3.

In this section, we divide each equivalence class defined in § 1 into finite groups. The division is closely related to the definitions in § 2. The relation is made clear by Lemmas 3.5 and 3.6 below.

A function $\mathfrak{G}(\mathfrak{A})$ is defined inductively as follows:

1. If $\mathfrak{A}$ is an atomic formula, then $\mathfrak{G}(\mathfrak{A})=\langle\ \rangle$.

2. $\mathfrak{G}(\neg\mathfrak{A})=\mathfrak{G}(\mathfrak{A})$. 3. $\mathfrak{G}(\mathfrak{A}\wedge\mathfrak{B})=\langle 0, \mathfrak{G}(\mathfrak{A}), \mathfrak{G}(\mathfrak{B})\rangle$.

4. $\mathfrak{G}(\forall x\mathfrak{A}(x))=\langle 1, \alpha, \mathfrak{G}(\mathfrak{A}(x))\rangle$, where $\alpha$ is a finite sequence of natural numbers such that: (i) The length of $\alpha$ is $\mathrm{d}(\mathfrak{A}(x))$. (ii) $\alpha(i)$ is the number of occurrences of $x$ in $\mathrm{g}(\mathfrak{A}(x))(i)$.

An equivalence relation $\mathfrak{A};\ x_1, \cdots, x_\nu \sim^* \mathfrak{B};\ y_1, \cdots, y_\nu$ is defined inductively as follows:

1. If $\mathfrak{A}$ and $\mathfrak{B}$ are atomic formulas with the same predicate symbol, then $\mathfrak{A};\ x_1, \cdots, x_\nu \sim^* \mathfrak{B};\ y_1, \cdots, y_\nu$.

2. If $\mathfrak{A};\ x_1, \cdots, x_\nu \sim^* \mathfrak{B};\ y_1, \cdots, y_\nu$, then $\neg\mathfrak{A};\ x_1, \cdots, x_\nu \sim^* \neg\mathfrak{B};\ y_1, \cdots, y_\nu$.

3. If $\mathfrak{A}_i;\ x_1, \cdots, x_\nu \sim^* \mathfrak{B}_i;\ y_1, \cdots, y_\nu (i=1, 2)$, then $\mathfrak{A}_1\wedge\mathfrak{A}_2;\ x_1, \cdots, x_\nu \sim^* \mathfrak{B}_1\wedge\mathfrak{B}_2;\ y_1, \cdots, y_\nu$.

4. If $x$ and $y$ do not occur in $x_1, \cdots, x_\nu$ and $y_1, \cdots, y_\nu$, respectively, and $\mathfrak{A}(x);\ x, x_1, \cdots, x_\nu \sim^* \mathfrak{B}(y);\ y, y_1, \cdots, y_\nu$, then $\forall x\mathfrak{A}(x);\ x_1, \cdots, x_\nu \sim^* \forall y\mathfrak{B}(y);\ y_1, \cdots, y_\nu$.

5. If $\mathfrak{A}(x_i);\ x_1, \cdots, x_\nu \sim^* \mathfrak{B}(y_i);\ y_1, \cdots, y_\nu$, then $\forall x_i\mathfrak{A}(x_i);\ x_1, \cdots, x_\nu \sim^* \forall y_i\mathfrak{B}(y_i);\ y_1, \cdots, y_\nu$.

A relation $s;\ x_1, \cdots, x_\nu \approx t;\ y_1, \cdots, y_\nu$ is defined as follows:

1. If $s=t=0$, or $s$ and $t$ are the same free variable, then $s;\ x_1, \cdots, x_\nu \approx t;\ y_1, \cdots, y_\nu$.

2. If $s=x_i$ and $t=y_i$, then $s;\ x_1, \cdots, x_\nu \approx t;\ y_1, \cdots, y_\nu$.

3. If $s$ and $t$ are the same bound variable and do not occur in $x_1, \cdots, x_\nu, y_1, \cdots, y_\nu$, then $s;\ x_1, \cdots, x_\nu \approx t;\ y_1, \cdots, y_\nu$.

4. If $s;\ x_1, \cdots, x_\nu \approx t;\ y_1, \cdots, y_\nu$, then $s';\ x_1, \cdots, x_\nu \approx t';\ y_1, \cdots, y_\nu$.

5. If $s_i;\ x_1, \cdots, x_\nu \approx t_i;\ y_1, \cdots, y_\nu (i=1, 2)$, then $s_1+s_2;\ x_1, \cdots, x_\nu \approx t_1+t_2;\ y_1, \cdots, y_\nu$.

The above relation is extended also for formulas in the following manner;

1. If $s_i;\ x_1, \cdots, x_\nu \approx t_i;\ y_1, \cdots, y_\nu (i=1, \cdots, \mu)$, then $P(s_1, \cdots, s_\mu);\ x_1, \cdots, x_\nu \approx P(t_1, \cdots, t_\mu);\ y_1, \cdots, y_\nu$.

2. If $\mathfrak{A};\ x_1, \cdots, x_\nu \approx \mathfrak{B};\ y_1, \cdots, y_\nu$, then $\neg\mathfrak{A};\ x_1, \cdots, x_\nu \approx \neg\mathfrak{B};\ y_1, \cdots, y_\nu$.

3. If $\mathfrak{A}_i;\ x_1, \cdots, x_\nu \approx \mathfrak{B}_i;\ y_1, \cdots, y_\nu (i=1, 2)$, then $\mathfrak{A}_1\wedge\mathfrak{A}_2;\ x_1, \cdots, x_\nu \approx \mathfrak{B}_1\wedge\mathfrak{B}_2;\ y_1, \cdots, y_\nu$.

4. If $x$ and $y$ do not occur in $x_1, \cdots, x_\nu$ and $y_1, \cdots, y_\nu$, respectively, and $\mathfrak{A}(x);\ x, x_1, \cdots, x_\nu \approx \mathfrak{B}(y);\ y, y_1, \cdots, y_\nu$, then $\forall x\mathfrak{A}(x);\ x_1, \cdots, x_\nu \approx \forall y\mathfrak{B}(y);\ y_1, \cdots, y_\nu$.

5. If $\mathfrak{A}(x_i);\ x_1, \cdots, x_\nu \approx \mathfrak{B}(y_i);\ y_1, \cdots, y_\nu$, then $\forall x_i\mathfrak{A}(x_i);\ x_1, \cdots, x_\nu \approx \forall y_i\mathfrak{B}(y_i), y_1 \cdots, y_\nu$.

LEMMA 3.1 *The number of elements of the set* $\{ [\mathfrak{A}; x_1, \cdots, x_\nu] \;/\; \deg(\mathfrak{A}) \leq m$ *and all predicate symbols in* $\mathfrak{A}$ *are among* $Q_1, \cdots, Q_\mu \}$ *is less than or equal to* $\mathrm{l}(m, \nu, \mu)$, *where* $[\mathfrak{A}; x_1, \cdots, x_\nu]$ *is the equivalence class defined by* $\sim^*$ *which contains* $\mathfrak{A}; x_1, \cdots, x_\nu$, *and* $\mathrm{l}(m, \nu, \mu)$ *is the function defined by*:

$$\begin{cases} 1. \quad \mathrm{l}(0, \nu, \mu) = \mu. \\ 2. \quad \mathrm{l}(m+1, \nu, \mu) = 2 \cdot \mathrm{l}(m, \nu, \mu) + (\mathrm{l}(m, \nu, \mu))^2 + \nu \cdot \mathrm{l}(m, \nu, \mu) + \mathrm{l}(m, \nu+1, \mu). \end{cases}$$

LEMMA 3.2 *The number of elements of the set*

$$\{ \mathfrak{G}(\mathfrak{A}) \mid \deg(\mathfrak{A}) \leq m, \ \mathrm{d}(\mathfrak{A}) \leq \nu \ \textit{and} \ \mathrm{b}(\mathfrak{A}) \leq \mu \}$$

*is less than or equal to* $\mathrm{k}(m, \nu, \mu)$, *where* $\mathrm{k}(m, \nu, \mu)$ *is the function defined as follows*;

$$\begin{cases} 1. \quad \mathrm{k}(0, \nu, \mu) = 1. \\ 2. \quad \mathrm{k}(m+1, \nu, \mu) = 2 \cdot \mathrm{k}(m, \nu, \mu) + (\mathrm{k}(m, \nu, \mu))^2 + (\mu+1)^\nu \cdot \mathrm{k}(m, \nu, \mu). \end{cases}$$

LEMMA 3.3 *If* $s; \approx t;$ , *then* $s = t$.

LEMMA 3.4 *If* $\mathfrak{A}; x_1, \cdots, x_\nu \sim^* \mathfrak{B}; y_1, \cdots, y_\nu$ *and* $1 \leq i \leq \nu$, *then* $\mathrm{e}(\mathfrak{A}, x_i) = \mathrm{e}(\mathfrak{B}, y_i)$ *and* $\mathrm{f}(\mathfrak{A}, x_i) = \mathrm{f}(\mathfrak{B}, y_i)$.

LEMMA 3.5 *Assume that all bound variables which occur free in* $\mathfrak{A}(\mathfrak{B})$ *occur in* $x_1, \cdots, x_\nu (y_1, \cdots, y_\nu)$, *and that*

$$\mathfrak{A}; x_1, \cdots, x_\nu \sim^* \mathfrak{B}; y_1, \cdots, y_\nu .$$

*Assume also that, for each* $i$ *and* $j$ *with* $i \leq \mathrm{d}(\mathfrak{A})$ *and* $1 \leq j \leq \nu$, *the number of occurrences of* $x_j$ *in* $\mathrm{g}(\mathfrak{A})(i)$ *and the number of occurrences of* $y_j$ *in* $\mathrm{g}(\mathfrak{B})(i)$ *are the same. If* $\mathfrak{G}(\mathfrak{A}) = \mathfrak{G}(\mathfrak{B})$, *then*

$$\mathrm{c}(\mathfrak{A}, \alpha, \xi); \ x_1, \cdots, x_\nu \approx \mathrm{c}(\mathfrak{B}, \alpha, \eta); \ y_1, \cdots, y_\nu ,$$

*where* $\alpha = \langle a_1, \cdots, a_\mu \rangle, \xi = \langle x_1, \cdots, x_\nu \rangle$ *and* $\eta = \langle y_1, \cdots, y_\nu \rangle$ *with* $\mu = \mathrm{d}(\mathfrak{A}) = \mathrm{d}(\mathfrak{B})$.

Lemma 3.3 is proved easily by induction on $s$, and Lemmas 3.4 and 3.5 are proved also by induction on $\mathfrak{A}$.

We define inductively an equivalence relation $\sim^*$ between proofs in $N^*$ in the following manner:

1. If $\mathfrak{P} = \mathfrak{B} \to \mathfrak{B}$, $\mathfrak{Q} = \mathfrak{C} \to \mathfrak{C}$, and $\mathfrak{B}; \sim^* \mathfrak{C};$, then $\mathfrak{P} \sim^* \mathfrak{Q}$.

2. If $\mathfrak{P} = \mathfrak{P}_0 / \Gamma \to \Delta$, $\mathfrak{Q} = \mathfrak{Q}_0 / \Pi \to \Sigma$, $\mathfrak{P}_0 \sim^* \mathfrak{Q}_0$, and the last inference rules of $\mathfrak{P}$ and $\mathfrak{Q}$ are the same type and not induction inference, or are induction inferences whose induction formulas are those of $L_0$, then $\mathfrak{P} \sim^* \mathfrak{Q}$.

3. If $\mathfrak{P} = \mathfrak{P}_1 \mathfrak{P}_2 / \Gamma \to \Delta$, $\mathfrak{Q} = \mathfrak{Q}_1 \mathfrak{Q}_2 / \Pi \to \Sigma$, $\mathfrak{P}_1 \sim^* \mathfrak{Q}_1$, $\mathfrak{P}_2 \sim^* \mathfrak{Q}_2$, and the last inference rules of $\mathfrak{P}$ and $\mathfrak{Q}$ are the same type, then $\mathfrak{P} \sim^* \mathfrak{Q}$.

4. If $\mathfrak{P} = \dfrac{\{\mathfrak{B}(0) \wedge \forall x(\mathfrak{B}(x) \supset \mathfrak{B}(x'))\} \supset \forall x \mathfrak{B}(x),\ \Gamma \to \Delta}{\Gamma \to \Delta}$,

$$\mathfrak{Q} = \frac{\{\mathfrak{C}(0) \wedge \forall y(\mathfrak{C}(y) \supset \mathfrak{C}(y'))\} \supset \forall y \mathfrak{C}(y),\ \Pi \to \Sigma}{\Pi \to \Sigma},$$

$\{\mathfrak{B}(0) \wedge \forall x(\mathfrak{B}(x) \supset \mathfrak{B}(x'))\} \supset \forall x \mathfrak{B}(x);\ \sim^* \{\mathfrak{C}(0) \wedge \forall y(\mathfrak{C}(y) \supset \mathfrak{C}(y'))\} \supset \forall y \mathfrak{C}(y)$ ; , $\mathfrak{G}(\{\mathfrak{B}(0) \wedge \forall x(\mathfrak{B}(x) \supset \mathfrak{B}(x'))\} \supset \forall x \mathfrak{B}(x)) = \mathfrak{G}(\{\mathfrak{C}(0) \wedge \forall y(\mathfrak{C}(y) \supset \mathfrak{C}(y'))\} \supset \forall y \mathfrak{C}(y))$, and $\mathfrak{P}_0 \sim^* \mathfrak{Q}_0$, then $\mathfrak{P} \sim^* \mathfrak{Q}$.

*Second division*: *We divide the set* $\{\mathfrak{P}_n / n \in \omega\}$ *by the above equialence relation* $\sim^*$. Note that the set is divided into finite groups because of Lemmas 3.1, 3.2 and the assumptions on $\mathfrak{P}_n (n \in \omega)$.

The following lemma is proved by induction from the end-sequent up to beginning sequents. In the induction step of the proof, we use Lemma 3.5 in the case where we consider induction inference.

LEMMA 3.6 *Assume that* $\mathfrak{P}_n \sim^* \mathfrak{P}_m$, *and that a sequent* $\mathfrak{S}$ *in* $\mathfrak{P}_n$ *corresponds to a sequent* $\mathfrak{S}'$ *in* $\mathfrak{P}_m$. *Let* $\Pi(a_1, \cdots, a_\nu), \Sigma(a_1, \cdots, a_\nu)$ *and* $\Pi'(b_1, \cdots, b_\mu), \Sigma'(b_1, \cdots, b_\mu)$ *be elements in* (c) *defined for* $\mathfrak{S}$ *and* $\mathfrak{S}'$, *respectively. Then* $\nu = \mu, a_1 = b_1, \cdots, a_\nu = b_\mu$, *and, for any formula* $\mathfrak{B}$ *in* $\Pi(\bar{a}), \Sigma(\bar{a})$, *there exists a formula* $\mathfrak{C}$ *in* $\Pi'(\bar{b}), \Sigma'(\bar{b})$ *corresponding to* $\mathfrak{B}$ *with* $\mathfrak{B};\ \approx \mathfrak{C};$ .

The following lemma is proved by induction from beginning sequents down to the end-sequent. We use Lemmas 3.3 and 3.6 in the basis of the proof where we consider beginning sequents. In the induction step of it, we use Lemma 3.4 in the case where we consider induction inference.

LEMMA 3.7 *Assume that* $\mathfrak{P}_n \sim^* \mathfrak{P}_m$, *and that a sequent* $\mathfrak{S}$ *in* $\mathfrak{P}_n$ *corresponds to a sequent* $\mathfrak{S}'$ *in* $\mathfrak{P}_m$. *If* $\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)$ *is defined for* $\mathfrak{S}$, *then the same* $\phi_{\mathfrak{S}}(a_1, \cdots, a_\nu)$ *is defined also for* $\mathfrak{S}'$.

Let $\Omega_1, \cdots, \Omega_\kappa$ be the finite groups of $\mathfrak{P}_n (n \in \omega)$ defined by the second division. Let $\phi_\nu(a)$ denote the same formula $\phi_{\mathfrak{S}}(a)$ defined for the end-sequents of the proofs in $\Omega_\nu$. By the discussion in the end of § 2,

$$\forall x(\phi_1(x) \vee \cdots \vee \phi_\kappa(x))$$

is valid, and

$$\forall x\{(\phi_1(x) \vee \cdots \vee \phi_\kappa(x)) \supset \mathfrak{A}(x)\}$$

is provable in $N$. Hence the desired theorem can be easily derived from the following famous result.

LEMMA 3.8 *Every valid formula of* $L_0$ *is provable in* $N$.