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EMBEDDING PRODUCTS IN SYMMETRIC PRODUCTS OF CONTINUA

By

Enrique Castañeda-Alvarado and Javier Sánchez-Martínez

Abstract. Let X be a continuum. For each natural number n, $F_n(X)$ is the n^{th}-symmetric product of X and X^n is the product of X with itself n times. In this paper we consider the problem of determining the continua X such that X^n can be embedded in $F_n(X)$. Moreover, we characterize finite graphs X for which X^2 is embeddable in $F_2(X)$.

1. Introduction

The symbol $\mathbf{N}$ will denote the set of positive integers. A *continuum* means a compact connected and metric space. Given a continuum X and $n \in \mathbf{N}$, we denote by X^n the product of X with itself n times with the product topology and by $F_n(X)$ the hyperspace of all nonempty subsets of X with at most n points, equipped with the Hausdorff metric (see [13, Definition 0.1, p. 1]). This is the so called *n^{th}-symmetric product of X*. It is known that $F_n(X)$ is a continuous image of X^n (see [2, p. 877]). Symmetric products were introduced by K. Borsuk and S. Ulam in [2]. They proved that, for $I = [0,1]$ and $n = 1,2,3$, $F_n(I)$ is homeomorphic to I^n; for $n \geq 4$, $F_n(I)$ is not homeomorphic to any subset of $\mathbf{R}^n$ and $F_2(S^1)$ is homeomorphic to Möbius strip, where S^1 is a simple closed curve. In [11], R. Molski proved that $F_2(I^2)$ is homeomorphic to the 4-cell and for $n \geq 3$ neither $F_n(I^2)$ nor $F_2(I^n)$ is homeomorphic to any subset of $\mathbf{R}^{2n}$. In [3], R. Bott corrected Borsuk's [1] statement that $F_3(S^1)$ is homeomorphic to $S^1 \times S^2$ by showing that $F_3(S^1)$ is homeomorphic to the 3-sphere S^3. Since $S^1 \times S^1$ can not be embedded in $F_2(S^1)$, a natural problem arises to determine continua X such

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that X^n can be embedded in $F_n(X)$. The paper consists of four section. In Section 2, we give necessary definitions. In Section 3, we characterize finite graphs X for which X^2 is embeddable in $F_2(X)$. Finally, in Section 4, we study some continua X for which X^n can be embedded in $F_n(X)$.

2. Definitions

Given a continuum Z and a subset A of Z, $\mathrm{cl}_Z(A)$, $\mathrm{int}_Z(A)$ and $\partial_Z(A)$ denote, respectively, the closure, interior and boundary of A in Z. The symbol $|A|$ denotes the cardinality of A.

By a *graph* we mean a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points. By an *edge* of a graph X we shall always mean one of those arcs. The end points of the edges of X are called *vertices* of X. Given a point $x \in X$ and $n \in \mathbf{N}$, the *order of X at x*, denoted by $ord(x, X)$, is n provided that for every $\varepsilon > 0$ there exists an open set U of X containing x such that $\mathrm{diam}(U) < \varepsilon$ and $\partial_X(U)$ consists of exactly n points (see [14, Lemma 9.7, p. 143]). For each vertex $v \in X$ we have either $ord(v, X) = 1$ if v is an *end point* of X or $ord(v, X) \geq 2$ otherwise. If $ord(v, X) \geq 3$, then v is called a *ramification point of X*. The set of all ramification points of X will be denoted by $R(X)$. By a *simple $n-od$* ($n \geq 3$) we mean a graph X with only one ramification point, exactly n end points and without circles. A simple $3-od$ will be called a simple *triod*. A *complete graph K_m* is a graph with exactly m vertices such that any two vertices are joined by an edge of the graph. Let V be the set of vertices of graph G; G is a *bipartite graph* with vertex clases V_1 and V_2 if $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and each edge of G joins a vertex of V_1 to a vertex of V_2. The graph G is said *bipartite complete* if each vertex of V_1 is joined to every vertex of V_2 by edges of G, if $|V_1| = m$ and $|V_2| = n$; G is denoted by $K_{m,n}$. Given a continuum X, the *cone over X* is the quotient space $X \times [0,1]/X \times \{1\}$ (see [9, Definition 5.1, p. 126]). The cone over X will be denoted by $\mathrm{cone}(X)$.

Given open subsets $U_1, \ldots, U_m$ of a continuum X, let

$$\langle U_1, \ldots, U_m \rangle = \left\{ A \in F_n(X) : A \subset \bigcup_{i=1}^{m} U_i \text{ and for each } i \in \{1, \ldots, m\},\ A \cap U_i \neq \emptyset \right\}.$$

It is known that the sets of the form $\langle U_1, \ldots, U_m \rangle$ form a basis for the topology of $F_n(X)$ called the *Vietoris topology* (see [13, Theorem 0.11, p. 9]), and that the Vietoris topology and the topology induced by the Hausdorff metric are the same (see [13, Theorem 0.13, p. 9]).

3. Embedding X^2 in $F_2(X)$

In this section, we study when X^2 can be embedded in $F_2(X)$, in the particular case when X is a graph. For every $n \geq 3$, T_n denotes a simple n-od.

Lemma 3.1. *Let $n \in \mathbf{N}$, with $n \geq 3$, then $T_n \times T_n$ can not be embedded in $F_2(T_n)$.*

Proof. By [5, Lemma 3.2, p. 59], $T_n \times T_n$ is homeomorphic to $\mathrm{cone}(K_{n,n})$. By [4, Lemma 2, p. 70], $F_2(T_n)$ is homeomorphic to $\mathrm{cone}(Z)$, where Z is the union of K_n and n pairwise disjoint arcs, each one of them intersecting K_n in exactly one of its vertices. Suppose, to the contrary, that there exists an embedding $h : \mathrm{cone}(K_{n,n}) \to \mathrm{cone}(Z)$. Let v and v' be the vertices of $\mathrm{cone}(K_{n,n})$ and $\mathrm{cone}(Z)$, respectively. By [5, Lemma 3.3, p. 60], each point $p \in \mathrm{cone}(Z) - \{v'\}$ has a basis of neighborhoods β in $\mathrm{cone}(Z)$ such that for each $U \in \beta$, U can be embedded in $\mathbf{R}^3$. So, $h(v) = v'$, which implies that the cylinder $K_{n,n} \times I$ embeds in the cylinder $Z \times I$. Looking at neighborhood bases of points of these two cylinders, we conclude that $K_{n,n}$ can be embedded in Z, therefore $2n = |R(K_{n,n})| \leq |R(Z)| = n$, but this is imposible. This contradiction concludes the proof of this lemma. □

The following result is obvious since $T_m \times [0,1]$ can be embedded in $\mathbf{R}^3$ while $T_k \times T_l$ can not, because $\mathrm{cone}(K_{k,l})$ can not be embedded in $\mathbf{R}^3$ (see [4, Lemma 4, p. 73]).

Lemma 3.2. *Let $k, l, m \in \mathbf{N}$, with $k, l, m \geq 3$. Then $T_k \times T_l$ can not be embedded in $T_m \times [0,1]$.*

Theorem 3.3. *If X is a graph with $|R(X)| \leq 1$, then X^2 can be embedded in $F_2(X)$ if and only if X is an arc.*

Proof. If X is an arc its second symmetric product is homeomorphic to X^2. Now, let X^2 embed in $F_2(X)$. If $|R(X)| = 0$, X is an arc or a simple closed curve (see [14, Proposition 9.5, p. 142]). Since $S^1 \times S^1$ can not be embedded in $F_2(S^1)$, X is an arc.

Suppose that $R(X) = \{p\}$ and that $h : X^2 \to F_2(X)$ is an embedding. Let $r = ord(p, X)$. So, $(p, p) \in X^2$ has a basis of neighborhoods β such that if $U \in \beta$ then U is homeomorphic to $T_r \times T_r$ (see [5, Lemma 3.4, p. 61]). Take

$\{a,b\} \in F_2(X)$ such that $h((p,p)) = \{a,b\}$. By [5, Lemma 3.3, p. 60], we have the following three cases:

Case I If $ord(a,X) \leq 2$ and $ord(b,X) \leq 2$, then $\{a,b\}$ has a basis of neighborhoods γ in $F_2(X)$, such that if $U \in \gamma$ then U is homeomorphic to $[0,1]^2$.

Case II If $ord(a,X) \leq 2$ and $b = p$, then $\{a,b\}$ has a basis of neighborhoods γ in $F_2(X)$, such that if $U \in \gamma$ then U is homeomorphic to $T_r \times [0,1]$.

Case III If $\{a,b\} = \{p\}$, then $\{a,b\}$ has a basis of neighborhoods γ in $F_2(X)$, such that if $U \in \gamma$ then U is homeomorphic to $F_2(T_r)$.

We conclude that $T_r \times T_r$ can be embedded in $[0,1]^2$, $T_r \times [0,1]$ or $F_2(T_r)$. This is a contradiction by Lemmas 3.2 and 3.1. Then X^2 can not be embedded in $F_2(X)$. □

Theorem 3.4. *Let X be a graph. If $R(X) \neq \emptyset$ then X^2 can not be embedded in $F_2(X)$.*

Proof. If $|R(X)| = 1$ it follows by Theorem 3.3. Suppose on the contrary that $R(X) = \{x_1, \dots, x_k\}$, with $k \geq 2$, and that there exists an embedding $h : X^2 \to F_2(X)$. Let $r_i = ord(x_i, X)$. It is clear that for each $i, j \in \{1, \dots, k\}$, $(x_i, x_j) \in X^2$ has a basis of neighborhoods β in X^2 such that if $U \in \beta$, then U is homeomorphic to $T_{r_i} \times T_{r_j}$.

By [5, Lemma 3.3, p. 60] and Lemma 3.2, for each $i, j \in \{1, \dots, k\}$, there exist $l, s \in \{1, \dots, k\}$ such that $h((x_i, x_j)) = \{x_l, x_s\}$. Therefore the sets $\{(x_i, x_j) \in X^2 : i, j \in \{1, \dots, k\}\}$ and

$$\{\{x_i, x_j\} \in F_2(X) : i, j \in \{1, \dots, k\}\}$$

have the same cardinality, which is impossible. □

Using the Theorems 3.3 and 3.4, we get the following result.

Corollary 3.5. *Let X be a graph. X^2 can be embedded in $F_2(X)$ if and only if X is an arc.*

4. α-Self-Homeomorphic Continua

Definition 4.1. Let α be a cardinal number and let X be a continuum. We say that X is **α-self-homeomorhic** if there exist α mutually disjoint proper subcontinua of X which are homeomorphic to X.

Remark 4.2. It is clear that every n-cell is m-self-homeomorphic for each $m, n \in \mathbf{N}$.

Proposition 4.3. *Let $n \in \mathbf{N}$ and X be a continuum. If X is n-self-homeomorphic then X^m can be embedded in $F_m(X)$ for each $m \leq n$.*

Proof. Let $m \leq n$. Since X is n-self-homeomorphic, X is m-self-homeomorphic. Therefore, there exist pairwise disjoint subcontinua $A_1, \dots, A_m$ of X, such that A_i is homeomorphic to X. Consider the map $f_m^X : X^m \to F_m(X)$ given by $f_m^X((x_1, \dots, x_m)) = \{x_1, \dots, x_m\}$ (see [2, p. 877]). So, $f_m^X|_{\prod_{i=1}^m A_i} : \prod_{i=1}^m A_i \to F_m(X)$ is an embedding. Since $\prod_{i=1}^m A_i$ is homeomorphic to X^m, we conclude that X^m can be embedded in $F_m(X)$. □

Proposition 4.4. *Let X be a continuum. X is 2-self-homeomorphic if and only if X is n-self-homeomorphic for each $n \geq 2$.*

Proof. It follows from Definition 4.1. □

As a consequence of propositions 4.3 and 4.4 we have the following result.

Corollary 4.5. *Let X be a continuum. If X is 2-self-homeomorphic then X^n can be embedded in $F_n(X)$ for each $n \in \mathbf{N}$.*

Corollary 4.6. *Let X be a graph. X is 2-self-homeomorphic if and only if X is an arc.*

Proof. It follows from Corollary 3.5 and Proposition 4.4. □

Corollary 4.7. *Let X be a continuum. If X contains a Hilbert cube, then X^n can be embedded in $F_n(X)$ for each $n \in \mathbf{N}$.*

Proof. It follows from Corollary 4.5 and [12, Theorem 1, p. 241]. □

The following definitions appear in [7, p. 217] and [6, p. 283–284].

Definition 4.8. A topological space X is **self-homeomorphic** (**strongly self-homeomorphic**, respectively) if for any open set $U \subset X$ there is a set $V \subset U$ (with nonempty interior, respectively) homeomorphic to X.

Definition 4.9. A topological space X is **pointwise self-homeomorphic** (**strongly pointwise self-homeomorphic**, respectively) at a point $x \in X$ if for any neighborhood U of x there is a set V (a neighborhood of x, respectively) such that $x \in V \subset U$ and V is homeomorphic to X. A space X is **pointwise self-homeomorphic** (**strongly pointwise self-homeomorphic**, respectively) if it is pointwise self-homeomorphic (strongly pointwise self-homeomorphic, respectively) at each point.

The following fact is easy to see, compare with [7, Theorem 2.5, p. 217].

Proposition 4.10. *Definitions* 4.1, 4.8 *and* 4.9 *satisfy*: $4.1 \Rightarrow 4.8 \Rightarrow 4.9$.

Concerning this proposition, in [7, Problems 6.21 and 6.23, p. 237] the authors asked whether X is pointwise self-homeomorphic if X is a self-homeomorphic or a strongly self-homeomorphic dendrite. A negative answer to both these questions is given in [15], where a dendrite is constructed which is strongly self-homeomorphic (at each of its points) but not pointwise self-homeomorphic (at some of its end points). In [8], the authors generalize the Pyrih's example given in [15].

Corollary 4.11. *There are uncontably many topologically different dendrites such that X^n can be embedded in $F_n(X)$ for each $n \in \mathbf{N}$.*

Proof. This is a consequence of Propositions 4.3, 4.10 and [7, Corollary 6.5, p. 230]. □

The following proposition on cartesian products is easy to see, compare with [7, Proposition 2.11, p. 219].

Proposition 4.12. *All types of self-homeomorphic spaces considered above are preserved by Cartesian products* (*of arbitrarily many factors*).

Other examples of continua such that X^n can be embedded in $F_n(X)$ are the Sierpiński universal curve, the Sierpiński triangle, the Menger curve, the familly of plane continua constructed in [6, Section 4] and by the previus proposition, cartesian products with factors being α-self-homeomorphic continua.

Example 4.13. There exists a continuum X such that for each $m, n \in \mathbf{N}$, X^n can be embedded in $F_n(X)$ but X is not m-self-homeomorphic.

Proof. Consider Y a continuum homeomorphic to the capital letter H. Let $X = Y \times I$ and $A_1, A_2, \ldots, A_n$ be pairwise disjoint cylinders in X, each one of them homeomorphic to X. Then $\langle A_1, \ldots, A_n \rangle$ is homeomorphic to X^n and is contained in $F_n(X)$, but X is not m-self-homeomorphic for every $m \in \mathbf{N}$. $\square$

Question 4.14. *Does there exist a dendrite X such that X^n can be embedded in $F_n(X)$ and X is not m-self-homeomorphic, for $m, n \in \mathbf{N}$?*

Question 4.15. *Does there exist a continuum X such that X^n can be embedded in $F_n(X)$ but X does not contain a proper subcontinuum Y homeomorphic to X?*

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Enrique Castañeda-Alvarado
Universidad Autónoma del Estado de México
Facultad de Ciencias, Instituto Literario No. 100
Col. Centro, C.P. 50000, Toluca
Estado de México, México
E-mail: eca@uaemex.mx

Javier Sánchez-Martínez
Benemérita Universidad Autónoma de Puebla
Facultad de Ciencias Físico-Matemáticas
Av. San Claudio y 18 Sur, Edificio 11A-109B
Col. San Manuel, Ciudad Universitaria
C.P. 72570, Puebla, Puebla
E-mail: jsanchezm@fcfm.buap.mx