TSUKUBA J. MATH.
Vol. 1 (1977), 149—156

A CORRESPONDENCE BETWEEN OBSERVABLE HOPE IDEALS AND LEFT COIDEAL SUBALGEBRAS

By
Hiroshi SHIGANO

1. Introduction. Let H be a commutative Hopf algebra over a field k with the antipode S. A Hopf ideal I of H is observable if it satisfies the following condition; for any finite dimensional right (resp. left) H/I—comodule V, there exists a right (resp. left) H-comodule W with the the structure map λ_W (resp. ρ_W) and an injective H/I—comodule map θ: $V\rightarrow W$, viewing W as a right (resp. left) H/I—comodule via

$$W \xrightarrow{\lambda_W} W\otimes H \xrightarrow{1\otimes\pi} W\otimes H/I$$

$$(\text{resp. } W \xrightarrow{\rho_W} H\otimes W \xrightarrow{\pi\otimes 1} H/I\otimes W),$$

where, in the following too, π: $H\longrightarrow H/I$ is a canonical Hopf algebra map and $\otimes$ means a tensor product over k. If G is an affine algebraic group defined over k and K is its closed subgroup defined over k, then $I=I(K)$, the ideal of the definition for K, is observable in $H=k[G]$, the coordinate ring of G over k, if and only if K is an observable subgroup of G in sense of [1].

A subalgebra of H which is also a left coideal of H is called a left coideal subalgebra.

In this paper, we give a bijective correspondence between observable Hopf ideals and left coideal subalgebras A which satisfies

$$(*) \qquad A=\mathrm{Ker}\,(H \underset{in_2}{\overset{in_1}{\rightrightarrows}} H\otimes_A H)$$

and a canonical construction of W from V. In the last section where we assume that a ground field k is algebraically closed and H is a finitely generated domain over k as an algebra (which we call an affine Hopf domain), we show that the condition (*) on A is equivalent to the fact that if $a\in A$ is a unit in H, then a is a unit in A. Moreover, in this case, A is finitely generated over k as an algebra (which we call an affine k-algebra as usual.) [2] shows that A is affine if and only

Received September 8, 1977. Revised November 15, 1977

if the observable subgroup K of G satisfies "*the codimention* 2 *condition on* G/K." Thus our result says that every observable subgroup automatically satisfies "*the codimention* 2 *condition on* G/K." As a corollary, we get if G is an affine algebraic group over k and K is any closed subgroup of G, then $k[G]^K$ is an affine k-algebra.

2. A bijective correspondence. The structure maps, $m, \mu, \Delta, \varepsilon$ and S of H are a multiplication, a unit, a comultiplication, a counit and an antipode respectively as usual.

For any Hopf ideal I of H, we set

$$L(I)=\mathrm{Ker}(H \underset{in}{\overset{\sigma_H}{\rightrightarrows}} H\otimes H/I)$$

where σ_H: $H\overset{\Delta}{\longrightarrow}H\otimes H\overset{1\otimes\pi}{\longrightarrow}H\otimes H/I$, and, for any left coideal subalgebra A of H,

$$J(A)=A^+H=\text{the ideal of } H \text{ generated by } A^+=A\cap\mathrm{Ker}\,\varepsilon\,.$$

Then we get that $L(I)$ is a left coideal subalgebra of H and $J(A)$ is a Hopf ideal of H. In fact, we have only to show that $L(I)$ is a left coideal and $J(A)$ is closed under the antipode. For any $h\in L(I)$, the subcoalgebra of H generated by h is of finite dimension. Let $h_1=h, h_2, \cdots, h_n$ be its k-basis. If we write

$$\Delta(h_i)=\sum_{j=1}^{n} h_j\otimes h_{ji},\quad h_{ji}\in H,$$

then

$$\sigma_H(h)=\sum_{j=1}^{n} h_j\otimes\pi(h_{j1})=h\otimes 1,$$

hence

$$\pi(h_{j1})=\begin{cases}1, & \text{if } j=1,\\ 0, & \text{otherwise}.\end{cases}$$

From the coassociativity of Δ, we get $\Delta(h_{j1})=\sum_{t=1}^{n} h_{jt}\otimes h_{t1}$, hence $\sigma_H(h_{j1})=\sum_{t=1}^{n} h_{jt}\otimes\pi(h_{t1}) =h_{j1}\otimes 1$. Therefore $h_{j1}\in L(I)$ for any j.

For any $a\in A^+$, we can write

$$\Delta(a)=a\otimes 1+\Sigma b_i\otimes c_i,\quad b_i\in H \text{ and } c_i\in A^+.$$

From the well-known equation $\mu\varepsilon=m(S\otimes 1)\Delta$, we get $S(a)=-\Sigma S(b_i)c_i$, hence $S(a)\in HA^+=J(A)$.

The following results are easy:

(1) $JL(I)\subset I$ and $LJ(A)\supset A$.

(2) If I_1 and I_2 are Hopf ideals such that $I_1\subset I_2$, then $L(I_1)\subset L(I_2)$. If A_1 and

A_2 are left coideal subalgebras such that $A_1 \subset A_2$, then $J(A_1) \subset J(A_2)$.

(3) $JLJ(A)=J(A)$ and $LJL(I)=L(I)$.

From these results, the mappings L and J give a bijective correspondence between the set $\mathcal{J}$ of Hopf ideals I of H such that $JL(I)=I$ and the set $\mathcal{L}$ of left coideal subalgebras A of H such that $LJ(A)=A$.

DEFINITION. *M is called a left (A, H)-Hopf module if M is a left A-module and a left H-comodule such that its H-comodule structure map*

$$\rho_M : M \longrightarrow H \otimes M$$

is A-linear, where $H \otimes M$ is viewed as a left A-module via

$$\rho_A = \Delta|_A : A \longrightarrow H \otimes A .$$

In the following we use the usual notation such as

$$\rho_M(x) = \sum_{(x)} x_{(1)} \otimes x_{(0)}$$

$$\Delta(h) = \sum_{(h)} h_{(2)} \otimes h_{(1)} \quad \text{etc} .$$

ρ_M is A-linear iff $\rho_M(ax) = \Sigma a_{(2)} x_{(1)} \otimes a_{(1)} x_{(0)}$ for any $x \in M$ and $a \in A$.

THEOREM 1. *Let M be a left (A, H)-Hopf module. We have an isomorphism of left (H, H)-Hopf module,*

$$\phi_M : H \otimes_A M \xrightarrow{\sim} H \otimes M / A^+ M, \quad h \otimes_A x \longmapsto \Sigma h x_{(1)} \otimes \bar{x}_{(0)} ,$$

where $H \otimes_A M$ and $H \otimes M / A^+ M$ are left H-comodules via

$$H \otimes_A M \longrightarrow H \otimes (H \otimes_A M), \quad h \otimes x \longmapsto \Sigma h_{(2)} x_{(1)} \otimes (h_{(1)} \otimes_A x_{(0)})$$

and

$$H \otimes M / A^+ M \xrightarrow{\Delta \otimes 1} H \otimes H \otimes H / A^+ M$$

respectively. In particular, since H and $B = LJ(A)$ are left (A, H)-Hopf modules, we get

$$\phi_H : H \otimes_A H \xrightarrow{\sim} H \otimes H / J(A), \text{ and}$$

$$\phi_B : H \otimes_A B \xrightarrow{\sim} H, \quad h \otimes b \longmapsto hb ,$$

PROOF. We have mutually inverse mappings

$$H \otimes ({}_A M) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} {}_A(H \otimes M)$$

where $\Psi(h \otimes x) = \Sigma h x_{(1)} \otimes x_{(0)}$ and $\Phi(h \otimes x) = \Sigma h S(x_{(1)}) \otimes x_{(0)}$. Notice that they are H-

linear and A-linear where A-module structures of them are indicated in the above diagram, moreover that the canonical mapping $H\otimes M \xrightarrow{1\otimes can.} H\otimes M/A^+M$ is not only H-linear but also A-linear viewing as ${}_A(H\otimes M)$ and ${}_A(H\otimes M/A^+M)={}_AH\otimes M/A^+M$. $H\otimes M \xrightarrow{\Psi} H\otimes M \xrightarrow{can.} H\otimes M/A^+M$ induces $\psi_M : H\otimes M \longrightarrow H\otimes M/A^+M$. $H\otimes M \xrightarrow{\Phi} H\otimes M \xrightarrow{can.} H\otimes_A M$ induces $H\otimes M/A^+M \longrightarrow H\otimes AM$ and it is obvious that they are H-linear, mutually inverse and also left H-comodule maps.

REMARK. If A is a sub-Hopf algebra of H, a (A, H)-Hopf module is introduced by [6]. (H, H)-Hopf module M is a Hopf module in sense of [5]. By this theorem,

$$\psi_M : M \xrightarrow{\sim} H\otimes_H M \xrightarrow{\sim} H\otimes M/H^+M$$

as (H, H)-Hopf modules. Let $M^H=\{x\in M|\rho_M(x)=1\otimes x\}$, then $\psi_M(M^H)=k\otimes{}_kM/H^+M$. Hence this theorem is an another form of the structure theorem of Hopf modules [5].

Notice that a similar argument is true for right coideal subalgebras of H. In particular, $S(A)$ is a right coideal subalgebra, hence we get $H\otimes_{S(A)}H\xrightarrow{\sim}H/J(A)\otimes H$.

COROLLARY. *$A\subset\mathcal{L}$, i.e. $A=LJ(A)$ iff $A=\mathrm{Ker}(H \underset{in_2}{\overset{in_1}{\rightrightarrows}} H\otimes_A H)$.*

In fact, we get a commutative diagram,

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \hookrightarrow & H & \xrightarrow{in_1-in_2} & H\otimes_A H \\
 & & \cap\downarrow & \circlearrowright & \downarrow id_H & \circlearrowright & \wr\downarrow \psi_H \\
0 & \longrightarrow & LJ(A) & \hookrightarrow & H & \xrightarrow{in-\sigma_H} & H\otimes H J(A)
\end{array}$$

where the first row is a zero sequence and the second one is exact.

In the rest of this section, we show that a Hopf ideal I is observable iff $I\subset\mathcal{J}$, i.e. $JL(I)=I$, and the canonical construction of a H-comodule W from a H/I-comodule V.

PROPOSITION 2. *If a Hopf ideal I is observable, then $I\in\mathcal{J}$.*

PROOF. Since $\sigma_H(I)\subset I\otimes H/I$, I is a right H/I-subcomodule of H. It is enough to show that $I\subset JL(I)$, hence $V\subset JL(I)$ for any finite dimensional right H/I-subcomodule V of I. Let $v_1, \cdots, v_n$ be the k-basis of V. If we write

$$\sigma_H(v_i)=\sum_{j=1}^{n} v_j\otimes b_{ji}, \quad b_{ji}\in H/J, \tag{1}$$

we get $\Delta(b_{ji})=\sum_{t=1}^{n} b_{jt}\otimes b_{ti}$ and $\varepsilon(b_{ji})=\delta_{ji}$ from the definition of a comodule. Let V^* be a dual space of V. V^* has the structure of a left H/I-comodule which is called a transposed one. If its dual basis is denoted by $v_1^*, \cdots, v_n^*$, then the structure map μ_{V^*} is given by

$$\mu_{V^*}(v_i^*)=\sum_{j=1}^{n} b_{ij}\otimes v_j^*,$$

where b_{ij}'s are those of (1). Since I is observable, there is a (finite dimensional) left H-comodule W and an injective left H/I-comodule map θ: $V^*\longrightarrow W$. If we take the k-basis of W as $w_1=\theta(v_1^*), \cdots, w_n=\theta(v_n^*), w_{n+1}, \cdots, w_N$ and write its structure map as

$$w_i \longmapsto \sum_{j=1}^{N} h_{ij}\otimes w_j,\ h_{ij}\in H,$$

then we get $\Delta(h_{ij})=\sum_{t=1}^{N} h_{it}\otimes h_{tj}$ and $\varepsilon(h_{ij})=\delta_{ij}$ as above. From the equation $(1\otimes\theta)\mu_{V^*}=(\pi\otimes 1)\Delta\theta$, we get

$$\sum_{j=1}^{n} b_{ij}\otimes\theta(v_j^*)=\sum_{j=1}^{N}\pi(h_{ij})\otimes w_j \quad (1\leq i\leq n),$$

hence

$$\pi(h_{ij})=\begin{cases} b_{ij} & (1\leq j\leq n) \\ 0 & (n<j\leq N). \end{cases} \tag{2}$$

If we show $\sum_{j=1}^{n} v_j S(h_{jt})\in L(I)^+ (1\leq t\leq N)$, then $v_i\in L(I)^+H=JL(I)(1\leq i\leq n)$, for

$$\sum_{t=1}^{n}\sum_{j=1}^{n} v_j S(h_{jt})h_{ti}=\sum_{j=1}^{n} v_j\varepsilon(h_{ji})=v_i \quad (1\leq i\leq n).$$

Now, from (2) and the equation $\mu\varepsilon=m(1\otimes S)\Delta$,

$$\begin{aligned}\sigma_H\left(\sum_{j=1}^{n} v_j S(h_{jt})\right)&=\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} v_i S(h_{st})\otimes b_{ij}\pi(S(h_{js}))\\ &=\sum_{s=1}^{n}\sum_{i=1}^{n} v_i S(h_{st})\otimes\sum_{j=1}^{n} b_{ij}S(b_{js})\\ &=\sum_{i=1}^{n} v_i S(h_{it})\otimes 1,\end{aligned}$$

and $\varepsilon(v_j)=0$ $(1\leq j\leq n)$. Therefore $\sum_{j=1}^{n} v_j S(h_{jt})\in L(I)^+$.

Notice that for a right comodule V over a coalgebra C, the structure map ρ_V: $V\longrightarrow V\otimes C$ is an injective right C-comodule map where $V\otimes C$ is a right C-comodule via $1\otimes\Delta$. If $\dim V=n<\infty$, then $V\otimes C\simeq\otimes^n C$ as a right C-comodule (which depends on the choice of the k-basis of V). Therefore to get the canonical construction of H-comodule W from H/I-comodule V, it is enough to show it when $V=H/I$.

Let A be any left coideal subalgebra of H. $H\otimes_A H$ is a right H-comodule via $1\otimes\Delta$ where $H\otimes H$ is viewed as a A-module through $A\xrightarrow{\Delta}H\otimes A$ and $H\otimes H\otimes H$, through $A\xrightarrow{\Delta^2}H\otimes H\otimes A$.

PROPOSITION 3. *Let*

$$\theta:\ H/J(A)\hookrightarrow H\otimes H/J(A)\ \underset{\sim}{\overset{\varphi_H^{-1}}{\longrightarrow}}\ H\otimes_A H,\ \textit{and}$$

$$\eta:\ H/J(A)\hookrightarrow H/J(A)\otimes H\ \overset{\sim}{\longrightarrow}\ H\otimes_{S(A)}H$$

then θ (*resp.* η) *is an injective right* (*resp. left*) $H/J(A)$-*comodule map. Therefore* $J(A)$ *is observable.*

In fact, paying attention to that φ_H is induced by Ψ as in the proof of theorem 1, we can show that θ is a required one by the routine calculation. The remark after theorem 1 suggests that a similar augument shows that η is a required one.

3. The condition $A=LJ(A)$.

In this section we assume that k is an algebraically closed field and H is an affine Hopf domain. Let A be a left coideal subalgebra of H which is an affine k-algebra. Let G be a connected affine algebraic group defined by H, K be a closed subgroup of G defined by $J(A)$, Y be an irreducible affine variety defined by A and p: $G\longrightarrow Y$ be a dominant morphism of affine varieties defined by the inclucion $A\longrightarrow H$. G acts morphically on Y and p is G-equivariant. It is well-known that $k[G/K]=k[G]^K=$the algebra of all functions on G constant on the cosets of K in G. It is easy to show $k[G]^K=LJ(A)$.

LEMMA. *Let the notations be as above, then* $p(G)$ *open in* Y *and* $p(G)\overset{\sim}{\longrightarrow}G/K$.

In fact, $p(G)$ contains an open dense subset U of Y, hence $p(G)=\bigcup_{x\in G}xU$ is open. Notice that p is really an open map. Now, if we show that p: $G\longrightarrow p(G)$ is a separable morphism whose fibres are cosets xK, then $G/K\overset{\sim}{\longrightarrow}p(G)$ from the universal property of G/K. Let e be the unit element of G, then $p^{-1}(p(e))$ is the variety defined by $H\otimes_A A/A^+\overset{\sim}{\longrightarrow}H/A^+H$. Hence $p^{-1}(p(e))=K$.

From the generic flatness, there exists $0\neq f\in A$ such that H_f is A_f-free. Since $H\otimes_A B\overset{\sim}{\longrightarrow}H, h\otimes b\longrightarrow hb$ by theorem 1, we get $B_f=A_f$. Hence their fields of quotients are equal. By the theorem 3 in [1], $k(G)^K$ coincides with the field of quotients of B. Clearly $k(G)$ is seprable over $k(G)^K$. Hence p is a separable morphism.

REMARK. K is an observable subgroup of G in sense of [1]. [1] shows that K is observable if and only if G/K is quasi-affine. This lemma and the following theorem show that there is the canonical embedding of G/K into an affine variety, namely, one defined by $k[G]^K = LJ(A)$.

THEOREM 4. *Let A be a left coideal subalgebra of H and $X(H)$ be the group of all the group-like elements of H (hence, the group of the rational characters of G). If $X(A)=X(H)\cap A$ is the subgroup of $X(H)$, then $A=LJ(A)$ and A is an affine k-algebra. For any left coideal subalgebra A of H, $LJ(A)$ satiesfies the hypothesis, hence $LJ(A)$ is always an affine k-algebra.*

We need the following lemma due to M.E. Sweedler:

LEMMA ([4], Collary 2.2). *Let H be as above, then $U(H)=U(k)\times X(H)$ and $X(H)$ is a finitely generated free abelian group, where $U(H)$ and $U(k)$ are the unit groups of H and k respectively.*

PROOF. Let $a_i \in A$ $(1\leq i\leq n)$ be a system of generators of an ideal $J(A)$. Since $X(H)$ is a finitely generated free abelian group, so is $X(A)$. Let x_j $(1\leq j\leq t)$ be a generators of $X(A)$. The left subcoideal of A generated by $a_i(1\leq i\leq n)$ and $x_j^{\pm 1}(1\leq j\leq t)$ is of finite dimension, hence the k-algebra generated by the left coideal is finitely generated over k as an algebra and also satisfies the hypothesis. Therefore we may assume A is an affine k-algebra. By the above lemma, $k[G/K]=S^{-1}A$, where

$$S=\{f\in A \mid f(y)\neq 0 \quad \text{for all } y\in p(G)\}.$$

In particular, $S\subset U(H)$. But by Sweedler's lemma and by the hypothesis on A, $S\subset U(A)$. Therefore $k[G/K]=A$.

COROLLARY. *Let G be an affine algebraic group over k and K be a closed subgroup of G. Then $k[G]^K$ is an affine k-algebra.*

In fact, we may assume that G is connected. Let $I=I(K)$ be an ideal of definition for K, then $L(I)=k[G]^K$. Since $LJL(I)=L(I)$ and $JL(I)$ is obsevable and satiesfies the hypothesis of theorem, we get the required result.

REMARKS. (1) In [2], the obsevable subgroup K of a connected affine algebraic group G is said to satisfy the codimension 2 condition on G/K iff $k[G]^K$ is affine. Hence our result shows that every obsevable subgroup automatically satisfies the codimension 2 condition on G/K.

(2) Let A be a left coideal subalgebra of H. Assume that H is faithfully flat over A. Then $0\longrightarrow A\longrightarrow H\xrightarrow{in_1-in_2} H\otimes_A H$ is exact, hence $A=LJ(A)$. By the theorem

4, A is an affine k-algebra. It follows from the faithfully flatness that $Y=p(G)$ in the lemma, hence G/K is affine. So we get the bijective correspondence between the set of left coideal subalgebras A of H such that H is faithfully flat over A and the set of closed subgroups K of G such that G/K is affine.

References

[1] Bialynicki-Birula, A., Hochschild, G. and Mostow, D.G.: "Extensions of representations of representations of algebraic linear groups." Amer. J. Math., vol. 85 (1963) pp. 131–144.

[2] Grosshans, F.: "Observable groups and Hilbert's fourteenth problem." Amer. J. Math. vol. 95 (1973) 229–253.

[3] Borel, A.: Linear Algebraic Groups. New York, W.A. Benjamine Inc., 1969.

[4] Sweedler, M.E.: "A unit theorem applied to Hopf algebras and Amitsur cohomology." Amer. J. Math., vol. 92 (1970)

[5] ———, Hopf Algebra, New York, W.A. Benjamine Inc., 1969.

[6] Takeuchi, M.: "A correspondence between Hopf ideals and sub-Hopf algebras. "Manuscripta Math. 7 (1972) pp. 251–270.